Construction of large graphs with no optimal surjective $L(2,1)$-labelings*

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Abstract

An $L(2,1)$-labeling of a graph G is a mapping $c : V(G) \to \{0,\ldots,K\}$ such that the labels of two adjacent vertices differ by at least two and the labels of vertices at distance two differ by at least one. A hole of c is an integer $h \in \{0,\ldots,K\}$ that is not used as a label for any vertex of G. The smallest integer K for which an $L(2,1)$-labeling of G exists is denoted by $\lambda(G)$. The minimum number of holes in an optimal labeling, i.e., a labeling with $K = \lambda(G)$, is denoted by $\rho(G)$. Georges and Mauro showed that $\rho(G) \le \Delta$, where Δ is the maximum degree of G, and conjectured that if $\rho(G) = \Delta$ and G is connected, then the order of G is at most $\Delta(\Delta+1)$. We disprove this conjecture by constructing graphs G with $\rho(G) = \Delta$ and order $\left\lfloor \frac{(\Delta+1)^2}{4} \right\rfloor (\Delta+1) \approx \Delta^3/4$.

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1 Introduction

$L(2,1)$-labelings of graphs form an important model for the frequency assignment problem [5]. An $L(2,1)$-*labeling* of a graph G is a labeling $c : V(G) \to \{0,\ldots,K\}$ of the vertices of G such that the labels of any two adjacent vertices differ by at least two and the labels of any two vertices at distance two are different. The smallest K for which there exists a proper labeling of G is denoted by $\lambda(G)$.

One of the most studied problems on $L(2,1)$-labelings is the famous Delta Square Conjecture of Griggs and Yeh [4]: they conjectured that $\lambda(G) \le \Delta(G)^2$ for every graph G where $\Delta(G)$ is the maximum degree of G. Though the conjecture has been verified for several special classes of graphs, including graphs of maximum degree two, chordal graphs [11] (see also [1, 8]), and hamiltonian cubic graphs [6, 7], it remains widely open. The original upper bound $\lambda(G) \le \Delta(G)^2 + 2\Delta(G)$ by Griggs and Yeh [4] has been improved to $\lambda(G) \le \Delta(G)^2 + \Delta(G)$ in [2] (an analogous bound in a more general setting of the channel assignment problem was proved by McDiarmid [10]). A recent more general result of the first two authors yields the present record $\lambda(G) \le \Delta(G)^2 + \Delta(G) - 1$.

In this paper, we focus on surjective $L(2,1)$-labelings that was first studied in a more detail by Georges and Mauro [3]. If c is a labeling of G, then a number h, $0 \le h \le K$, is a *hole*, if there is no vertex v of G with $c(v) = h$. The minimum number of holes in an $L(2,1)$-labeling of G with $K = \lambda(G)$ is denoted by $\rho(G)$. Georges and Mauro [3] established that $\rho(G)$ never exceeds the maximum degree $\Delta(G)$ of G. In [3], Georges and Mauro also posed (among others) the following conjectures on $L(2,1)$-labelings and holes:

Conjecture 1. *If G is a r-regular graph and $\rho(G) \ge 1$, then $\rho(G)$ divides r.*

Conjecture 2. *If G is a connected graph with maximum degree $\Delta(G)$ and $\rho(G) = \Delta(G)$, then the order of G does not exceed $\Delta(\Delta + 1)$.*

Conjecture 3. *If G is a graph with $\lambda(G) > 2\Delta(G)$, then $\rho(G) = 0$. In other words, if G is a graph with $\rho(G) > 0$, then $\lambda(G) \le 2\Delta(G)$.*

In this paper, we focus on Conjecture 2. We provide a construction of connected r-regular graphs G of order $(r+1)\left\lfloor\frac{(r+1)^2}{4}\right\rfloor \approx r^3/4$ with $\rho(G) = r$ (Corollary 6). This shows that Conjecture 2 does not hold for $\Delta \ge 3$. Note that Conjecture 2 trivially holds for $\Delta = 1$ since the only graph G

satisfying the assumptions of the conjecture is K_2. In [3], it was shown that Conjecture 2 also holds for $\Delta = 2$.

2 Previous results

In this section, we survey results obtained by Georges and Mauro [3] on the structure of graphs G with $\rho(G) = \Delta(G)$. The following theorem shows that the structure of such graphs is very restricted:

Theorem 1. *If G is a graph with $\rho(G) = \Delta(G)$, then G is a Δ-regular graph with $\lambda(G) = 2\Delta$. Moreover, for every optimum $L(2,1)$-labeling c, i.e., a labeling using the labels $0, \ldots, \lambda(G)$, the following holds:*

- *every odd integer between 0 and $\lambda(G)$ is a hole of c,*
- *the cardinality of the preimage in c of every even number between 0 and $\lambda(G)$ is the same, and*
- *the subgraph of G induced by the preimages of any two even numbers is a perfect matching (union of disjoint edges).*

In particular, there exists an integer $t > 0$ such that the order of G is $(\Delta + 1)t$.

In [3], Georges and Mauro constructed connected Δ-regular graphs G with $\rho(G) = \Delta$ of order $(\Delta + 1)t$ for every $t = 1, \ldots, \Delta$. They conjectured that the number t (under the assumption that G is connected) cannot exceed Δ (this is equivalent to Conjecture 2 stated in Section 1).

We now recall a construction of an r-regular graph Ω_r of order $r(r+1)$ from [3]. Consider a union of r vertex disjoint cliques of order r and number the vertices of each clique from 1 to r. Add to the graph r new vertices and join the i-th of them to the vertices of the cliques numbered with i. The resulting graph is Ω_r. Clearly, Ω_r is a connected r-regular graph. It can be shown that $\lambda(\Omega_r) = 2r$ and $\rho(\Omega_r) = r$.

In order to show that $\lambda(\Omega_r) = 2r$ and $\rho(\Omega_r) = r$ for the graph Ω_r, Georges and Mauro [3] showed that Ω_r has a special property which we call the neighborhood property in this paper. Assume that G is a connected r-regular graph of order $(r+1)t$. We say that G has the *t-neighborhood property* if the following holds for any two (disjoint) sets V and W of vertices of G: if neither V nor W contains two vertices at distance at most two and no vertex of V is adjacent to a vertex of W, then $|V| + |W| \leq t$.

We finish this section with the following proposition whose proof is implicitly contained in [3]. We include its short proof for the sake of completeness.

Proposition 2. *If G is a connected r-regular graph of order $(r+1)t$ with $\lambda(G) \le 2r$ that has the t-neighborhood property, then $\lambda(G) = 2r$ and $\rho(G) = r$.*

Proof. Let us consider an $L(2,1)$-labeling of G with $0, \dots, 2r$ and let V_i, $i = 0, \dots, 2r$, be the set of the vertices labelled with i. Since G has the t-neighborhood property, it holds

$$|V_i| + |V_{i+1}| \le t \tag{1}$$

for every $i = 0, \dots, 2r-1$.

First, we show that $V_i = \emptyset$ for all odd i's. Let i_0 be an odd integer between 0 and $2r$ and let $\mu = |V_{i_0}|$. We now bound the sum $|V_0| + \dots + |V_{2r}|$ using (1):

$$\sum_{i=0}^{2r} |V_i| = \sum_{i=0}^{(i_0-1)/2} (|V_{2i}| + |V_{2i+1}|) + \sum_{i=(i_0+1)/2}^{r} (|V_{2i-1}| + |V_{2i}|) - |V_{i_0}| \le$$

$$(r+1)t - \mu \ .$$

Since the sets $V_0, \dots, V_{2r}$ partition the vertex set of G, the sum of their sizes is $(r+1)t$. Therefore, $\mu = 0$. Since the choice of i_0 was arbitrary, $V_i = \emptyset$ for all odd i's as claimed.

Note that $|V_i| \le t$ for every $i = 0, \dots, 2r$ by (1). Since the sum $|V_0| + \dots + |V_{2r}|$ is equal to $(r+1)t$ and the set V_i is empty for every odd i, it must hold that $|V_i| = t$ for every $i = 0, 2, 4, \dots, 2r$. The statement of the proposition now readily follows. □

3 Construction

In this section, we present our construction of graphs of order $\Theta(\Delta^3)$ with $\rho = \Delta$ (the exact parameters of the constructed graphs can be found in Theorem 5). First, we describe the considered graphs in Subsection 3.1. In Subsection 3.2, we analyze their properties. Finally, we slightly generalize our construction to obtain additional graphs with similar properties in Subsection 3.3.

3.1 The Graph

In this subsection, we construct an $(\alpha+\beta-1)$-regular connected graph $\Gamma_{\alpha,\beta}$ of order $(\alpha+\beta)\alpha\beta$ with $\rho(\Gamma_{\alpha,\beta}) = \Delta(\Gamma_{\alpha,\beta}) = \alpha+\beta-1$. The vertex set of $\Gamma_{\alpha,\beta}$ is comprised of two sets V_g and V_r:

$$\begin{aligned} V_g &= \{[a,b,\overline{a}] \mid 1 \le a \le \alpha, 1 \le b \le \beta \ \& \ 1 \le \overline{a} \le \alpha\}, \text{ and} \\ V_r &= \{[a,b,\overline{b}] \mid 1 \le a \le \alpha, 1 \le b \le \beta \ \& \ 1 \le \overline{b} \le \beta\}. \end{aligned}$$

Note that $|V_g| = \alpha^2\beta$ and $|V_r| = \alpha\beta^2$. The vertices of V_g are later referred to as *green* and those of V_r as *red*.

We now describe the edge set of $\Gamma_{\alpha,\beta}$. Two distinct green vertices $[a,b,\overline{a}]$ and $[a',b',\overline{a}']$ are joined by an edge if $b=b'$ and $\overline{a}=\overline{a}'$. Similarly, two distinct red vertices $[a,b,\overline{b}]$ and $[a',b',\overline{b}']$ are joined by an edge if $a=a'$ and $\overline{b}=\overline{b}'$. A green vertex $[a,b,\overline{a}]$ and a red vertex $[a',b',\overline{b}']$ are joined by an edge if $a=a'$ and $b=b'$.

Notice the following: the subgraph of $\Gamma_{\alpha,\beta}$ induced by the green vertices is comprised of $\alpha\beta$ cliques of order α, the subgraph induced by the red vertices of $\alpha\beta$ cliques of order β, and the spanning subgraph containing edges between the red and green vertices is comprised of $\alpha\beta$ complete bipartite graphs isomorphic to $K_{\alpha,\beta}$. It is not hard to verify that the graph $\Gamma_{\alpha,\beta}$ is connected, its order is $(\alpha+\beta)\alpha\beta$ and it is $(\alpha+\beta-1)$-regular. Examples of graphs $\Gamma_{\alpha,\beta}$ for some (small) values of α and β are given in Figure 1. Note that the graph $\Gamma_{\alpha,1}$ is isomorphic to the graph Ω_α. Also note that the graphs $\Gamma_{\alpha,\beta}$ and $\Gamma_{\beta,\alpha}$ are isomorphic for all $\alpha,\beta \ge 1$.

3.2 Analysis

In this subsection, we analyze properties of the graphs $\Gamma_{\alpha,\beta}$. First, we show an upper bound on $\lambda(\Gamma_{\alpha,\beta})$:

Proposition 3. *For every $\alpha,\beta \ge 1$, the number $\lambda(\Gamma_{\alpha,\beta})$ does not exceed $2\alpha+2\beta-2$.*

Proof. We partition green vertices into α independent sets $V_1,\ldots,V_\alpha$ and red vertices into β independent sets $W_1,\ldots,W_\beta$. A green vertex $[a,b,\overline{a}]$ is contained in the set V_i where i is congruent to $a+\overline{a}$ modulo α. A red vertex $[a,b,\overline{b}]$ is contained in the set W_i where i is congruent to $b+\overline{b}$ modulo β.

Clearly, the sets $V_1,\ldots,V_\alpha$ and $W_1,\ldots,W_\beta$ are independent. We claim that the distance between any two vertices contained in the same set is at

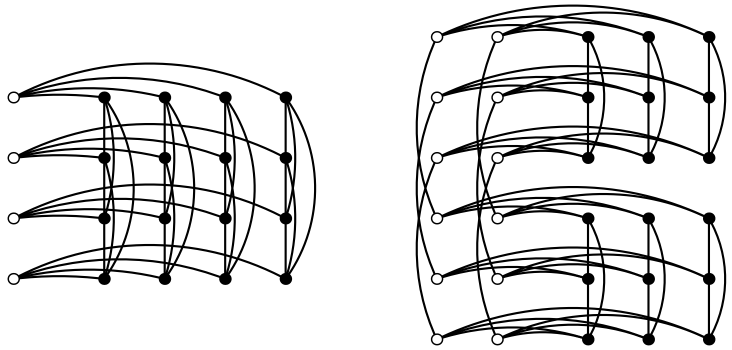

Figure 1: The graphs $\Gamma_{1,4}$ and $\Gamma_{2,3}$. Green vertices are depicted by empty circles and red vertices by full ones.

least three: assume the opposite. By symmetry, it is enough to consider the case when V_1 contains two distinct green vertices $[a, b, \overline{a}]$ and $[a', b', \overline{a}']$ at distance two. If the common neighbor of $[a, b, \overline{a}]$ and $[a', b', \overline{a}']$ is a green vertex, then $b = b'$ and $\overline{a} = \overline{a}'$. By the definition of V_1, it follows that $a = a'$ and the vertices $[a, b, \overline{a}]$ and $[a', b', \overline{a}']$ are not distinct. On the other hand, if their common neighbor is a red vertex, then $a = a'$ and $b = b'$. The definition of V_1 now yields that $\overline{a} = \overline{a}'$ and the vertices $[a, b, \overline{a}]$ and $[a', b', \overline{a}']$ are not distinct as supposed.

We now construct an $L(2,1)$-labeling of $\Gamma_{\alpha,\beta}$. Label the vertices of V_i, $i = 1, \ldots, \alpha$, by the number $2i - 2$ and the vertices of W_i, $i = 1, \ldots, \beta$, by the number $2\alpha + 2i - 2$. The obtained labeling is a proper $L(2,1)$-labeling of $\Gamma_{\alpha,\beta}$. In particular, $\lambda(\Gamma_{\alpha,\beta}) \leq 2\alpha + 2\beta - 2$. $\square$

We now establish the key proeprty of graphs $\Gamma_{\alpha,\beta}$:

Lemma 4. *For every $\alpha, \beta \geq 1$, the graph $\Gamma_{\alpha,\beta}$ has the $\alpha\beta$-neighborhood property.*

Proof. Fix $\alpha \geq 1$ and $\beta \geq 1$ and let us consider two sets V_1 and V_2 of vertices of $\Gamma_{\alpha,\beta}$. Assume that V_1 contains no two vertices at distance at most two, V_2 no two vertices at distance at most two, and no two vertices of V_1 and V_2 are adjacent. We show that $|V_1| + |V_2| \leq \alpha\beta$. The statement of the lemma would then follow.

Let us construct two auxiliary matrices M_g and M_r of type $\alpha \times \beta$. For a, $1 \le a \le \alpha$, and b, $1 \le b \le \beta$, the entry $M_g[a,b]$ is the number of green vertices of the form $[a,b,\overline{a}]$, $1 \le \overline{a} \le \alpha$, contained in $V_1 \cup V_2$. Similarly, the entry $M_r[a,b]$ is the number of red vertices of the form $[a,b,\overline{b}]$, $1 \le \overline{b} \le \beta$, contained in $V_1 \cup V_2$. Next, several properties of the matrices M_g and M_r are established. We formulate the properties as a series of claims:

Claim 4.1. *All the entries of the matrices M_g and M_r are integers 0, 1 or 2.*

For fixed numbers a, $1 \le a \le \alpha$, and b, $1 \le b \le \beta$, all the green vertices $[a,b,\overline{a}]$, $1 \le \overline{a} \le \alpha$, of $\Gamma_{\alpha,\beta}$ are at distance two. In particular, at most one of them is contained in the set V_1 and at most one of them in V_2. Hence, $M_g[a,b] \le 2$. A symmetric argument applies to M_r.

Claim 4.2. *For every a, $1 \le a \le \alpha$, and b, $1 \le b \le \beta$, at most one of the entries $M_g[a,b]$ and $M_r[a,b]$ is non-zero.*

If $M_g[a,b] > 0$, then there is a green vertex $[a,b,\overline{a}]$, $1 \le \overline{a} \le \alpha$, that is contained in $V_1 \cup V_2$. Since every red vertex $[a,b,\overline{b}]$, $1 \le \overline{b} \le \beta$, is adjacent to the green vertex $[a,b,\overline{a}]$ contained in $V_1 \cup V_2$, no red vertex of the form $[a,b,\overline{b}]$, $1 \le \overline{b} \le \beta$, is contained in $V_1 \cup V_2$. Hence, $M_r[a,b]$ is equal to zero. An analogous argument yields that if $M_r[a,b] > 0$, then $M_g[a,b] = 0$.

Claim 4.3. *If $M_g[a,b] = 2$ for a, $1 \le a \le \alpha$, and b, $1 \le b \le \beta$, then $M_r[a',b] = M_r[a,b'] = 0$ for every a', $1 \le a' \le \alpha$, and b', $1 \le b' \le \beta$.*

Let $\overline{a}_1$ and $\overline{a}_2$, $1 \le \overline{a}_1, \overline{a}_2 \le \alpha$, be two distinct integers such that both green vertices $[a,b,\overline{a}_1]$ and $[a,b,\overline{a}_2]$ are contained in $V_1 \cup V_2$. Since the distance between the vertices $[a,b,\overline{a}_1]$ and $[a,b,\overline{a}_2]$ is two, one of them is contained in V_1 and the other in V_2. By symmetry, we can assume that $[a,b,\overline{a}_1] \in V_1$ and $[a,b,\overline{a}_2] \in V_2$.

Let us consider integers a', $1 \le a' \le \alpha$, and $\overline{b}$, $1 \le \overline{b} \le \beta$. If $a = a'$, then $M_r[a',b] = 0$ by Claim 4.2. In the rest, we consider the case $a \ne a'$. The green vertices $[a',b,\overline{a}_1]$ and $[a',b,\overline{a}_2]$ are neighbors of the green vertices $[a,b,\overline{a}_1]$ and $[a,b,\overline{a}_2]$, respectively. Since the red vertex $[a',b,\overline{b}]$ is a neighbor of both the green vertices $[a',b,\overline{a}_1]$ and $[a',b,\overline{a}_2]$, the vertex $[a',b,\overline{b}]$ can be included neither in V_1 nor in V_2. Since the choice of $\overline{b}$ was arbitrary, $M_r[a',b]$ must be equal to zero for every $a' \ne a$, $1 \le a \le \alpha$. A symmetric argument yields that $M_r[a,b'] = 0$ for every b', $1 \le b' \le \beta$.

Claim 4.4. *If $M_r[a,b] = 2$ for a, $1 \le a \le \alpha$, and b, $1 \le b \le \beta$, then $M_g[a',b] = M_g[a,b'] = 0$ for every a', $1 \le a' \le \alpha$, and b', $1 \le b' \le \beta$.*

The proof is analogous to the proof of Claim 4.3.

Claim 4.5. *For every a, $1 \le a \le \alpha$, the sum of the entries of M_r on the a-th row is at most β.*

For every $\overline{b}$, $1 \le \overline{b} \le \beta$, the vertices $[a,b,\overline{b}]$, $1 \le b \le \beta$, form a clique in $\Gamma_{\alpha,\beta}$. Hence, at most one of them can be contained in $V_1 \cup V_2$. Since there are β possible choices of $\overline{b}$, there are at most β red vertices with the first coordinate equal to a in $V_1 \cup V_2$.

Claim 4.6. *For every b, $1 \le b \le \beta$, the sum of the entries of M_g on the b-th column is at most α.*

The proof is analogous to the proof of Claim 4.5.

We now continue the main part of the proof of Lemma 4. Let A_g be the set of all integers a such that $M_g[a,b] = 2$ for some b. Similarly, B_g is the set of all b's such that $M_g[a,b] = 2$ for some a. Analogously, A_r and B_r are sets of all integers a and b such that $M_r[a,b] = 2$. In addition, let M be the matrix that is the sum of the matrices M_r and M_g, i.e., $M = M_r + M_g$. Note that the sum of all the entries of M is $|V_1| + |V_2|$. By Claims 4.3 and 4.4, it holds that $M_g[a,b] = M_r[a,b] = M[a,b] = 0$ for all $[a,b] \in A_r \times B_g$ and $[a,b] \in A_g \times B_r$. On the other hand, by the definitions of the sets A_g, B_g, A_r and B_r, if $[a,b] \notin (A_g \cup A_r) \times (B_g \cup B_r)$, $M_g[a,b] \le 1$, $M_r[a,b] \le 1$, and at most one of $M_g[a,b]$ and $M_r[a,b]$ is non-zero by Claim 4.2. We conclude that $M[a,b] = M_g[a,b] + M_r[a,b] \le 1$ for every $[a,b] \notin (A_g \cup A_r) \times (B_g \cup B_r)$.

In order to finish the proof, we distinguish four cases according the cardinalities of the sets A_g, B_g, A_r and B_r:

- $|A_g| \le |A_r|$ and $|B_g| \ge |B_r|$
  Observe first that the following holds:

$$\begin{aligned} |B_r| &\le |B_g| \\ |B_r|(|A_r| - |A_g|) &\le |B_g|(|A_r| - |A_g|) \\ |A_g||B_g| + |A_r||B_r| &\le |A_g||B_r| + |A_r||B_g| \qquad (2) \end{aligned}$$

  By Claim 4.1, it holds that $M[a,b] \le 2$ for every $[a,b] \in (A_g \times B_g) \cup (A_r \times B_r)$. Since $M[a,b] = 0$ for all $[a,b] \in (A_g \times B_r) \cup (A_r \times B_g)$, the sum of the entries $M[a,b]$ for $[a,b] \in (A_g \cup A_r) \times (B_g \cup B_r)$ is at most the following (the first inequality follows from (2)):

$$\begin{aligned} 2(|A_g||B_g| + |A_r||B_r|) &\leq |A_g||B_g| + |A_r||B_r| + |A_g||B_r| + |A_g||B_r| \\ &\leq (|A_g| + |A_r|)(|B_g| + |B_r|). \end{aligned}$$

Since $M[a,b] \leq 1$ for every $[a,b] \notin (A_g \cup A_r) \times (B_g \cup B_r)$, the sum of the entries of the matrix M is at most $\alpha\beta$ as desired.

- $|A_g| \geq |A_r|$ and $|B_g| \leq |B_r|$
  This case is symmetric to the first one.

- $|A_g| \leq |A_r|$ and $|B_g| \leq |B_r|$
  For $a \in A_r$, all the entries of M_g on the a-th row are zero by Claim 4.3. In particular, the entries of M and M_r on the a-th row coincide. Hence, the sum of the entries of the matrix M on the rows $a \in A_r$ is at most $|A_r|\beta$ by Claim 4.5. The sum of the entries $M[a,b]$ with $[a,b] \in A_g \times (B_g \cup B_r)$ is at most $2|A_g||B_g| \leq |A_g|(|B_g| + |B_r|)$ by Claim 4.2. Finally, all the remaining entries of M are at most one. We infer that the sum of all the entries of M does not exceed $\alpha\beta$.

- $|A_g| \geq |A_r|$ and $|B_g| \geq |B_r|$
  An argument that is similar to that in the previous case and that involves Claim 4.6 applies.

Since the sum of the entries of M is equal to $|V_1| + |V_2|$, we conclude that $|V_1| + |V_2| \leq \alpha\beta$. □

The following theorem readily follows from Propositions 2 and 3 and Lemma 4:

Theorem 5. *For every $\alpha, \beta \geq 1$, the graph $\Gamma_{\alpha,\beta}$ has the following properties:*

- *the order of $\Gamma_{\alpha,\beta}$ is $(\alpha+\beta)\alpha\beta$,*
- *the graph $\Gamma_{\alpha,\beta}$ is connected,*
- *the graph $\Gamma_{\alpha,\beta}$ is $(\alpha+\beta-1)$-regular, and*
- *its hole number $\rho(\Gamma_{\alpha,\beta})$ is $\alpha+\beta-1$.*

An immediate corollary of Theorem 5 is the following:

Corollary 6. *For every $r \geq 1$, there exists an r-regular connected graph G of order $(r+1)\left\lfloor\frac{(r+1)^2}{4}\right\rfloor \approx r^3/4$ with $\rho(G) = r$.*

Proof. Set $\alpha = \lfloor r/2 \rfloor + 1$ and $\beta = \lceil r/2 \rceil$, and consider the graph $\Gamma_{\alpha,\beta}$. Note that $\alpha\beta(\alpha+\beta) = (r+1)\left\lfloor\frac{(r+1)^2}{4}\right\rfloor$. □

3.3 Generalization

In this subsection, we slightly generalize our construction. If G is a graph with the vertex set $V(G)$, then $G^{[s]}$ is the graph whose vertex set is $V(G) \times \{1,\ldots,s\}$ and two distinct vertices $[v,i]$ and $[v',i']$ of $G^{[s]}$ are joined by an edge if $v = v'$ or $v \neq v'$ and vv' is an edge of G. Clearly, if G is an r-regular graph of order n, then $G^{[s]}$ is an $(rs+s-1)$-regular graph of order ns. Note that $G^{[s]}$ is the lexicographic products of G and the complete graph of order s.

We now formulate the following lemma:

Lemma 7. *Let G be a connected r-regular graph of order $(r+1)t$. If G has the t-neighborhood property, then $G^{[s]}$ has also the t-neighborhood property for every $s \geq 1$.*

Proof. Let V and W be two sets of vertices of $G^{[s]}$ such that the distance between any two vertices in each of the sets is at least two and no vertex of V is adjacent to a vertex of W. Let V' be the set of vertices v of G such that $[v,i] \in V$ for some i, $1 \leq i \leq s$. Similarly, W' is the set of vertices w such that $[w,i] \in W$. Note that the sets V' and W' are disjoint, $|V| = |V'|$ and $|W| = |W'|$. Moreover, V' and W' do not contain any two vertices at distance two and no vertex of V' is ajdacent to a vertex of W'. Since G has the t-neighborhood property, $|V'| + |W'| \leq t$. Hence, $|V| + |W| \leq t$. Because the choice of V and W was arbitrary, $G^{[s]}$ has the t-neighborhood property. □

Fix $\alpha, \beta \geq 1$ and $s \geq 2$. Consider the labeling of $\Gamma_{\alpha,\beta}$ with the labels $0, 2, \ldots, 2\alpha + 2\beta - 2$ constructed in Proposition 3. We now construct an $L(2,1)$-labeling of $\Gamma^{[s]}_{\alpha,\beta}$. If v is a vertex of $\Gamma_{\alpha,\beta}$ that is labeled with γ, then a vertex $[v,i]$, $i = 1,\ldots,s$, of $\Gamma^{[s]}_{\alpha,\beta}$ is labeled with $\gamma + 2(i-1)(\alpha+\beta)$. The obtained labeling is a proper $L(2,1)$-labeling of $\Gamma^{[s]}_{\alpha,\beta}$. Hence, $\lambda(\Gamma^{[s]}_{\alpha,\beta}) \leq 2s(\alpha+\beta) - 2$.

The following theorem readily follows from Lemmas 4 and 7:

Theorem 8. *For every $\alpha, \beta, s \geq 1$, the graph $\Gamma^{[s]}_{\alpha,\beta}$ has the following properties:*

- *the order of $\Gamma^{[s]}_{\alpha,\beta}$ is $(\alpha+\beta)\alpha\beta s$,*
- *the graph $\Gamma^{[s]}_{\alpha,\beta}$ is connected,*
- *the graph $\Gamma^{[s]}_{\alpha,\beta}$ is $((\alpha+\beta)s-1)$-regular, and*
- *its hole number $\rho(\Gamma^{[s]}_{\alpha,\beta})$ is $(\alpha+\beta)s-1$.*

Note that Theorem 8 yields a construction of connected r-regular graphs G of order $(r+1)t$ for some (but not all) numbers t between r and $\left\lfloor \frac{(r+1)^2}{4} \right\rfloor$.

4 Conclusion

We conclude the paper with several problems in the spirit of Conjecture 2. The first problem that comes to one's mind is the following:

Problem 1. *Is it true that there exists a function $f(r)$ with the following property: if G is a connected r-regular graph of order $(r+1)t$ with $\rho(G) = r$, then $t \leq f(r)$? Does there exist a polynomial $f(r)$ with this property?*

Georges and Mauro [3] constructed connected r-regular graphs of order $(r+1)t$ for every $t = 1, \ldots, r$. We constructed such graphs for some numbers t larger than r, but we were not able to construct such graphs for all $t = 1, \ldots, \left\lfloor \frac{(r+1)^2}{4} \right\rfloor$. This leads us to the following problem:

Problem 2. *Assume that G is a connected r-regular graph of order $(r+1)t_0$ with $\rho(G) = r$. Is it true that for every $t = 1, \ldots, t_0$, there exists a connected r-regular graph of order $(r+1)t$ with $\rho(G) = r$? In particular, is this true for $t_0 = \left\lfloor \frac{(r+1)^2}{4} \right\rfloor$?*

In the case of cubic graphs, we are aware of constructions of connected cubic graphs G of orders 4, 8, 12 and 16 with $\rho(G) = 3$. We have a computer-assisted proof that there is no such cubic graph of order 20. If the answer to Problem 2 were positive, then the answer to the following problem would also be positive:

Problem 3. *Is it true that there is no connected cubic graph G with $\rho(G) = 3$ whose order is at least 20?*

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