

Drawing a Graph in a Hypercube*

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Abstract

A *d-dimensional hypercube drawing* of a graph represents the vertices by distinct points in $\{0, 1\}^d$, such that the line-segments representing the edges do not cross. We study lower and upper bounds on the minimum number of dimensions in hypercube drawing of a given graph. This parameter turns out to be related to Sidon sets and antimagic injections.

1 Introduction

Two-dimensional graph drawing [5, 14], and to a lesser extent, three-dimensional graph drawing [6, 16, 25] have been widely studied in recent years. Much less is known about graph drawing in higher dimensions. For research in this direction, see references [3, 8, 24, 25]. This paper studies drawings of graphs in which the vertices are positioned at the points of a hypercube.

We consider undirected, finite, and simple graphs G with vertex set $V(G)$ and edge set $E(G)$. Consider an injection $\lambda : V(G) \rightarrow \{0, 1\}^d$. For each edge $vw \in E(G)$, let $\lambda(vw)$ be the open line-segment with endpoints $\lambda(v)$ and $\lambda(w)$. Two distinct edges $vw, xy \in E(G)$ *cross* if $\lambda(vw) \cap \lambda(xy) \neq \emptyset$. We say λ is a *d-dimensional hypercube drawing* of G if no two edges of G cross. A *d-dimensional hypercube drawing* is said to have *volume* 2^d . That is, the volume is the total number of points in the hypercube, and is a measure of the efficiency of the drawing. Let $\text{vol}(G)$ be the minimum volume of a hypercube drawing of a graph G . This paper studies lower and upper bounds on $\text{vol}(G)$.

The remainder of the paper is organised as follows. In Section 2 we review material on Sidon sets and so-called antimagic injections of graphs. In Section 3 we explore the relationship

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between hypercube drawings and antimagic injections. This enables lower and upper bounds on $\text{vol}(K_n)$ to be proved. In Section 4, we present a simple algorithm for computing an antimagic injection that gives upper bounds on the volume of hypercube drawings in terms of the degeneracy of the graph. In Section 5 we prove a relationship between antimagic injections and queue layouts of graphs that enables an \mathcal{NP} -completeness result to be concluded. In Section 6 we relate antimagic injections of graphs to the bandwidth and pathwidth parameters. Finally, in Section 7 we give an asymptotic bound on the volume of hypercube drawings. The proof is based on the Lovász Local Lemma.

2 Sidon Sets and Antimagic Injections

A set $S \subseteq \mathbb{Z}^+$ is called *Sidon* if $a + b = c + d$ implies $\{a, b\} = \{c, d\}$ for all $a, b, c, d \in S$. See the recent survey by O’Bryant [20] for results and numerous references on Sidon sets. A graph in which self-loops are allowed (but no parallel edges) is called a *pseudograph*. For a pseudograph G , an injection $f : V(G) \rightarrow \mathbb{Z}^+$ is *antimagic* if $f(v) + f(w) \neq f(x) + f(y)$ for all distinct edges $vw, xy \in E(G)$; see [1, 11, 26]. Let $[k] := \{1, 2, \dots, k\}$. Let $\text{mag}(G)$ be the minimum k such that the pseudograph G has an antimagic injection $f : V(G) \rightarrow [k]$.

Let K_n^+ be the complete pseudograph; that is, every pair of vertices are adjacent and there is one loop at every vertex. Clearly an antimagic injection of K_n^+ is nothing more than a Sidon set of cardinality n . It follows from results by Singer [21] and Erdős and Turán [10] (see Bollobás and Pikhurko [1]) that

$$\text{mag}(K_n) = (1 + o(1))n^2 \text{ and } \text{mag}(K_n^+) = (1 + o(1))n^2 . \quad (1)$$

Note the following simple lower bound.

Lemma 1. *Every pseudograph G satisfies $\text{mag}(G) \geq \max\{|V(G)|, \frac{1}{2}(|E(G)| + 3)\}$.*

Proof. That $\text{mag}(G) \geq |V(G)|$ follows from the definition. Let $\lambda : V(G) \rightarrow [k]$ be an antimagic injection of G . For every edge $vw \in E(G)$, $\lambda(v) + \lambda(w)$ is a distinct integer in $\{3, 4, \dots, 2k - 1\}$. Thus $|E(G)| \leq 2k - 3$ and $k \geq \frac{1}{2}(|E(G)| + 3)$. \square

3 Hypercube Drawings

Consider the maximum number of edges in a hypercube drawing. The following observation is a special case of a result by Bose *et al.* [2] regarding the volume of grid drawings, where the bounding box is unrestricted.

Lemma 2 ([2]). *The maximum number of edges in a d -dimensional hypercube drawing is $3^d - 2^d$.*

Trivially, $\text{vol}(G) \geq |V(G)|$. For dense graphs, we have the following improved lower bound.

Lemma 3. *Every n -vertex m -edge graph G satisfies $\text{vol}(G) \geq (n + m)^{1/\log_2 3} = (n + m)^{0.631\dots}$.*

Proof. Suppose that G has a d -dimensional hypercube drawing. By Lemma 2 and since $n \leq 2^d$, we have $n + m \leq 3^d$. That is, $d \geq \log_2(n + m)/\log_2 3$, and the volume $2^d \geq (n + m)^{1/\log_2 3}$. \square

Now we characterise when two edges cross.

Lemma 4. *Consider an injection $\lambda : V(G) \rightarrow \{0, 1\}^d$ for some graph G . Two distinct edges $vw, xy \in E(G)$ cross if and only if $\lambda(v) + \lambda(w) = \lambda(x) + \lambda(y)$.*

Proof. Suppose that $\lambda(v) + \lambda(w) = \lambda(x) + \lambda(y)$. Then $\frac{1}{2}(\lambda(v) + \lambda(w)) = \frac{1}{2}(\lambda(x) + \lambda(y))$. That is, the midpoint of $\lambda(vw)$ equals the midpoint of $\lambda(xy)$. Hence vw and xy cross. (Note that this idea is used to prove the upper bound in Lemma 2, since the number of midpoints is at most $3^d - 2^d$.) Conversely, suppose that vw and xy cross. Since all vertex coordinates are 0 or 1, the point of intersection between $\lambda(vw)$ and $\lambda(xy)$ is the midpoint of both edges. That is, $\frac{1}{2}(\lambda(v) + \lambda(w)) = \frac{1}{2}(\lambda(x) + \lambda(y))$, and $\lambda(v) + \lambda(w) = \lambda(x) + \lambda(y)$. \square

Loosely speaking, Lemma 4 implies that a hypercube drawing of G can be thought of as an antimagic injection of G into a set of boolean vectors (where vector addition is *not* modulo 2). Moreover, from an antimagic injection we can obtain a hypercube drawing, and vice versa.

Lemma 5. *Every graph G satisfies $\text{vol}(G) \leq 2^{\lceil \log_2 \text{mag}(G) \rceil} < 2 \text{mag}(G)$.*

Proof. Let $k := \text{mag}(G)$, and let $f : V(G) \rightarrow [k]$ be an antimagic injection of G . For each vertex $v \in V(G)$, let $\lambda(v)$ be the $\lceil \log_2 k \rceil$ -bit binary representation of $f(v)$. Suppose that edges vw and xy cross. By Lemma 4, $\lambda(v) + \lambda(w) = \lambda(x) + \lambda(y)$. For each $1 \leq i \leq \lceil \log_2 k \rceil$, the sum of the i -th coordinates of v and w equals the sum of the i -th coordinates of x and y . Thus $f(v) + f(w) = f(x) + f(y)$, which is the desired contradiction. Therefore no two edges cross, and λ is a $\lceil \log_2 k \rceil$ -dimensional hypercube drawing of G . \square

Lemma 6. *Every graph G satisfies $\text{mag}(G) \leq \text{vol}(G)^{\log_2 3} = \text{vol}(G)^{1.585\dots}$.*

Proof. Let $\lambda : V(G) \rightarrow \{0, 1\}^d$ be a hypercube drawing of G , where $d = \log_2 \text{vol}(G)$. For each vertex $v \in V(G)$, define an integer $f(v)$ so that $\lambda(v)$ is the base-3 representation of $f(v)$. Now $\lambda(v) + \lambda(w) \in \{0, 1, 2\}^d$. Thus $\lambda(v) + \lambda(w) = \lambda(x) + \lambda(y)$ if and only if $f(v) + f(w) = f(x) + f(y)$. Since edges do not cross in λ and by Lemma 4, f is an antimagic injection of G into $[3^d] = [3^{\log_2 \text{vol}(G)}] = \lceil \text{vol}(G)^{\log_2 3} \rceil$. \square

Consider the minimum volume of a hypercube drawing of the complete graph K_n .

Lemma 7. *Let $V = \{\vec{v}_1, \vec{v}_2, \dots, \vec{v}_n\}$ be a set of binary d -dimensional vectors. Then V is the vertex set of a hypercube drawing of K_n if and only if $\vec{v}_i + \vec{v}_j \neq \vec{v}_k + \vec{v}_\ell$ for all distinct pairs $\{i, j\}$ and $\{k, \ell\}$.*

Proof. Suppose that V is the vertex set of a hypercube drawing of K_n . Since no two edges cross, by Lemma 4, $\vec{v}_i + \vec{v}_j \neq \vec{v}_k + \vec{v}_\ell$ for all distinct pairs $\{i, j\}$ and $\{k, \ell\}$ with $i \neq j$ and $k \neq \ell$. If $i = j$ and $k = \ell$, then $\vec{v}_i + \vec{v}_j \neq \vec{v}_k + \vec{v}_\ell$ because distinct vertices are mapped to distinct points. If $i = j$ and $k \neq \ell$, then $\vec{v}_i + \vec{v}_j \neq \vec{v}_k + \vec{v}_\ell$, as otherwise the midpoint of the edge $v_k v_\ell$ would coincide with the vertex v_i , which is clearly impossible. Hence $\vec{v}_i + \vec{v}_j \neq \vec{v}_k + \vec{v}_\ell$ for all distinct pairs $\{i, j\}$ and $\{k, \ell\}$. The converse result follows immediately from Lemma 4. \square

Sets of binary vectors satisfying Lemma 7 were first studied by Lindström [17, 18], and more recently by Cohen *et al.* [4]. Their results can be interpreted as follows, where the lower bound is by Cohen *et al.* [4], and the upper bound follows from (1) and Lemma 5.

Theorem 1. *Every complete graph K_n satisfies $\text{vol}(K_n) < (2+o(1))n^2$, and $\text{vol}(K_n) > n^{1.7384\dots}$ for large enough n .* \square

4 Degeneracy

Wood [26] proved that every n -vertex m -edge graph G with maximum degree Δ satisfies $\text{mag}(G) < (\Delta(m - \Delta) + n)$. Thus Lemma 5 implies that

$$\text{vol}(G) < 2(\Delta(m - \Delta) + n) . \quad (2)$$

This result of Wood [26] is proved using a greedy algorithm. We can obtain a more precise result as follows. The *degeneracy* of a graph G is the maximum, taken over all induced subgraphs H of G , of the minimum degree of H .

Lemma 8. *Every n -vertex m -edge graph G with degeneracy d satisfies $\text{mag}(G) \leq n + dm$, and thus $\text{vol}(G) < 2n + 2dm$.* \square

Proof. We proceed by induction on n' with the hypothesis that “every induced subgraph H of G on n' vertices has $\text{mag}(H) \leq n' + dm$.” If $n' = 1$ the result is trivial. Let H be an induced subgraph of G on $n' \geq 2$ vertices. Then H has a vertex v of degree at most d in H . By induction, $H \setminus v$ has an antimagic injection $\lambda : V(H \setminus v) \rightarrow [n' - 1 + dm]$. Now

$$\begin{aligned} & \left| \{ \lambda(x) : x \in V(H \setminus v) \} \cup \{ \lambda(x) + \lambda(y) - \lambda(w) : xy \in E(H \setminus v), vw \in E(H) \} \right| \\ & \leq |V(H \setminus v)| + \deg_H(v) \cdot |E(H \setminus v)| \\ & \leq n' - 1 + dm . \end{aligned}$$

Thus there exists an $i \in [n' + dm]$ such that $\lambda(x) \neq i$ for all $x \in V(H \setminus v)$, and $\lambda(x) + \lambda(y) - \lambda(w) \neq i$ for all edges $xy \in E(H \setminus v)$ and $vw \in E(H)$. Let $\lambda(v) := i$. Thus $\lambda(v) \neq \lambda(x)$ for all $x \in V(H)$, and $\lambda(v) + \lambda(w) \neq \lambda(x) + \lambda(y)$ for all edges $xy \in E(H)$ and $vw \in E(G)$. Thus λ is an antimagic injection of H into $[n' + dm]$, and $\text{mag}(H) \leq n' + dm$. By induction, $\text{mag}(G) \leq n + dm$. \square

Planar graphs G are 5-degenerate, and thus satisfy $\text{mag}(G) < 16n$ and $\text{vol}(G) < 32n$ by Lemmata 5 and 8. More generally, Kostochka [15] and Thomason [22, 23] independently proved that a graph G with no K_k minor is $\mathcal{O}(k\sqrt{\log k})$ -degenerate, and thus satisfy $\text{mag}(G) \in \mathcal{O}(k^2(\log k)n)$ and $\text{vol}(G) \in \mathcal{O}(k^2(\log k)n)$ by Lemmata 5 and 8. As we now show, a large clique minor does not necessarily force up $\text{mag}(G)$ or $\text{vol}(G)$. Let K'_n be the graph obtained from K_n by subdividing every edge once. Say K'_n has $n' := n + \binom{n}{2}$ vertices. Clearly K'_n is 2-degenerate. It follows from Lemma 8 that $\text{mag}(K'_n) \leq 5n' + o(n')$ and $\text{vol}(K'_n) \leq 10n' + o(n')$, yet K'_n contains a $(\sqrt{2n'} + o(n'))$ -clique minor.

5 Queue Layouts and Complexity

Let G be a graph. A bijection $\sigma : V(G) \rightarrow [|V(G)|]$ is called a *vertex ordering* of G . Consider edges $vw, xy \in E(G)$ with no common endpoint. Without loss of generality $\sigma(v) < \sigma(w)$, $\sigma(x) < \sigma(y)$ and $\sigma(v) < \sigma(x)$. We say vw and xy are *nested* in σ if $\sigma(v) < \sigma(x) < \sigma(y) < \sigma(w)$. A *queue* in σ is a set of edges $Q \subseteq E(G)$ such that no two edges in Q are nested in σ . A *k-queue layout* of G consists of a vertex ordering σ of G , and a partition of $E(G)$ into k queues in σ . Heath *et al.* [12, 13] introduced queue layouts; see Dujmović and Wood [7] for references and a summary of known results.

Lemma 9. *If a graph G has a 1-queue layout, then $\text{mag}(G) = |V(G)|$.*

Proof. Let $\sigma : V(G) \rightarrow [|V(G)|]$ be the vertex ordering in a 1-queue layout of G . If for distinct edges $vw, xy \in E(G)$, we have $\sigma(v) + \sigma(w) = \sigma(x) + \sigma(y)$, then vw and xy are nested. Since no two edges are nested in a 1-queue layout, σ is an antimagic injection of G , and $\text{mag}(G) \leq |V(G)|$. \square

Heath and Rosenberg [13] proved that it is \mathcal{NP} -complete to determine if a given graph has a 1-queue layout. Thus, Lemma 9 implies:

Corollary 1. *Testing whether $\text{mag}(G) = |V(G)|$ is \mathcal{NP} -complete.* \square

It has been widely conjectured that it is \mathcal{NP} -complete to recognise graphs that admit certain types of magic and antimagic injections. Corollary 1 is the first result in this direction that we are aware of.

Open Problem 1. Dujmović and Wood [7] proved that a k -queue graph G on n vertices has less than $2kn$ edges and is $4k$ -degenerate. By Lemma 8, $\text{mag}(G) \in \mathcal{O}(k^2n)$ and $\text{vol}(G) \in \mathcal{O}(k^2n)$. Can these bounds be improved to $\mathcal{O}(kn)$?

6 Bandwidth and Pathwidth

Let P_n^k be the k -th power of a path. That is, P_n^k is the graph with vertex set $\{v_0, v_1, \dots, v_{n-1}\}$ and edge set $\{v_i v_j : 1 \leq |i - j| \leq k\}$. Now P_n^k has $kn - \frac{1}{2}k(k+1)$ edges. By Lemma 1, $\text{mag}(P_n^k) \geq \frac{1}{2}(kn - \frac{1}{2}k(k+1) + 3)$. The following upper bound is a generalisation of the construction of a Sidon set by Erdős and Turán [10].

Lemma 10. *For every prime p , $\text{mag}(P_n^p) \leq p(2n - 1)$.*

Proof. If $p = 2$ then $\text{mag}(P_n^2)$ has a 1-queue layout, and $\text{mag}(P_n^2) = n$ by Lemma 9. Now assume that $p > 2$. Let $\lambda(v_i) := 1 + 2pi + (i^2 \bmod p)$ for every vertex v_i , $0 \leq i \leq n - 1$. Clearly λ is an injection into $[p(2n - 1)]$. Suppose on the contrary, that there are distinct edges $v_i v_\ell$ and $v_j v_k$ with $\lambda(v_i) + \lambda(v_\ell) = \lambda(v_j) + \lambda(v_k)$. Without loss of generality, $i < j < k < \ell \leq i + p$. Then

$$2pi + (i^2 \bmod p) + 2p\ell + (\ell^2 \bmod p) = 2pj + (j^2 \bmod p) + 2pk + (k^2 \bmod p) .$$

That is,

$$2p(i + \ell - j - k) = (j^2 \bmod p) + (k^2 \bmod p) - (i^2 \bmod p) - (\ell^2 \bmod p) .$$

Now $|(j^2 \bmod p) + (k^2 \bmod p) - (i^2 \bmod p) - (\ell^2 \bmod p)| \leq 2(p-1)$. Thus $i + \ell - j - k = 0$, and

$$(i^2 \bmod p) + (\ell^2 \bmod p) = (j^2 \bmod p) + (k^2 \bmod p) .$$

Thus

$$i^2 + \ell^2 \equiv j^2 + k^2 \pmod{p} . \quad (3)$$

Let $a := j - i$ and $b := k - i$. Then $0 < a < b < p$. Since $i + \ell = j + k$, we have $\ell = i + a + b$. Rewriting (3),

$$i^2 + (i + a + b)^2 \equiv (i + a)^2 + (i + b)^2 \pmod{p} .$$

Hence $2ab \equiv 0 \pmod{p}$. Since p is prime and $p > 2$, $a \equiv 0 \pmod{p}$ or $b \equiv 0 \pmod{p}$, which is a contradiction since $0 < a < b < p$. Hence $\lambda(v_i) + \lambda(v_\ell) \neq \lambda(v_j) + \lambda(v_k)$, and λ is antimagic. \square

The *bandwidth* of an n -vertex graph G is the minimum k such that G is a subgraph of P_n^k . By Bertrand's postulate there is a prime $p \leq 2k$. Thus Lemmata 5 and 10 imply:

Corollary 2. *Every n -vertex graph G with bandwidth k has $\text{mag}(G) \leq 2k(2n-1)$ and $\text{vol}(G) < 4k(2n-1)$.* \square

We have the following technical lemma.

Lemma 11. *Let G be a graph. Let $f_V : V(G) \rightarrow [t] \times [r]$ be an injection. Define a function $f_E : E(G) \rightarrow \binom{[t]}{2} \times [2r]$ as follows. For every edge $vw \in E(G)$ with $f_V(v) = (a, i)$ and $f_V(w) = (b, j)$, let $f_E(vw) := (\{a, b\}, i + j)$. If f_E is also an injection, then $\text{mag}(G) \leq (2 + o(1))t^2r$.*

Proof. Singer [21] proved that there is a Sidon set $\{s_1, s_2, \dots, s_t\} \in [(1 + o(1))t^2]$. For every vertex $v \in V(G)$ with $f(v) = (a, i)$, let $\lambda(v) := 2r(s_a - 1) + i$. Since f is an injection, λ is an injection into $[(2 + o(1))t^2r]$. We claim that λ is antimagic. Suppose on the contrary that there are distinct edges $vw, xy \in E(G)$ with $\lambda(v) + \lambda(w) = \lambda(x) + \lambda(y)$. Say $f(v) = (a, i)$, $f(w) = (b, j)$, $f(x) = (c, k)$, and $f(y) = (d, \ell)$. Then

$$2r(s_a - 1) + i + 2r(s_b - 1) + j = 2r(s_c - 1) + k + 2r(s_d - 1) + \ell . \quad (4)$$

That is, $2r(s_a + s_b - s_c - s_d) = k + \ell - i - j$. Now $|k + \ell - i - j| < 2r$. Thus $s_a + s_b = s_c + s_d$. Since $\{s_1, s_2, \dots, s_t\}$ is Sidon, $\{a, b\} = \{c, d\}$. By (4), $i + j = k + \ell$. Hence, $f_E(vw) = f_E(xy)$, which is a contradiction since f_E is an injection by assumption. Thus $\lambda(v) + \lambda(w) \neq \lambda(x) + \lambda(y)$, and λ is antimagic. Hence $\text{mag}(G) \leq (2 + o(1))t^2r$. \square

Let \mathcal{S} be a set of closed intervals in \mathbb{R} . Associated with \mathcal{S} , is the *interval graph* with vertex set \mathcal{S} such that two vertices are adjacent if and only if the corresponding intervals have a non-empty intersection. The *pathwidth* of a graph G is the minimum k such that G is a spanning subgraph of an interval graph with no clique on $k + 2$ vertices.

Theorem 2. *Every n -vertex graph G with pathwidth k satisfies $\text{mag}(G) \leq (8 + o(1))kn$ and $\text{vol}(G) \leq (16 + o(1))kn$. For all k and $n \geq k + 1$, there exist n -vertex graphs G with pathwidth k and $\text{mag}(G) \geq \frac{1}{2}kn - \mathcal{O}(k^2)$.*

Proof. Dujmović *et al.* [6] proved that there is an injection f satisfying Lemma 11 with $t = 2k + 2$ and $r = \lceil n/k \rceil$. In fact, they proved the stronger result that for all edges $vw, xy \in E(G)$ with $f(v) = (a, i)$, $f(w) = (b, j)$, $f(x) = (a, k)$, $f(y) = (b, \ell)$, if $i < k$ then $j \leq \ell$ (which implies that $i + j < k + \ell$). By Lemma 11, $\text{mag}(G) \leq (2 + o(1))(2k + 2)^2 r = (8 + o(1))kn$. By Lemma 5, $\text{vol}(G) \leq (16 + o(1))kn$. For the lower bound, let $G = P_n^k$ for example. Then G has pathwidth k and $kn - \frac{1}{2}k(k + 1)$ edges. By Lemma 1, $\text{mag}(G) \geq \frac{1}{2}kn - \mathcal{O}(k^2)$. \square

Open Problem 2. Lemma 8 implies that graphs G of treewidth k satisfy $\text{mag}(G) \in \mathcal{O}(k^2n)$ and $\text{vol}(G) \in \mathcal{O}(k^2n)$. Can these bounds be improved to $\mathcal{O}(kn)$? Note that Wood [26] proved that every tree G satisfies $\text{mag}(G) = |V(G)|$, which implies that $\text{vol}(G) < 2|V(G)|$ by Lemma 5.

7 An Asymptotic Upper Bound

Our upper bounds on $\text{vol}(G)$ have thus far been obtained as corollaries of upper bounds on $\text{mag}(G)$. The next theorem, which improves upon (2), only applies to hypercube drawings. In fact, the method used only gives a $\mathcal{O}(n + \Delta m)$ bound on $\text{mag}(G)$.

Theorem 3. *Every n -vertex m -edge graph G with maximum degree Δ satisfies*

$$\text{vol}(G) \leq \mathcal{O}(n + (\Delta m)^{1/\log_2 8/3}) = \mathcal{O}(n + (\Delta m)^{0.707\dots}) .$$

Theorem 3 is proved using the Local Lemma of Erdős and Lovász [9] (see [19]).

Lemma 12 ([9]). *Let $\mathcal{E} = \{A_1, A_2, \dots, A_n\}$ be a set of ‘bad’ events in some probability space, such that each event A_i is mutually independent of $\mathcal{E} \setminus (\{A_i\} \cup \mathcal{D}_i)$ for some $\mathcal{D}_i \subseteq \mathcal{E}$. Suppose there is a set $\{x_i \in [0, 1) : 1 \leq i \leq n\}$, such that for all i ,*

$$\mathbf{P}(A_i) \leq x_i \cdot \prod_{A_j \in \mathcal{D}_i} (1 - x_j) . \tag{5}$$

Then

$$\mathbf{P} \left(\bigwedge_{i=1}^n \overline{A_i} \right) \geq \prod_{i=1}^n (1 - x_i) > 0 .$$

That is, with positive probability, no event in \mathcal{E} occurs.

Proof of Theorem 3. Let d be a positive integer, to be specified later. For each vertex $v \in V(G)$, let $\lambda(v)$ be a point in $\{0, 1\}^d$ chosen randomly and independently. (One can think of this process as d fair coin tosses for each vertex.) We now set up an application of Lemma 12. For all pairs of distinct vertices $v, w \in V(G)$, let $A_{v,w}$ be the event that $\lambda(v) = \lambda(w)$. For all disjoint edges $vw, xy \in E(G)$, let $B_{vw,xy}$ be the event that vw and xy cross.

We will apply Lemma 12 to prove that with positive probability, no event occurs. Hence there exists λ such that no event occurs. No A -event means that λ is an injection. No B -event means that no edges cross. Thus λ is a d -dimensional hypercube drawing.

Observe that $\mathbf{P}(A_{v,w}) = (\frac{1}{2})^d$. It is easily seen that $\mathbf{P}(B_{vw,xy}) \leq (\frac{1}{2})^d$. Below we prove that $\mathbf{P}(B_{vw,xy}) = (\frac{3}{8})^d$. The idea here is that it is unlikely that some edges are involved in a crossing. For example, the actual edges of the hypercube cannot be in a crossing.

Let $M := \{(x_1, x_2, \dots, x_d) : x_i \in \{0, 1, 2\}, 1 \leq i \leq d\}$. Consider an edge $vw \in E(G)$. Clearly $\lambda(v) + \lambda(w) \in M$. The i -coordinate of $\lambda(v) + \lambda(w)$ equals 1 if and only if the i -coordinates of $\lambda(v)$ and $\lambda(w)$ are distinct, which occurs with probability $\frac{1}{2}$. The i -coordinate of $\lambda(v) + \lambda(w)$ equals 0 if and only if the i -coordinates of $\lambda(v)$ and $\lambda(w)$ both equal 0, which occurs with probability $\frac{1}{4}$. The i -coordinate of $\lambda(v) + \lambda(w)$ equals 2 if and only if the i -coordinates of $\lambda(v)$ and $\lambda(w)$ both equal 1, which occurs with probability $\frac{1}{4}$.

Let M_k be the subset of M consisting of those points with exactly k coordinates equal to 1. Thus, for every edge $vw \in E(G)$ and point $p \in M_k$,

$$\mathbf{P}(\lambda(v) + \lambda(w) = p) = (\frac{1}{2})^k (\frac{1}{4})^{d-k} = 2^{k-2d} .$$

Hence for all disjoint edges $vw, xy \in E(G)$ and points $p \in M_k$,

$$\mathbf{P}(\lambda(v) + \lambda(w) = \lambda(x) + \lambda(y) = p) = 2^{2k-4d} .$$

Now $|M_k| = \binom{d}{k} 2^{d-k}$. Thus,

$$\mathbf{P}(\lambda(v) + \lambda(w) = \lambda(x) + \lambda(y) \in M_k) = \binom{d}{k} 2^{d-k} \cdot 2^{2k-4d} = \binom{d}{k} 2^{k-3d} .$$

Thus by Lemma 4,

$$\mathbf{P}(B_{vw,xy}) = \mathbf{P}(\lambda(v) + \lambda(w) = \lambda(x) + \lambda(y)) = \sum_{k=0}^d \binom{d}{k} 2^{k-3d} = \left(\frac{3}{8}\right)^d .$$

The base of the natural logarithm e satisfies the following well-known inequality for all $y > 0$:

$$\frac{1}{e} < \left(1 - \frac{1}{y+1}\right)^y . \quad (6)$$

Now define

$$d := \left\lceil \max \left\{ \log_2 e(4n+1), \log_{8/3} e^2(4\Delta m+1) \right\} \right\rceil . \quad (7)$$

For each A -event, let $x_A := 1/(4n+1)$. For each B -event, let $x_B := 1/(4\Delta m+1)$. Thus $0 < x_A < 1$ and $0 < x_B < 1$, as required.

Each vertex is involved in at most n A -events, and at most Δm B -events. An A -event involves two vertices, and is thus dependent on at most $2n$ other A -events, and at most $2\Delta m$ B -events. A B -event involves four vertices, and is thus dependent on at most $4n$ A -events, and on at most $4\Delta m$ other B -events. We first verify (5) for each event $A_{v,w}$. By (6),

$$x_A (1 - x_A)^{2n} (1 - x_B)^{2\Delta m} = \frac{1}{4n+1} \left(1 - \frac{1}{4n+1}\right)^{2n} \left(1 - \frac{1}{4\Delta m+1}\right)^{2\Delta m} \geq \frac{1}{e(4n+1)} .$$

By the definition of d in (7), $\frac{1}{e(4n+1)} \geq \frac{1}{2^d}$, and thus

$$x_A (1 - x_A)^{2n} (1 - x_B)^{2\Delta m} \geq \left(\frac{1}{2}\right)^d = \mathbf{P}(A_{v,w}) .$$

Now we verify (5) for each event $B_{vw,xy}$. By (6),

$$x_B (1 - x_A)^{4n} (1 - x_B)^{4\Delta m} = \frac{1}{4\Delta m + 1} \left(1 - \frac{1}{4n + 1}\right)^{4n} \left(1 - \frac{1}{4\Delta m + 1}\right)^{4\Delta m} \geq \frac{1}{e^2(4\Delta m + 1)} .$$

Note that (7) implies that $(\frac{8}{3})^d \geq e^2(4\Delta m + 1)$. Thus,

$$x_B (1 - x_A)^{4n} (1 - x_B)^{4\Delta m} \geq \left(\frac{3}{8}\right)^d = \mathbf{P}(B_{vw,xy}) .$$

By Lemma 12, there is a d -dimensional hypercube drawing of G . The volume 2^d is $\mathcal{O}(n + (\Delta m)^{1/\log_2 8/3})$. This completes the proof of Theorem 3. \square

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