

VERTEX PARTITIONS OF CHORDAL GRAPHS

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ABSTRACT. A k -tree is a chordal graph with no $(k + 2)$ -clique. An ℓ -tree-partition of a graph G is a vertex partition of G into ‘bags’, such that contracting each bag to a single vertex gives an ℓ -tree (after deleting loops and replacing parallel edges by a single edge). We prove that for all $k \geq \ell \geq 0$, every k -tree has an ℓ -tree-partition in which every bag induces a connected $\lfloor k/(\ell + 1) \rfloor$ -tree. An analogous result is proved for oriented k -trees.

1. INTRODUCTION

Let G be an (undirected, simple, finite) graph with vertex set $V(G)$ and edge set $E(G)$. The neighbourhood of a vertex v of G is denoted by $N(v) = \{w \in V(G) : vw \in E(G)\}$. A *chord* of a cycle C is an edge not in C whose endpoints are both in C . G is *chordal* if every cycle on at least four vertices has a chord. A k -clique ($k \geq 0$) is a set of k pairwise adjacent vertices. A k -tree is a chordal graph with no $(k + 2)$ -clique. The *tree-width* of G , denoted by $\text{tw}(G)$, is the minimum k such that G is a subgraph of a k -tree. It is well known that G is a k -tree if and only if $V(G) = \emptyset$, or G has a vertex v such that $G \setminus v$ is a k -tree, and $N(v)$ is a k' -clique for some $k' \leq k$.

Let G and H be graphs. The elements of $V(H)$ are called *nodes*. Let $\{H_x \subseteq V(G) : x \in V(H)\}$ be a set of subsets of $V(G)$ indexed by the nodes of H . Each set H_x is called a *bag*. The pair $(H, \{H_x \subseteq V(G) : x \in V(H)\})$ is an H -partition of G if:

- \forall vertices v of G , \exists node x of H with $v \in H_x$, and
- \forall distinct nodes x and y of H , $H_x \cap H_y = \emptyset$, and
- \forall edge vw of G , either
 - \exists node x of H with $v \in H_x$ and $w \in H_x$, or
 - \exists edge xy of H with $v \in H_x$ and $w \in H_y$.

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For brevity we say H is a partition of G . A k -tree-partition is an H -partition for some k -tree H . A tree-partition is a 1-tree-partition. Tree-partitions were independently introduced by Seese [13] and Halin [12], and have since been investigated by a number of authors [2, 3, 6, 7, 12, 13]. The main property of tree-partitions that has been studied is the maximum cardinality of a bag, called the *width* of the tree-partition. The minimum width over all tree-partitions of a graph G is the *tree-partition-width*¹ of G , denoted by $\text{tpw}(G)$. A graph with bounded degree has bounded tree-partition-width if and only if it has bounded tree-width [7]. In particular, for every graph G , Seese [13] proved that $\text{tw}(G) \leq 2 \text{tpw}(G) - 1$, and Ding and Oporowski [6] proved that $\text{tpw}(G) \leq 24 \text{tw}(G) \max\{\Delta(G), 1\}$, where $\Delta(G)$ is the maximum degree of G . See [1, 5, 8, 9] for other results related to tree-width and vertex partitions.

Tree-partition-width is not bounded above by any function solely of tree-width. For example, wheel graphs have bounded tree-width and unbounded tree-partition-width, as observed by Bodlaender and Engelfriet [3]. Thus, it seems unavoidable that the maximum degree appears in an upper bound on the tree-partition-width. This fact, along with other applications, motivated Dujmović *et al.* [10, 11] to study the structure of the bags in a tree-partition. In this paper we continue this approach, and prove the following result (in Section 2).

Theorem 1. *Let k and ℓ be integers with $k \geq \ell \geq 0$. Let $t = \lfloor k/(\ell + 1) \rfloor$. Every k -tree G has an ℓ -tree-partition in which each bag induces a connected t -tree in G .*

It is easily seen that Theorem 1 is tight for $G = K_{k+1}$ and for all ℓ . Note that Theorem 1 can be interpreted as a statement about chromomorphisms (see [15, 16]).

Dujmović *et al.* [10, 11] proved that every k -tree has a tree-partition in which each bag induces a $(k - 1)$ -tree. Thus Theorem 1 with $\ell = 1$ improves this result. That said, the tree-partition of Dujmović *et al.* [10, 11] has a number of additional properties that were important for the intended application. We generalise these additional properties in Section 3. The price paid is that each bag may now induce a $(k - \ell)$ -tree, thus matching the result of Dujmović *et al.* [10, 11] for $\ell = 1$. Note that the proof of Dujmović *et al.* [10, 11] uses a different construction to the one given here.

2. PROOF OF THEOREM 1

We proceed by induction on $|V(G)|$. If $V(G) = \emptyset$, then the result holds with $V(H) = \emptyset$ regardless of k and ℓ . Now suppose that $|V(G)| \geq 1$. Thus G has a vertex v such that $G \setminus v$ is a k -tree, and $N(v)$ is a k' -clique for some $k' \leq k$. By induction, $G \setminus v$ has an ℓ -tree-partition H in which each bag induces a connected t -tree. Let $C = \{x \in V(H) : N(v) \cap H_x \neq \emptyset\}$. Since $N(v)$ is a clique, C is a clique of H (by the definition of H -partition). Since H is an ℓ -tree, $|C| \leq \ell + 1$.

¹Tree-partition-width has also been called *strong tree-width* [3, 13].

Case 1. $|C| \leq \ell$: Add one new node y to H adjacent to each node $x \in C$. Since C is a clique of H and $|C| \leq \ell$, H remains an ℓ -tree. Let $H_y = \{v\}$. The other bags remain unchanged. Since $t \geq 0$, H_y induces a connected t -tree ($= K_1$) in G . Thus H is now a partition of G in which each bag induces a connected t -tree in G .

Case 2. $|C| = \ell + 1$: There is a node $y \in C$ such that $|N(v) \cap H_y| \leq t$, as otherwise $|N(v)| \geq (t + 1)|C| = (\lfloor k/(\ell + 1) \rfloor + 1)(\ell + 1) \geq k + 1$. Add v to the bag H_y . Let $u \in N(v) \cap H_y$. Every neighbour of v not in H_y is adjacent to u (in $G \setminus v$). Thus H is a partition of G . H_y induces a connected t -tree in G , since $H_y \setminus \{v\}$ induces a connected t -tree in $G \setminus v$, and the neighbourhood of v in H_y is a clique of at least one and at most t vertices. The other bags do not change. Thus each bag of H induces a connected t -tree in G . \square

3. ORIENTED PARTITIONS

Let G be an oriented graph with arc set $A(G)$. Let \widehat{G} be the underlying undirected graph of G . The in- and out-neighbourhoods of a vertex v of G are respectively denoted by $N^-(v) = \{u \in V(G) : uv \in A(G)\}$ and $N^+(v) = \{w \in V(G) : vw \in A(G)\}$. It is easily seen that an (undirected) graph G is a k -tree if and only if there is an acyclic orientation of G such that for every vertex v of G , $N^-(v)$ is a k' -clique for some $k' \leq k$. An oriented graph with this property is called an *oriented k -tree*. Let G and H be oriented graphs. An *oriented H -partition* of G is an \widehat{H} -partition of \widehat{G} such that for every arc xy of H , and for every edge vw of \widehat{G} with $v \in H_x$ and $w \in H_y$, vw is oriented from v to w . This concept is similar to an oriented homomorphism (see [4, 14] for example).

Theorem 2. *Let k and ℓ be integers with $k \geq \ell \geq 0$. Let $t = k - \ell$. Every oriented k -tree G has an oriented ℓ -tree partition H in which each bag induces a weakly connected oriented t -tree in G . Moreover, for every node x of H , the set of vertices $Q(x) = \bigcup_{v \in H_x} (N^-(v) \setminus H_x)$ is a k' -clique of G for some $k' \leq k$.*

The construction in the proof of Theorem 2 only differs from that of Theorem 1 in the choice of the node y in Case 2.

Proof. We proceed by induction on $|V(G)|$. If $V(G) = \emptyset$, then the result holds with $V(H) = \emptyset$ regardless of k and ℓ . Now suppose that $|V(G)| \geq 1$. Since G is acyclic, there is a vertex v of G such that $N^+(v) = \emptyset$, $N^-(v)$ is a k' -clique for some $k' \leq k$, and $G \setminus v$ is an oriented k -tree. By induction, there is an oriented ℓ -tree-partition H of $G \setminus v$ in which each bag induces a weakly connected oriented t -tree in $G \setminus v$. Moreover, for every node x of H , $Q(x)$ is a k' -clique for some $k' \leq k$. Let $C = \{x \in V(H) : N^-(v) \cap H_x \neq \emptyset\}$. Since $N^-(v)$ is a clique, C is a clique of H . Since H is an oriented ℓ -tree, $|C| \leq \ell + 1$.

Case 1. $|C| \leq \ell$: Add one new node y to H adjacent to each node $x \in C$. Orient each new edge from x to y . Obviously H remains acyclic. Since C is a clique of H and $|C| \leq \ell$, H remains an oriented ℓ -tree. Let $H_y = \{v\}$. The other bags are

unchanged. Since $t \geq 0$, H_y induces a weakly connected oriented t -tree ($= K_1$) in G . All edges of G that are incident to a vertex in H_y are oriented into the vertex in H_y . Thus H is now an oriented partition of G in which each bag induces a weakly connected oriented t -tree in G . Now $Q(y) = N^-(v)$, which is a k' -clique for some $k' \leq k$. $Q(x)$ is unchanged for nodes $x \neq y$. Hence the theorem is satisfied.

Case 2. $|C| = \ell + 1$: The clique C induces an acyclic tournament in H . Let y be the sink of this tournament. Since $|N^-(v) \cap H_x| \geq 1$ for every node $x \in C \setminus \{y\}$, $|N^-(v) \cap H_y| \leq k' - (|C| - 1) \leq k - \ell = t$. Add v to the bag H_y .

Consider a neighbour u of v . Since $N^+(v) = \emptyset$, uv is oriented from u to v . Say $u \in H_z$ with $z \neq y$. Then z is in the clique C . Thus zy is an edge of H . Since y is a sink of C , zy is oriented from z to y . Thus H is now an oriented partition of G . H_y induces a weakly connected oriented t -tree in G , since $H_y \setminus \{v\}$ induces an oriented t -tree in $G \setminus v$, and the in-neighbourhood of v in H_y is a clique of at least one and at most t vertices. The other bags do not change. Thus each bag of H induces a weakly connected oriented t -tree in G .

$Q(y)$ is not changed by the addition of v to H_y , as there is at least one vertex $u \in N^-(v) \cap H_y$, and any vertex in $N^-(v) \setminus H_y$ is also in $N^-(u) \setminus H_y$. For nodes $x \neq y$, $Q(x)$ is unchanged by the addition of v to H_y , since v is not in the in-neighbourhood of any vertex. Hence the theorem is satisfied. \square

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