VERTEX PARTITIONS OF CHORDAL GRAPHS

DAVID R. WOOD

ABSTRACT. A *k-tree* is a chordal graph with no $(k+2)$-clique. An *ℓ-tree-partition* of a graph G is a vertex partition of G into 'bags', such that contracting each bag to a single vertex gives an ℓ-tree (after deleting loops and replacing parallel edges by a single edge). We prove that for all $k \geq \ell \geq 0$, every k-tree has an ℓ-tree-partition in which every bag induces a connected $\lfloor k/(\ell+1) \rfloor$-tree. An analogous result is proved for oriented k-trees.

1. INTRODUCTION

Let G be an (undirected, simple, finite) graph with vertex set $V(G)$ and edge set $E(G)$. The neighbourhood of a vertex v of G is denoted by $N(v) = \{w \in V(G) : vw \in E(G)\}$. A *chord* of a cycle C is an edge not in C whose endpoints are both in C. G is *chordal* if every cycle on at least four vertices has a chord. A k-clique ($k \geq 0$) is a set of k pairwise adjacent vertices. A *k-tree* is a chordal graph with no $(k+2)$-clique. The *tree-width* of G, denoted by $\mathsf{tw}(G)$, is the minimum k such that G is a subgraph of a k-tree. It is well known that G is a k-tree if and only if $V(G) = \emptyset$, or G has a vertex v such that $G \setminus v$ is a k-tree, and $N(v)$ is a k'-clique for some $k' \leq k$.

Let G and H be graphs. The elements of $V(H)$ are called *nodes*. Let $\{H_x \subseteq V(G) : x \in V(H)\}$ be a set of subsets of $V(G)$ indexed by the nodes of H. Each set H_x is called a *bag*. The pair $(H, \{H_x \subseteq V(G) : x \in V(H)\})$ is an *H-partition* of G if:

- $\forall$ vertices v of G, $\exists$ node x of H with $v \in H_x$, and
- $\forall$ distinct nodes x and y of H, $H_x \cap H_y = \emptyset$, and
- $\forall$ edge vw of G, either
  - $\exists$ node x of H with $v \in H_x$ and $w \in H_x$, or
  - $\exists$ edge xy of H with $v \in H_x$ and $w \in H_y$.

Date: December 22, 2004.

1991 *Mathematics Subject Classification.* 05C15 (Coloring of graphs and hypergraphs).

Key words and phrases. graph, chordal graph, k-tree, vertex partition, H-partition, tree-partition, tree-width.

Research completed in the Department of Applied Mathematics and the Institute for Theoretical Computer Science at Charles University, Prague, Czech Republic. Supported by project LN00A056 of the Ministry of Education of the Czech Republic, and by the European Union Research Training Network COMBSTRU (Combinatorial Structure of Intractable Problems).

For brevity we say H is a partition of G. A *k-tree-partition* is an H-partition for some k-tree H. A *tree-partition* is a 1-tree-partition. Tree-partitions were independently introduced by Seese [13] and Halin [12], and have since been investigated by a number of authors [2, 3, 6, 7, 12, 13]. The main property of tree-partitions that has been studied is the maximum cardinality of a bag, called the *width* of the tree-partition. The minimum width over all tree-partitions of a graph G is the *tree-partition-width*[1] of G, denoted by $\mathsf{tpw}(G)$. A graph with bounded degree has bounded tree-partition-width if and only if it has bounded tree-width [7]. In particular, for every graph G, Seese [13] proved that $\mathsf{tw}(G) \leq 2\,\mathsf{tpw}(G) - 1$, and Ding and Oporowski [6] proved that $\mathsf{tpw}(G) \leq 24\,\mathsf{tw}(G)\max\{\Delta(G), 1\}$, where $\Delta(G)$ is the maximum degree of G. See [1, 5, 8, 9] for other results related to tree-width and vertex partitions.

Tree-partition-width is not bounded above by any function solely of tree-width. For example, wheel graphs have bounded tree-width and unbounded tree-partition-width, as observed by Bodlaender and Engelfriet [3]. Thus, it seems unavoidable that the maximum degree appears in an upper bound on the tree-partition-width. This fact, along with other applications, motivated Dujmović *et al.* [10, 11] to study the structure of the bags in a tree-partition. In this paper we continue this approach, and prove the following result (in Section 2).

Theorem 1. *Let k and ℓ be integers with $k \geq \ell \geq 0$. Let $t = \lfloor k/(\ell+1) \rfloor$. Every k-tree G has an ℓ-tree-partition in which each bag induces a connected t-tree in G.*

It is easily seen that Theorem 1 is tight for $G = K_{k+1}$ and for all ℓ. Note that Theorem 1 can be interpreted as a statement about chromomorphisms (see [15, 16]).

Dujmović *et al.* [10, 11] proved that every k-tree has a tree-partition in which each bag induces a $(k-1)$-tree. Thus Theorem 1 with $\ell = 1$ improves this result. That said, the tree-partition of Dujmović *et al.* [10, 11] has a number of additional properties that were important for the intended application. We generalise these additional properties in Section 3. The price paid is that each bag may now induce a $(k-\ell)$-tree, thus matching the result of Dujmović *et al.* [10, 11] for $\ell = 1$. Note that the proof of Dujmović *et al.* [10, 11] uses a different construction to the one given here.

2. Proof of Theorem 1

We proceed by induction on $|V(G)|$. If $V(G) = \emptyset$, then the result holds with $V(H) = \emptyset$ regardless of k and ℓ. Now suppose that $|V(G)| \geq 1$. Thus G has a vertex v such that $G \setminus v$ is a k-tree, and $N(v)$ is a k'-clique for some $k' \leq k$. By induction, $G \setminus v$ has an ℓ-tree-partition H in which each bag induces a connected t-tree. Let $C = \{x \in V(H) : N(v) \cap H_x \neq \emptyset\}$. Since $N(v)$ is a clique, C is a clique of H (by the definition of H-partition). Since H is an ℓ-tree, $|C| \leq \ell + 1$.

[1]Tree-partition-width has also been called *strong tree-width* [3, 13].

Case 1. $|C| \leq \ell$: Add one new node y to H adjacent to each node $x \in C$. Since C is a clique of H and $|C| \leq \ell$, H remains an ℓ-tree. Let $H_y = \{v\}$. The other bags remain unchanged. Since $t \geq 0$, H_y induces a connected t-tree ($= K_1$) in G. Thus H is now a partition of G in which each bag induces a connected t-tree in G.

Case 2. $|C| = \ell + 1$: There is a node $y \in C$ such that $|N(v) \cap H_y| \leq t$, as otherwise $|N(v)| \geq (t+1)|C| = (\lfloor k/(\ell+1) \rfloor + 1)(\ell + 1) \geq k + 1$. Add v to the bag H_y. Let $u \in N(v) \cap H_y$. Every neighbour of v not in H_y is adjacent to u (in $G \setminus v$). Thus H is a partition of G. H_y induces a connected t-tree in G, since $H_y \setminus \{v\}$ induces a connected t-tree in $G \setminus v$, and the neighbourhood of v in H_y is a clique of at least one and at most t vertices. The other bags do not change. Thus each bag of H induces a connected t-tree in G. □

3. Oriented Partitions

Let G be an oriented graph with arc set $A(G)$. Let $\widehat{G}$ be the underlying undirected graph of G. The in- and out-neighbourhoods of a vertex v of G are respectively denoted by $N^-(v) = \{u \in V(G) : uv \in A(G)\}$ and $N^+(v) = \{w \in V(G) : vw \in A(G)\}$. It is easily seen that an (undirected) graph G is a k-tree if and only if there is an acyclic orientation of G such that for every vertex v of G, $N^-(v)$ is a k'-clique for some $k' \leq k$. An oriented graph with this property is called an *oriented k-tree*. Let G and H be oriented graphs. An *oriented H-partition* of G is an $\widehat{H}$-partition of $\widehat{G}$ such that for every arc xy of H, and for every edge vw of $\widehat{G}$ with $v \in H_x$ and $w \in H_y$, vw is oriented from v to w. This concept is similar to an oriented homomorphism (see [4, 14] for example).

Theorem 2. *Let k and ℓ be integers with $k \geq \ell \geq 0$. Let $t = k - \ell$. Every oriented k-tree G has an oriented ℓ-tree partition H in which each bag induces a weakly connected oriented t-tree in G. Moreover, for every node x of H, the set of vertices $Q(x) = \bigcup_{v \in H_x} (N^-(v) \setminus H_x)$ is a k'-clique of G for some $k' \leq k$.*

The construction in the proof of Theorem 2 only differs from that of Theorem 1 in the choice of the node y in Case 2.

Proof. We proceed by induction on $|V(G)|$. If $V(G) = \emptyset$, then the result holds with $V(H) = \emptyset$ regardless of k and ℓ. Now suppose that $|V(G)| \geq 1$. Since G is acyclic, there is a vertex v of G such that $N^+(v) = \emptyset$, $N^-(v)$ is a k'-clique for some $k' \leq k$, and $G \setminus v$ is an oriented k-tree. By induction, there is an oriented ℓ-tree-partition H of $G \setminus v$ in which each bag induces a weakly connected oriented t-tree in $G \setminus v$. Moreover, for every node x of H, $Q(x)$ is a k'-clique for some $k' \leq k$. Let $C = \{x \in V(H) : N^-(v) \cap H_x \neq \emptyset\}$. Since $N^-(v)$ is a clique, C is a clique of H. Since H is an oriented ℓ-tree, $|C| \leq \ell + 1$.

Case 1. $|C| \leq \ell$: Add one new node y to H adjacent to each node $x \in C$. Orient each new edge from x to y. Obviously H remains acyclic. Since C is a clique of H and $|C| \leq \ell$, H remains an oriented ℓ-tree. Let $H_y = \{v\}$. The other bags are unchanged. Since $t \geq 0$, H_y induces a weakly connected oriented t-tree ($= K_1$) in G. All edges of G that are incident to a vertex in H_y are oriented into the vertex in H_y. Thus H is now an oriented partition of G in which each bag induces a weakly connected oriented t-tree in G. Now $Q(y) = N^-(v)$, which is a k'-clique for some $k' \leq k$. $Q(x)$ is unchanged for nodes $x \neq y$. Hence the theorem is satisfied.

Case 2. $|C| = \ell + 1$: The clique C induces an acyclic tournament in H. Let y be the sink of this tournament. Since $|N^-(v) \cap H_x| \geq 1$ for every node $x \in C \setminus \{y\}$, $|N^-(v) \cap H_y| \leq k' - (|C| - 1) \leq k - \ell = t$. Add v to the bag H_y.

Consider a neighbour u of v. Since $N^+(v) = \emptyset$, uv is oriented from u to v. Say $u \in H_z$ with $z \neq y$. Then z is in the clique C. Thus zy is an edge of H. Since y is a sink of C, zy is oriented from z to y. Thus H is now an oriented partition of G. H_y induces a weakly connected oriented t-tree in G, since $H_y \setminus \{v\}$ induces an oriented t-tree in $G \setminus v$, and the in-neighbourhood of v in H_y is a clique of at least one and at most t vertices. The other bags do not change. Thus each bag of H induces a weakly connected oriented t-tree in G.

$Q(y)$ is not changed by the addition of v to H_y, as there is at least one vertex $u \in N^-(v) \cap H_y$, and any vertex in $N^-(v) \setminus H_y$ is also in $N^-(u) \setminus H_y$. For nodes $x \neq y$, $Q(x)$ is unchanged by the addition of v to H_y, since v is not in the in-neighbourhood of any vertex. Hence the theorem is satisfied. □

Acknowledgements

Thanks to Matthew DeVos, Vida Dujmović, Attila Pór, and Ricardo Strausz for stimulating discussions.

References

[1] Noga Alon, Guoli Ding, Bogdan Oporowski, and Dirk Vertigan. Partitioning into graphs with only small components. *J. Combin. Theory Ser. B*, 87(2):231–243, 2003.

[2] Hans L. Bodlaender. A note on domino treewidth. *Discrete Math. Theor. Comput. Sci.*, 3(4):141–150, 1999.

[3] Hans L. Bodlaender and Joost Engelfriet. Domino treewidth. *J. Algorithms*, 24(1):94–123, 1997.

[4] Oleg V. Borodin, Dmitry Fon-Der-Flaass, Alexandr V. Kostochka, André Raspaud, and Éric Sopena. On deeply critical oriented graphs. *J. Combin. Theory Ser. B*, 81(1):150–155, 2001.

[5] Matt DeVos, Guoli Ding, Bogdan Oporowski, Daniel P. Sanders, Bruce Reed, Paul Seymour, and Dirk Vertigan. Excluding any graph as a minor allows a low tree-width 2-coloring. *J. Combin. Theory Ser. B*, 91(1):25–41, 2004.

[6] Guoli Ding and Bogdan Oporowski. Some results on tree decomposition of graphs. *J. Graph Theory*, 20(4):481–499, 1995.

[7] Guoli Ding and Bogdan Oporowski. On tree-partitions of graphs. *Discrete Math.*, 149(1-3):45–58, 1996.

[8] Guoli Ding, Bogdan Oporowski, Daniel P. Sanders, and Dirk Vertigan. Partitioning graphs of bounded tree-width. *Combinatorica*, 18(1):1–12, 1998.

[9] Guoli Ding, Bogdan Oporowski, Daniel P. Sanders, and Dirk Vertigan. Surfaces, tree-width, clique-minors, and partitions. *J. Combin. Theory Ser. B*, 79(2):221–246, 2000.

[10] Vida Dujmović, Pat Morin, and David R. Wood. Layout of graphs with bounded tree-width. *SIAM J. Comput.*, to appear.

[11] Vida Dujmović and David R. Wood. Tree-partitions of k-trees with applications in graph layout. In Hans Bodlaender, ed., *Proc. 29th Workshop on Graph Theoretic Concepts in Computer Science (WG'03)*, vol. 2880 of *Lecture Notes in Comput. Sci.*, pp. 205–217. Springer, 2003.

[12] Rudolf Halin. Tree-partitions of infinite graphs. *Discrete Math.*, 97:203–217, 1991.

[13] Detlef Seese. Tree-partite graphs and the complexity of algorithms. In Lothar Budach, ed., *Proc. International Conf. on Fundamentals of Computation Theory*, vol. 199 of *Lecture Notes in Comput. Sci.*, pp. 412–421. Springer, 1985.

[14] Éric Sopena. The chromatic number of oriented graphs. *J. Graph Theory*, 25(3):191–205, 1997.

[15] Ricardo Strausz. *On Separoids.* Ph.D. thesis, Universidad Nacional Autónoma de México, México, 2003.

[16] Ricardo Strausz. Representation of separoids and a Tverberg-type problem. In Martin Mareš, ed., *Proc. 9th Midsummer Combinatorial Workshop*, vol. 686 of *KAM-DIMATIA Series*, pp. 42–51. Department of Applied Mathematics, Charles University, Prague, Czech Republic, 2004.

School of Computer Science and School of Mathematics and Statistics, McGill University, Montréal, Canada.

E-mail address: wood@cs.mcgill.ca