Short answers to exponentially long questions: Extremal aspects of homomorphism duality

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Abstract

We prove that there exists a constant k such that for every n there exist directed core graphs H with at least 2^n vertices such that a directed graph G is H-colourable if and only if every subgraph of G with at most $kn\log(n)$ vertices is H-colourable. Our examples show that in general the "duals of relational structures" in the sense of [11] can have superpolynomial size. The construction given in [11] gives a double exponential upper bound for such a construction. Here we improve this to an exponential upper bound.

1 Introduction

A *homomorphism* between two directed graphs G and H is a map ϕ from the vertex set of G to that of H such that $(\phi(x), \phi(y))$ is an arc of H whenever (x, y) is an arc of G. We write $G \to H$ when there exists a homomorphism from G to H. For a fixed target H, the H*-colouring problem* is the following decision problem:

> **H-colouring problem**
> Instance: A directed graph G.
> Question: Does there exists a homomorphism from G to H?

The complexity of the H-colouring problem depends on H. A complete classification seems out of reach for the moment, but the dichotomy conjecture of [2] (see also [1]) states that every H-colouring problem is polynomial or NP-complete.

Here we concentrate on a subclass of the polynomial H-colouring problems, namely those for which there exists a constant $m(H)$ such that the following holds.

> For every directed graph G, there exists a homomorphism from G to H if and only if every subgraph G' of G with at most $m(H)$ vertices admits a homomorphism to H.

The H-colouring problem can then be reduced to a polynomial search for an obstruction to a homomorphism among the subgraphs of G with at most

$m(H)$ vertices. The best know example of this situation is the relation between the transitive tournament on n vertices and the directed path with n forward edges (see [3, 4, 14, 16]): A directed graph G admits a homomorphism to the former if and only if it admits no homomorphism from the latter, hence it is sufficient to look for an obstruction among the subgraphs of G with at most $n+1$ vertices.

More generally, for any H-colouring problem considered here, there is only a finite list $O_1, O_2, \ldots, O_m$ of directed graphs with at most $m(H)$ vertices which do not admit a homomorphism to H. According to [11, Theorems 2.9, 3.1], the "minimal" obstructions among these are directed trees $T_1, \ldots, T_\ell$. For each tree T_i, there exists a "dual" directed graph D_i with the following property:

> For every directed graph G, there exists a homomorphism from G to D_i if and only if there exist no homomorphism from T_i to G;

and H is homomorphically equivalent to the product of these duals.

The construction given in [11] for the dual of a tree T gives a directed graph D which could have as many as $2^{2^{|V(T)|}}$ vertices, yielding $m(D) \simeq \lg(\lg(|V(D)|))$. However in the example cited above where T is the directed path with n forward arcs and D is the transitive tournament with n vertices, we have $m(D) = |V(D)|+1$. Indeed in all known cases, the dual D of T can be dismantled to a structure with the same order of magnitude as T. Thus questions arises as to whether polynomial constructions would be possible instead of the double exponential construction of [11].

In this paper, we answer these question by giving a new construction which always gives a dual with at most $2^{n \lg(n)}$ vertices for a tree with n vertices. The new construction is conceptually simpler and yields new insights in the structure of duals (see [15]). On the other hand, we can also exhibit trees with n vertices whose dual must have at least $2^{\Omega(n/\lg(n))}$ vertices, indicating that the new construction is close to optimal.

Our construction will be presented in the general context of relational structures, that is, the original context of [11] which is also the natural context of constraint satisfaction problems [1, 2]. Incidentally, we note that it is a specification of relational examples that led to the discovery of the examples mentioned above. We give the necessary terminology in the following section. The new construction of duals is given in Section 3 and the examples with large duals are given in Section 4. We will conclude with a few comments concerning the bound $m(H)$.

2 Relational structures

Let $\Delta = (\delta_i; i \in I)$ be a sequence of positive integers. A *relational structure of type* Δ (or Δ-structure) is a pair $A = (X, (R_i; i \in I))$ where X is a finite set and R_i is a δ_i-nary relation on X (that is, $R_i \subset X^{\delta_i}$). We will denote $\underline{A}$ the base set of A (that is, $\underline{A} = X$ when $A = (X, (R_i; i \in I))$).

Given a type Δ and Δ-structures $A = (X, (R_i; i \in I))$ and $A' = (X', (R'_i; i \in I))$ a *homomorphism* from A to A' is a mapping $f : X \mapsto X'$ such that for every $i \in I$ we have

$$(f(x_1), f(x_2), \ldots, f(x_{\delta_i})) \in R'_i \text{ whenever } (x_1, x_2, \ldots, x_{\delta_i}) \in R_i.$$

We write $A \to A'$ if there exists a homomorphism from A to A'.

For a Δ-structure H, the H-colouring problem is defined just as in the case of directed graphs:

> **H-colouring problem**
> Instance: A Δ-structure A.
> Question: Does there exists a homomorphism from A to H?

Two Δ-structures H and H' are called *homomorphically equivalent* if $H \to H'$ and from $H' \to H$; we then write $H \leftrightarrow H'$. Note that when $H \leftrightarrow H'$, we have $A \to H$ if and only if $A \to H'$ hence the H-colouring problem is equivalent to the H'-colouring problem.

A Δ-structure is called a *core* if it is not homomorphically equivalent to any Δ-structure on a smaller base set. Clearly, any Δ-structure is homomorphically equivalent to at least one core. It can be shown (see [11]) that two homomorphically equivalent cores are isomorphic, hence the core of any Δ-structure H is well defined up to isomorphism. In studying H-colouring problems, we can restrict our attention to the case where H is a core without loss of generality. Indeed, the parameter $m(H)$ presented in the introduction is not very interesting when H is not a core.

3 A construction of duals

Let A be a relational structure of type $\Delta = (\delta_i; i \in I)$. We define the *incidence graph* $\mathrm{Inc}(A)$ of A as the bipartite graph with parts $\underline{A}$ and

$$\mathrm{Block}(A) = \{(i, (a_1, \ldots, a_{\delta_i})) : i \in I, (a_1, \ldots, a_{\delta_i}) \in R_i(A)\},$$

and edges $[a, (i, (a_1, \ldots, a_{\delta_i}))]$ such that $a \in (a_1, \ldots, a_{\delta_i})$. (Here we write $x \in (x_1, \ldots, x_n)$ when there exists an index k such that $x = x_k$.) A is called a Δ-*tree* when $\mathrm{Inc}(A)$ is a tree.

A Δ-structure D is called a *dual* of A if for every Δ-structure X, there exists a homomorphism $\phi : X \mapsto D$ if and only if there is no homomorphism $\phi : A \mapsto X$. In [11], it was shown that a structure A admits a dual if and only if A is a Δ-tree[1]. Note that any two duals D, D' of A are necessarily homomophically equivalent. Therefore it is possible to define *the* dual of a Δ-tree A up to homomorphic equivalence. In [11] a construction for duals of Δ-trees is presented, using gaps and exponentiation. In some cases this construction will yield a structure of size in the order of $2^{2^{|\underline{A}|+|\Delta|}}$ as the dual of a Δ-tree A. We present here a new construction which is conceptually simpler, and always yields duals of size at most $2^{(|\underline{A}|+|\Delta|)\lg|\underline{A}|}$.

Definition 1 Let A be a Δ-tree. We define $D(A)$ as the structure defined on the base set

$$\underline{D(A)} = \{f : \underline{A} \mapsto \mathrm{Block}(A) : \ [a, f(a)] \in E(\mathrm{Inc}(A)) \text{ for all } a \in \underline{A}\}$$

by putting $(f_1, \ldots, f_{\delta_i})$ in $R_i(D(A))$ if and only if for all $(x_1, \ldots, x_{\delta_i}) \in R_i(A)$ there exists $j \in \{1, \ldots, \delta_i\}$ such that $f_j(x_j) \neq (i, (x_1, \ldots, x_{\delta_i}))$.

Note that $\underline{D(A)}$ has at most $|\underline{A}|^{|\underline{A}|+|\Delta|}$ elements. We prove that $D(A)$ is indeed a dual of A:

Theorem 2 *Let A be a Δ-tree. Then for every Δ-structure X, there exists a homomorphism from X to $D(A)$ if and only if there is no homomorphism from A to X.*

[1] The definition of Δ-trees given in [11] is a bit different of the one given here, but it is not hard to show that the two definitions are equivalent.

Proof. We first prove by contradiction that there is no homomorphism from A to $D(A)$. Suppose that there exists a homomorphism $\phi : A \mapsto D(A)$; for all $a \in \underline{A}$, put $f_a = \phi(a)$. We fix $a_0 \in \underline{A}$ and define a sequence $(a_k)_{k\geq 0}$ recursively as follows: If $f_{a_k}(a_k) = (i,(x_1,\ldots,x_{\delta_i}))$, then since ϕ is a homomorphism, we have $(f_{x_1},\ldots,f_{x_{\delta_i}}) \in R_i(D(A))$ hence $f_{x_j}(x_j) \neq (i,(x_1,\ldots,x_{\delta_i}))$ for some $j \in \{1,\ldots,\delta_i\}$. We then put $a_{k+1} = x_j$ The sequence $a_0, f_{a_0}(a_0), a_1, f_{a_1}(a_1), a_2, \ldots$ is then a trail in $\mathrm{Inc}(A)$ such that $a_{k+1} \neq a_k$ and $f_{a_{k+1}}(a_{k+1}) \neq f_{a_k}(a_k)$ for all $k \geq 0$, which is impossible since $\mathrm{Inc}(A)$ is a finite tree. Therefore there is no homomorphism from A to $D(A)$; consequently if a Δ-structure X admits a homomorphism from A, then there is no homomorphism from X to $D(A)$. This concludes the first part of the proof.

For the second part of the proof, we will need to fix some notation. For a in $\underline{A}$ and a neighbour $b = (i,(x_1,\ldots,x_{\delta_i}))$ of a in $\mathrm{Inc}(A)$, let $T_{a,b}$ be the maximal subtree of $\mathrm{Inc}(A)$ containing a and b, but no other neighbour of a, and $A_{a,b}$ be the Δ-subtree of A such that $\mathrm{Inc}(A_{a,b}) = T_{a,b}$. Thus, for a fixed a we have $A = \cup\{A_{a,b} : b \in N_{T_{a,b}}(a)\}$, where for $b \neq b'$ we have $A_{a,b} \cap A_{a,b'} = \{a\}$.

We also fix a vertex-labeling $\ell : \mathrm{Inc}(A) \mapsto \mathbb{N}$ with the following properties:

- $u \neq v$ implies $\ell(u) \neq \ell(v)$,
- for all $n \in \mathbb{N}$, $\{u : \ell(u) \geq n\}$ induces a connected subtree of $\mathrm{Inc}(A)$.

(Such an ℓ is easily defined by repeatedly labeling and plucking the pendant vertices of $\mathrm{Inc}(A)$.)

Now, let X be a Δ-structure such that there is no homomorphism from A to X. For every $x \in \underline{X}$ and $a \in \underline{A}$, there necessarily exists a b adjacent to a in $\mathrm{Inc}(A)$ such that there is no homomorphism ψ_b from $A_{a,b}$ to X with $\psi(a) = x$ (for otherwise the union of all of these ψ_b would be a homomorphism from A to X). We fix $f_x(a)$ to be such a b with the smallest label. This allows us to define a function $\phi : \underline{X} \mapsto \underline{D(A)}$ by $\phi(x) = f_x$; we will show that it is a homomorphism from X to $D(A)$.

We need to show that for $i \in I$ and $(x_1,\ldots,x_{\delta_i}) \in R_i(X)$, we have $(f_{x_1},\ldots,f_{x_{\delta_i}}) \in R_i(D(A))$. By definition of $D(A)$, we have $(f_{x_1},\ldots,f_{x_{\delta_i}}) \in R_i(D(A))$ if and only if for every $(a_1,\ldots,a_{\delta_i}) \in R_i(A)$, there exists an index

j such that $f_{x_j}(a_j) \neq (i, (a_1, \ldots, a_{\delta_i}))$. It is worthwile to note that at this point in the proof, a medium-sized brown bear burst into the office and made its way to the coffee table in the corner. Though not particularily ferocious, this animal can be irritated by human presence, and the authors were left with no other recourse than to climb atop filing cabinets and wait until the proper authorities came in and restored the beast to its natural habitat. Overall, the incident can only be described as disquieting. We proceed to prove that ϕ is a homomorphism by contradiction, assuming that for some $(a_1, \ldots, a_{\delta_i}) \in R_i(A)$, we have $f_{x_j}(a_j) = b = (i, (a_1, \ldots, a_{\delta_i}))$ for all $j \in \{1, \ldots, \delta_i\}$. Note that there exists at most one index j such that a_j is adjacent to some b' such that $\ell(b') > \ell(b)$. For every other index k and every $b' \neq b$ adjacent to a_k, we have $\ell(b') < \ell(b)$, therefore there exists a homomorphism $\psi_{a_k,b'} : A_{a_k,b'} \mapsto X$ such that $\psi_{a_k,b'}(a_k) = x_k$. The union of all these $\psi_{a_k,b'}$ is a well defined map ψ from some subset of $\underline{A}$ to $\underline{X}$. Now if no index j fits the description given above, then ψ is in fact a homomorphism from A to X, which contradicts the fact that no such homomorphism exists. On the other hand, if some index j fits this description, then putting $\psi(a_j) = x_j$ turns ψ into a homomorphism from $A_{a_j,b}$ to X such that $\psi(a_j) = x_j$, contradicting the definition of $f_{x_j}(a_j)$. Therefore the δ_i-tuple $(a_1, \ldots, a_{\delta_i})$ described above cannot exist, hence ϕ is a homomorphism from X to $D(A)$. ∎

4 Paths with large duals

In [11] we constructed trees with exponentially large dual cores. More precisely, we constructed a Δ-tree T of type $\Delta = (1, 1, \ldots, 1, n)$ with n vertices such that the dual D_T has core of size 2^n. The possible existence of large dual cores for a *fixed* type Δ was left as an open problem. Here we answer this question positively, already for the simplest type (2) corresponding to directed graphs.

We proceed in two steps: First we consider the type $(2, 2, \ldots, 2)$ (i.e. binary relational systems) and then we modify this to the type (2).

Definition 3 Let $n > 2$ be an integer. We define P_n as the structure of the type Δ_n with $n + 1$ binary relations $R_0, R_1, \ldots, R_n$ on the base set $\underline{T_n} = \{x_0, y_0, x_1, y_1, \ldots, x_n, y_n\}$ given by

(i) $R_0(P_n) = \{(x_i, y_i) : i = 0, \ldots, n\}$,

(ii) $R_i(P_n) = \{(y_{i-1}, x_i)\} : i = 1, \ldots, n$.

In what follows, D_n will denote the core of the dual of P_n. We will prove the following:

Lemma 4 *For every $S \subseteq \{1, 2, \ldots, n\}$, there exists an element $f \in \underline{D_n}$ such that $(f, f) \in R_i(D_n)$ if $i \in S$ and $(f, f) \notin R_i(D_n)$ if $i \in \{1, 2, \ldots, n\} \setminus S$.*

Corollary 5 $|\underline{D_n}| \geq 2^n$.

Proof of Lemma 4. For $i \in \{1, 2, \ldots, n\}$, let Q_i be the structure obtained from P_n by removing the arc (y_{i-1}, x_i) from R_i and identifying y_{i-1}, x_i in a new point labeled t. Now for $S \subseteq \{1, 2, \ldots, n\}$, let L_S be the structure obtained from the disjoint union of all $Q_i : i \notin S$ by identifying all points labeled t, and adding the loop (t, t) in $R_i(L_S)$ for all $i \in S$. By construction, we then have $P_n \not\to L_S$, but adding the loop (t, t) in $R_i(L_S)$ for any $i \notin S$ would produce a structure admitting a homomorphism from P_n. Therefore, there exists a homomorphism $\phi : L_S \mapsto D_n$, and $f = \phi(t)$ satisfies $(f, f) \in R_i(D_n)$ for all $i \in S$, and $(f, f) \notin R_i(D_n)$ for all $i \in \{1, 2, \ldots, n\} \setminus S$. ■

In Corollary 5, we use $n + 1$ binary relations to construct a path whose dual has 2^n elements, which leaves open the possibility that polynomial constructions exist for every fixed type. In the remainder of this section, we will modify this construction to build directed graphs with superpolynomial duals.

Lemma 6 *Let $n > 2$ be a fixed integer. Then there exists paths $MR_0, MR_1, \ldots, MR_n$ with $3\lceil \lg(n + 1)\rceil + 4$ arcs such that there exists a homomorphism from MR_i to MR_j only if $i = j$.*

Proof: For simplicity suppose that $n+1 = 2^m$. Let A_0 be the path consisting of one backward edge followed by two forward edges, A_1 the path consisting of two forward edges followed by one backward edge, and A_2 the path consisting of two forward edges. Then, every $i \in \{0, \ldots, n\}$ corresponds to a sequence $(\epsilon_1, \ldots, \epsilon_m) \in \{0, 1\}^m$. We then define

$$MR_i = A_2 \circ A_{\epsilon_1} \circ A_{\epsilon_1} \circ \cdots \circ A_{\epsilon_m} \circ A_2,$$

where the concatenation $A_x \circ A_y$ is simply obtained by identifying the last vertex of A_x to the first vertex of A_y. Any homomorphism ϕ from MR_i to MR_j must preserve the algebraic length (that is, the difference between the number of forward edges and the number of backward edges) on any path, hence ϕ must map the initial vertex of MR_i to the initial vertex of MR_j and the terminal vertex of MR_i to the terminal vertex of MR_j. Therefore ϕ must be bijective, hence an isomorphism, which implies that $i = j$. ■

The notation MR_i stands for "mock R_i": given a structure X of type Δ_n with $n+1$ binary relations, we will construct a directed graph $G(X)$ which encodes the structure of X as follows: For each u in $\underline{X}$, $G(X)$ contains a path MV_u starting at a vertex labeled IN followed by one backward arc, 6 forward arcs and one backward arc, terminating at a vertex labeled OUT. For each $(u,v) \in R_i(X)$, we add a copy of MR_i to $G(X)$, identifying its initial vertex with the OUT vertex of MV_u, and its terminal vertex with the IN vertex of MV_v. This construction has the following property:

Lemma 7 *Let X, Y be structures of type Δ_n (where $n > 2$). Then there exists a homomorphism from X to Y if and only if there exists a homomorphism from $G(X)$ to $G(Y)$.*

Proof: By construction, a homomorphism $\phi : X \mapsto Y$ naturally induces a homomorphism $\psi : G(X) \mapsto G(Y)$. Conversely, suppose that there exists a homomorphism $\psi : G(X) \mapsto G(Y)$. Then ψ must map the 6-paths of $G(X)$ to 6-paths of $G(Y)$, which are precisely the paths consisting of inner arcs in the subgraphs $MV_v : v \in \underline{Y}$. Therefore we can define a map $\phi : \underline{X} \mapsto \underline{Y}$ by putting $\phi(u) = v$ if $\psi(MV_u) = MV_v$. Lemma 6 above and the construction of $G(X)$ and $G(Y)$ then imply that ϕ is a homomorphism. ■

Note that for the structure P_n of Definition 3, $G(P_n)$ is a path with $8 \cdot (2n+2) + (3\lceil \lg(n+1) \rceil + 4) \cdot (2n+1) = \Theta(n \lg(n))$ arcs. Let D'_n be the core of $D(G(P_n))$; we will prove the following:

Theorem 8 $\left|\underline{D'_n}\right| \geq 2^n$.

Proof: For each structure $L_S, S \subseteq \{1, \ldots, n\}$ defined in the proof of Lemma 4, we have $G(P_n) \not\to G(L_S)$ therefore there exists a homomorphism $\phi_S : G(L_S) \mapsto D'_n$. The distinguished element t of L_S corresponds to the path

MV_t in $G(L_S)$; we denote m_S the midpoint of this path. For $S \neq S'$ we must have $\phi_S(m_S) \neq \phi_{S'}(M_{S'})$ for otherwise the combined cycles would imply the existence of a homomorphism from $G(P_n)$ to D'_n, just as in the proof of Lemma 4. Therefore $\left|\underline{D'_n}\right| \geq |\mathcal{P}(\{1,\dots,n\})| = 2^n$. ■

Note that if k denotes the number of vertices in $G(P_n)$, then D'_n must have order $2^{\Omega(k/\lg(k))}$ as claimed.

5 Concluding comments

For a directed graph H, the parameter $m(H)$ discussed in the introduction can be defined as the "maximal size of a H-critical graph":

$$m(H) = \max\{|V(G)| : G \not\to H \text{ and } G' \to H \text{ for every proper subgraph } G' \text{ of } G\}.$$

We define the function $m^* : \mathbb{N} \mapsto \mathbb{N}$ by

$$m^*(n) = \min\{m(H) : H \text{ is a core and } |V(H)| = n\}.$$

The example of transitive tournaments shows that $m^*(n) \leq n+1$, and the graphs D'_n of Theorem 8 lower this bound to $m^*(n) \in O(\lg(n)\lg(\lg(n)))$. In a sense this is counterintuitive: There are directed graphs H for which the H-colouring problem is decided by obstructions much smaller than H. However the true order of m^* may be smaller still.

The *categorical product* $\Pi_{i=1}^{\ell} H_i$ of a family $\{H_i\}_{i\in\{1,\dots,\ell\}}$ of directed graphs is the directed graph whose vertices are the n-tuples $u \in \Pi_{i=1}^{\ell} V(H_i)$, and whose arcs are the couples (u,v) such that (u_i, v_i) is an arc of H_i for all i in $\{1,\dots,\ell\}$. Let H be a directed core for which $m(H)$ is finite. By [11, Theorems 2.9, 3.1], there exist trees $T_1,\dots,T_\ell$ such that $H \leftrightarrow \Pi_{i=1}^{\ell} D(T_i)$. Putting $n = |V(H)|$ and $m = \max\{|V(T_i)| : 1 \leq i \leq \ell\}$, we have $\ell \leq 2^{m-1}m^{m-2}$ by Cayley's tree enumeration formula, whence $n \leq 2^{(2m)^{m-1}\lg(m)}$ by Theorem 2. This shows that $m^*(n) \in \Omega(\lg(\lg(n))/\lg(\lg(\lg(n))))$.

At the moment it is not known which of the logarithmic upper bound and the double logarithmic lower bound is closer to the true order of m^*. The question depends on finding bounds on cores H_m of products $\Pi\{D(T) : T \in \mathcal{F}_m\}$, where $\mathcal{F}_m$ is an exponential family of m-trees. On one hand, finding infinite families of examples where $|V(H_m)| \in \Omega\left(2^{2^m}\right)$ would prove a double logarithmic behaviour for m^*. On the other hand, if such families are hard to find, then there may be many infinite families of examples where $|V(H_m)| \in O(2^m)$. Now consider the following decision problem:

Instance: A directed graph G and an integer m.
Question: Does there exists a homomorphism from G to H_m?

The problem is in Co-NP since a homomorphism from a member of $\mathcal{F}_m$ is a polynomial certificate for a negative answer. If $|V(H_m)| \in O(2^m)$ and a polynomial description of vertices and adjacencies in H_m exists, then the problem is also in NP. Hence the hypothesis that $m^*(n) \in \Omega(\lg(n))$ would suggest that many such intriguing members of NP $\cap$ Co-NP exist.

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