A Theorem About a Contractible and Light Edge

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Abstract

In 1955 Kotzig proved that every planar 3-connected graph contains an edge such that sum of degrees of its endvertices is at most 13. Moreover, if the graph does not contain 3-vertices, then this sum is at most 11. Such an edge is called light. The well-known result of Steinitz that the 3-connected planar graphs are precisely the skeletons of 3-polytopes, gives an additional trump to Kotzig's theorem. On the other hand, in 1961, Tutte proved that every 3-connected graph, distinct from K_4, contains a contractible edge. In this paper, we strengthen Kotzig's theorem by showing that every 3-connected planar graph distinct from K_4 contains an edge which is both light and contractible. A consequence is that every 3-polytope can be constructed from the Tetrahedron by a sequence of splittings of vertices of degree at most 11.

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1 Light Edges

Throughout the paper, we consider 3-connected planar graphs without loops and multiple edges. The *weight* of an edge is the sum of the degrees of its endvertices. It is well-known that every planar graph contains a vertex of degree at most 5. Kotzig [5] proved a similar result for the edge weight:

Theorem 1 (Kotzig) *Every 3-connected planar graph G contains an edge of weight at most* 13*. Moreover, if G has minimum degree at least 4, then G contains an edge of weight at most* 11*.*

Regarding the above theorem, an edge of a 3-connected planar graph is called *light* if it satisfies the requirements of the above theorem. In particular, if the graph has minimum degree ≥ 4, then an edge is light only if it is of weight ≤ 11.

The bounds of 13 and 11 from Kotzig's theorem are the best possible in the sense that there exist a planar 3-connected graph G_1 such that each edge of G_1 has weight at least 13, and a planar 3-connected graph G_2 of minimum degree 4 such that each edge of G_2 has weight at least 11. As for G_1, consider a copy of the Icosahedron and insert into each face a vertex and connect it with the three vertices of the face. As for G_2, consider any fulleren where no two vertices of degree 5 are adjacent.

The well-known theorem of Steinitz [9, 10] states that the 3-connected planar graphs are precisely the skeletons of 3-dimensional polytopes. This gives an additional importance to Theorem 1.

Kotzig's Theorem has been generalized in many directions. It served as a starting point for looking for other subgraphs of small weight in plane graphs, which later developed the subject into Light Graph Theory: let $\mathcal{H}$ be a family of graphs, and let H be a connected graph such that infinitely many members of $\mathcal{H}$ contain a subgraph isomorphic to H. Let $\mathcal{H}_H$ be the subfamily of graphs in $\mathcal{H}$ that contain H as a subgraph. We say that H is a *light* graph in the family $\mathcal{H}$ if there exists a constant c for that each graph $G \in \mathcal{H}_H$ contains a subgraph $K \cong H$ such that $d_G(v) \leq c$ for every vertex $v \in K$. Just to mention a few results from Light Graph Theory: Fabrici and Jendrol' [2] proved that only paths are light in the family of all 3-connected plane graphs; the same holds also for the family of all 3-connected plane graphs of minimum degree 4 (see [3]). A survey on light graphs in various families of plane, projective plane, and general graphs can be found in the paper by Jendrol' and Voss [4].

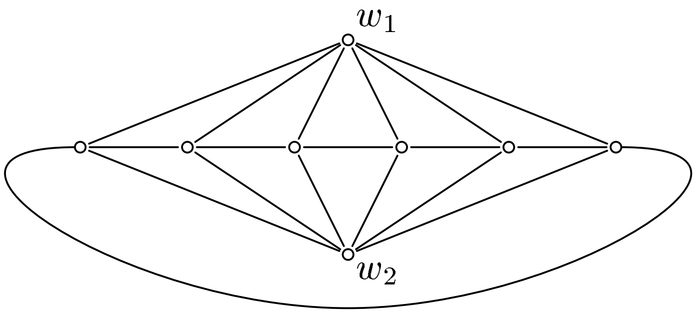

Figure 1: A double wheel

1.1 Light Edge Avoiding Prescribed Triangle

Here we prove the existence of a light edge which avoids vertices of a prescribed triangle face.

Lemma 1 *Let $G \neq K_4$ be a plane 3-connected graph with an outer-face $O = x_1x_2x_3$. Let δ' be the minimum degree of vertices of G distinct from x_1, x_2 and x_3. Let d be 13 if $\delta' = 3$, and 11 otherwise. Then G contains an edge of weight at most d not incident with x_1, x_2 and x_3.*

Proof: Suppose that the statement of the lemma is false and G is a counterexample on n vertices. Obviously, $n \geq 5$. In addition, we may assume that G has the maximum number of edges among all such graphs.

We claim that *every face incident with x_1, x_2, or x_3 is a triangle.* Otherwise, we may assume that x_1 lies on a face f' of length ≥ 4. Next insert an edge between x_1 and some vertex of f' that is not adjacent to x_1. This is always possible since G is 3-connected. Let G' be the resulting graph. Notice that if G is of minimum degree ≥ 4, then G' is also of minimum degree ≥ 4. Hence G' is a counterexample to the lemma with the same number of vertices but it has more edges than G, a contradiction.

By the above claim, it easily follows that at most one of x_1, x_2 and x_3 is of degree 3. Thus, we may assume that $d(x_1) \geq 4$ and $d(x_2) \geq 4$. Furthermore, $d(x_3) \geq 3$ since G is 3-connected.

Consider the double wheel W of order 8 depicted in Figure 1. Let w_1 and w_2 be the vertices of W of degree 6. We construct the graph W_G by gluing a copy of G in each face of W in such a way that the vertex x_3 of the copy is identified with either w_1 or w_2. Notice that W_G is a planar

3-connected graph and that each vertex of W has degree ≥ 12 in W_G (this follows from the assumption on the degrees of vertices x_1, x_2 and x_3 in G).

By Kotzig's Theorem, the graph W_G contains a light edge e_w. This edge is not incident with any vertex of the copy of W, since these vertices are of degree ≥ 12. Thus, e_w corresponds to an edge e of G, that is not incident with x_1, x_2 and x_3. Notice that if $\delta' \geq 4$, then W_G has minimum degree ≥ 4. This implies that weight of e satisfies requirements of the lemma.

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2 Contractible Edges

A subset S of vertices of a graph G is a *cut*, if the graph $G-S$ is disconnected and S is a minimal set with this property. In addition, if S is of size k, then it is called a *k-cut*. A graph G is *k-connected* if it has at least $k+1$ vertices and it has no cuts of size $< k$.

Let $e = ab$ be an edge in a 3-connected graph G. Let G/e be the graph obtained by identifying the vertices a and b into a new vertex w, and removing the arising loop and multiple edges (in order to obtain a simple graph). We say that G/e is obtained from G by contracting the edge e. Similarly, we say that G is obtained from G/e by splitting w. If G/e is a 3-connected graph, then we say that the edge e is *contractible*, and otherwise that e is *non-contractible*. It is easy to see that e is non-contractible if and only if G has a 3-cut S such that $\{a, b\} \subseteq S$.

Tutte [11] proved that every 3-connected graph, distinct from K_4, contains a contractible edge and as a consequence, it follows:

Theorem 2 (Tutte) *A graph G is 3-connected if and only if there exists a sequence $G_0, \ldots, G_n$ of graphs with the following properties:*

(a) $G_0 = K_4$ and $G_n = G$;

(b) G_{i+1} has an edge xy with $d(x), d(y) \geq 3$ and $G_i = G_{i+1}/xy$, for every $i < n$.

In fact, every 3-connected graph on ≥ 5 vertices has more then just one contractible edge, see the survey of Kriesell [6] for results of this kind.

Notice that if G is a 3-connected planar graph and S is a 3-cut, then $G-S$ comprises of precisely two components: there cannot be more than two, otherwise we obtain a subdivision of $K_{3,3}$ in G. Let these two components be denoted by $G_1(S)$ and $G_2(S)$. Let $G_i^*(S)$ be the subgraph of G induced by $V(G_1(S)) \cup S$. Thus, $G_i(S) = G_i^*(S) - S$, for $i \in \{1,2\}$. Observe that if $x, y \in S$ are non-adjacent, then there exists precisely one face incident with both of them. When the graph G is clear from the context, its face which contains these two vertices is denoted by $f_{x,y}$.

A triangle $v_1v_2v_3$ in a graph is called *separating* if $\{v_1, v_2, v_3\}$ is a cut. If $v_1v_2v_3$ is a separating triangle in G, then each of the edges v_1v_2, v_1v_3 and v_2v_3 is obviously non-contractible. But it is not necessarily true that every non-contractible edge of G belongs to a separating triangle. However, in this section we show that unless G contains a light contractible edge, we may extend G to a supergraph that satisfies this condition by adding new non-contractible edges and without creating any new contractible edges, see Lemma 8.

The proofs of the following three folklore lemmas can be found in [1, 7, 8]:

Lemma 2 *Let G be a 3-connected graph of order at least five. Suppose x is a 3-vertex of G whose neighbors are a, b and c. If ab is an edge of G, then xc is contractible.*

If H is a subgraph of G, then we denote by G/H the graph constructed from G by contracting all edges of H.

Lemma 3 *Let x be a 3-vertex in a 3-connected graph $G \neq K_4$. If xa and xb are two non-contractible edges of G, then a and b are adjacent vertices of degree 3. Moreover, $G^* = G/axb$ is 3-connected.*

Lemma 4 *Let G be a 3-connected graph and let $C = x_1x_2x_3$ be a 3-cycle of G with all vertices of degree 3. An edge e of G/C is contractible if and only if its corresponding edge e in G is contractible.*

Now, we are ready to prove the following lemma about smallest possible 3-connected graphs without a light contractible edge.

Lemma 5 *Suppose that $G \neq K_4$ is a 3-connected planar with the smallest possible number $n \geq 5$ of vertices such that every light edge is non-contractible. Then, G does not contain a 3-cycle whose all vertices are of degree 3.*

Proof: Suppose first that $n < 7$. Then degree of each vertex of G is at most 5, and therefore each edge of G is light. Since every 3-connected graph on at least 5 vertices contains a contractible edge, the graph G contains a light contractible edge.

Therefore we may assume that $n \geq 7$. Suppose that $C = x_1x_2x_3$ is a 3-cycle of G such that all vertices of C are of degree 3.

Let y_i be the neighbor of x_i that does not belong to C. Note that the vertices y_1, y_2 and y_3 are mutually distinct, since G is 3-connected. Let $G^* = G/C$ and let w be the vertex of G^* into which C is contracted. By Lemma 3, the graph G^* is 3-connected. Then w is a 3-vertex whose neighbors are y_1, y_2 and y_3. Also notice that each edge e^* of G^* has the same weight as the corresponding edge e of G. Lemma 4 claims that e^* is contractible in G^* if and only if e is contractible in G. This implies that every light edge of G^* is non-contractible. Since G^* has at least 5 vertices, it contradicts the minimality of G.

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The following two lemmas describe the structure of a graph containing a non-contractible edge xy that becomes contractible after a new edge bc is added to the graph.

Lemma 6 *Let G be a planar 3-connected graph, xy a non-contractible edge of G, and b and c two non-adjacent vertices of G that lie on a common face. Suppose that xy is contractible in $G \cup \{bc\}$. Then for each vertex z such that $S = \{x, y, z\}$ is a 3-cut the following four claims hold:*

(a) b and c are distinct from x, y and z, and belong to distinct components of $G - S$;

(b) z belongs to $f_{b,c}$, and precisely one of x and y belongs to $f_{b,c}$ (let this vertex be denoted by w);

(c) $f_{b,c} = w \cdots b \cdots z \cdots c \cdots w$ in the clockwise or anti-clockwise orientation;

(d) w and z are non-adjacent.

Proof: Let z be an arbitrary vertex of G such that $S = \{x, y, z\}$ is a 3-cut. Since xy is contractible in $G \cup \{bc\}$ but not in G, it follows that b and c belong to distinct components of $G - S$. Therefore the vertices b and c are distinct from x, y and z.

Since S is a cut and the edge bc connects the two components of $G - S$, it follows that b, c and z belong to a common face. Moreover, one of x and

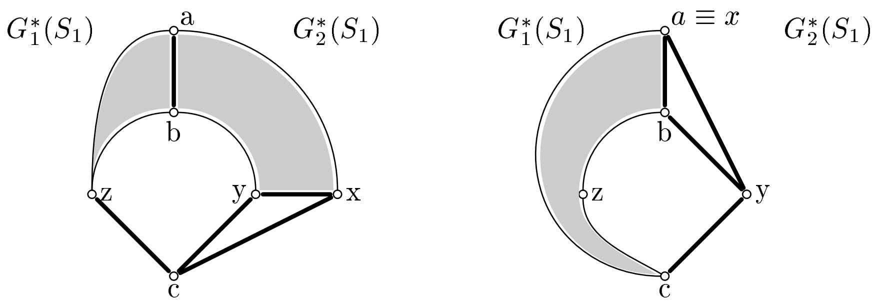

Figure 2: Configurations in Lemma 7

y lie on that face as well (but not both since no face may contain all three vertices of a 3-cut in G). The order of the vertices w, z, b and c that appear around the face must be as described in the claim, because b and c belong to distinct components of $G - S$. Since G is 3-connected, it follows that w and z are non-adjacent.

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Lemma 7 *Let ab and xy be two non-contractible edges and let $S_1 = \{a, b, c\}$ and $S_2 = \{x, y, z\}$ be two 3-cuts of G. Suppose that the edge xy is contractible in $G \cup \{bc\}$. Then, the following two claims hold:*

- *(i) If $a \notin \{x, y\}$, then c is a vertex of degree 3 with $N(c) = \{z, x, y\}$ and cxy is a 3-face.*
- *(ii) If $a = x$, then y is a vertex of degree 3 with $N(y) = \{a, b, c\}$ and aby is a 3-face.*

Proof: First notice that $G \cup \{bc\}$ is a planar graph, since the vertices b and c lie on a common face in G. Also notice that b and c are non-adjacent in G. By Lemma 6, the vertices b and c belong to different components of $G - S_2$ and they are distinct from x, y and z. By the same lemma, without loss of generality, we may assume that y and z are non-adjacent and lie on the same face with b and c (i.e., $f_{b,c} = f_{y,z}$) and $f_{y,z} = y \cdots c \cdots z \cdots b \cdots y$. We may also assume that z is a vertex of $G_1^* = G_1^*(S_1)$ and x, y are vertices of $G_2^* = G_2^*(S_1)$. Consider now both the cases separately and see Figure 2 for illustration:

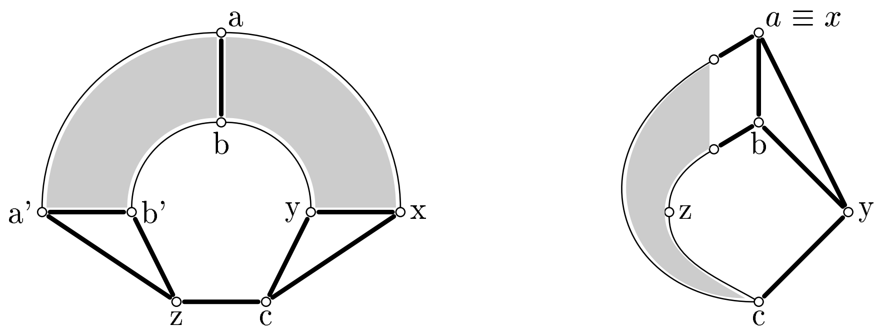

Figure 3: Configurations in Lemma 8

(*i*) Observe that z is a cut-vertex in G_1^*, which separates a, b from c; otherwise we infer that S_2 is not a cut of G. Since G is a 3-connected it follows that c is adjacent only to z in G_1^*.

Similarly one can show that $\{x, y\}$ is a cut in G_2^*, which also separates a, b from c. To show the minimality of $\{x, y\}$ observe that if x or y is a vertex-cut in G_2^*, then $\{x, z\}$ or $\{y, z\}$ is a 2-cut in G.

If there is a vertex adjacent to c in G_2^* which is distinct from x and y, then $\{c, x, y\}$ is a 3-cut in $G \cup \{bc\}$ but this is a contradiction with the assumption that xy is a contractible edge in $G \cup \{bc\}$. Since $\{x, y\}$ is a cut in G_2^*, both x and y are adjacent to c. Thus, x, y are the only neighbors of c in G_2^*. This implies that cxy is a 3-face and $N(c) = \{z, x, y\}$.

(*ii*) Since $\{x, y, b\}$ is not a 3-cut in $G \cup \{bc\}$, we infer that aby is a 3-face. Similarly, since $\{x, y, c\}$ is not a 3-cut in $G \cup \{bc\}$, it follows that cy is an edge of G, and hence $N(y) = \{a, b, c\}$.

■

Now we are ready to show that in a maximal graph which does not contain a light contractible edge, every non-contractible edge belongs to a separating 3-cycle.

Lemma 8 *Suppose that there exists a planar graph on $n \geq 5$ vertices such that each of its light edges is non-contractible. Suppose that G is such a* *graph on n vertices with maximum number of edges. Then, every non-contractible edge of G belongs to a separating 3-cycle.*

Proof: Suppose that the claim is false and G is a counterexample with minimum number of vertices $n \geq 5$. Let ab be a non-contractible edge which does not belong to a separating cycle and let $S = \{a, b, c\}$ be a 3-cut of G. Without loss of generality, we may assume that b and c are non-adjacent.

Consider the graph $G \cup \{bc\}$. By the maximality of $|E(G)|$ the graph $G \cup \{bc\}$ contains a light contractible edge xy. Obviously the edge xy is distinct from bc, since bc is non-contractible. The edge xy is light in G as well, thus it must be non-contractible in G. Let $\{x, y, z\}$ be a 3-cut of G. We may assume that $x, y \in V(G_2^*(S))$ and $z \in V(G_1^*(S))$. By Lemma 6, we may assume that b, y, c and z belong to a common face. Consider now the following two cases and see Figure 3 for illustration:

- **Case 1:** $a \notin \{x, y\}$. Then, by Lemma 7(a), we may assume that c is a vertex of degree 3 with neighbors x, y and z. By the maximality of G, the graph $G \cup \{yz\}$ must contain a light contractible edge $e = a'b'$. Notice that this edge is non-contractible in G. By Lemma 6 one of the endvertices of e must be incident with $f_{y,z}$, say b'. Observe that the only 3-cut that shows non-contractibility of e is $\{a', b', c\}$. If e belongs to $G_2^*(S)$, then $\{a', b', z\}$ is a cut of $G \cup \{yz\}$ which separates a or b from y, x or c, and $a'b'$ would be non-contractible in $G \cup \{yz\}$. Therefore we may assume that e belongs to $G_1^*(S)$. See the left graph of Figure 3. In particular the edges $a'b'$ and xy are not incident. Hence, by Lemma 7(a), z is a 3-vertex and $za'b'$ is a 3-face of G. Finally, Lemma 2 implies that zc is a contractible edge of weight 6.

- **Case 2:** $a \in \{x, y\}$, say $a = x$. We assume that no choice of x, y, z, a, b and c may satisfy Case 1. By Lemma 7(b), y is a 3-vertex with neighbors a, b, c and aby is a face. Due to the maximality of G, the graph $G \cup \{yz\}$ contains a light contractible edge $a'b'$. The edge $a'b'$ must be non-contractible in G and distinct from ay. Excluding Case 1, the edge $a'b'$ must be incident with the edge ay. However, adding the edge yz does not affect contractibility of any edge incident with y or z, therefore the edge $a'b'$ must be incident with a.

  We may assume that $a = a'$. Notice that b' is a vertex of $G_1^*(S)$ and by Lemma 7(b), we conclude that ayb' is a 3-face and b' is of degree 3. Hence $b' = b$ or $b' = c$.

If $b' = c$, then the edge cy has weight 6 and by Lemma 2 it is contractible in G.

Consider now the case $b = b'$ and see the right graph of Figure 3. The vertex b has degree 3 and the edge by has weight 6. If by is a non-contractible edge, then b is a vertex of degree 3 and incident with two non-contractible edges ab and by. Lemma 3 implies that a is also of degree 3. Thus, G contains a 3-cycle with each vertex of degree 3, but Lemma 5 excludes such a subgraph in G. Thus we conclude that by is a contractible light edge in G. This finishes the proof.

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3 Contractible Light Edge

If C is a cycle of a plane graph G, then $\mathrm{Int}(C)$ denotes the subgraph of G induced by the vertices and edges of G which lie on C or in its interior. We are now ready to prove the theorem.

Theorem 3 *Every 3-connected planar graph, distinct from K_4, contains a light and contractible edge.*

Before we proceed with the proof of the theorem, let us emphasize that this result strengthens Theorem 1, i.e., we show precisely the same bounds on the weight of contractible edges.

Proof: Suppose that the theorem is false and G is a counterexample with the minimum number of vertices $n \geq 5$. Thus, every light edge of G is non-contractible. We may additionally assume that G has the maximum number of edges between all such graphs of order n.

By Lemma 8, every non-contractible edge of G belongs to a separating 3-cycle. Since G is 3-connected, it follows that every vertex that belongs to a separating 3-cycle is of degree ≥ 4. Therefore every 3-vertex is incident only to contractible edges. This implies that every 3-vertex of G is adjacent only to vertices of degree ≥ 11. In order to complete the proof consider the following two possibilities:

Suppose first that *every separating 3-cycle C of G satisfies* $\mathrm{Int}(C) = K_4$. By Theorem 1, the graph G contains a light edge $e = uv$. This edge e does not lie on a separating 3-cycle; otherwise u and v are adjacent with a 3-vertex, and each of them is of degree ≥ 11 by the argument in the above paragraph. Thus we conclude that e is a contractible light edge.

Suppose now that *G has a separating 3-cycle C such that* $\mathrm{Int}(C) \neq K_4$. We may additionally assume that $C = x_1x_2x_3$ is chosen so that $G' := \mathrm{Int}(C)$ has the smallest possible number of vertices. The graph G' has at least 5 vertices. By the choice of C, each separating 3-cycle C' of G' satisfies $\mathrm{Int}(C') = K_4$. By Lemma 1, G' contains an edge e' that is not incident with x_1, x_2 and x_3 such that e' is light in G. Applying a similar argument as in the previous paragraph, one can observe that e' is also contractible. This establishes the theorem.

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Theorems 2 and 3 imply the following result:

Corollary 1 *Every 3-polytope G can be constructed from Tetrahedron by a sequential splittings of vertices of degree at most* 11.

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