On the Chromatic Number of the Visibility Graph of a Set of Points in the Plane*

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December 22, 2004

Abstract

The visibility graph $\mathcal{V}(P)$ of a point set $P \subseteq \mathbb{R}^2$ has vertex set P, such that two points $v, w \in P$ are adjacent whenever there is no other point in P on the line segment between v and w. We study the chromatic number of $\mathcal{V}(P)$. We characterise the 2- and 3-chromatic visibility graphs. It is an open problem whether the chromatic number of a visibility graph is bounded by its clique number. Our main result is a super-polynomial lower bound on the chromatic number (in terms of the clique number).

1 Introduction

Let $P \subseteq \mathbb{R}^2$ be a set of points in the plane. Let $\overline{vw}$ denote the closed line-segment between points $v \in \mathbb{R}^2$ and $w \in \mathbb{R}^2$. Two distinct points $v, w \in P$ are *visible* with respect to P if $P \cap \overline{vw} = \{v, w\}$. The *visibility graph* $\mathcal{V}(P)$ of P has vertex set P, where two distinct points $v, w \in P$ are adjacent if and only if they are visible with respect to P.

A k*-colouring* of a graph $G = (V, E)$ is a function $f : V \to C$ for some set C of k *colours*, such that $f(v) \neq f(w)$ for every edge $vw \in E$. We say G is k*-colourable*. The *chromatic number* $\chi(G)$ is the minimum k such that G is k-colourable. The *clique number* $\omega(G)$ is the maximum k such that G has a k-clique.

*The research of A. Pór and D. Wood was completed in the Department of Applied Mathematics and the Institute for Theoretical Computer Science at Charles University, Prague, Czech Republic. Supported by project LN00A056 of the Ministry of Education of the Czech Republic, and by the European Union Research Training Network COMBSTRU (Combinatorial Structure of Intractable Problems).

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This paper studies the chromatic number of visibility graphs. We begin with an interesting example.

Proposition 1. *Let $P = \{(x,y) : x, y \in \mathbb{Z}\}$ be the integer lattice. Then $\chi(\mathcal{V}(P)) = 4$.*

Proof. Let $f((x,y)) = (x \bmod 2, y \bmod 2)$ for all $(x,y) \in P$. For any two points (x_1,y_1) and (x_2,y_2) in P for which $f((x_1,y_1)) = f((x_2,y_2))$, both $|x_1 - x_2|$ and $|y_1 - y_2|$ are even. Thus the midpoint of the segment $\overline{(x_1,y_1)(x_2,y_2)}$ is in P, and (x_1,y_1) and (x_2,y_2) are not visible. Hence f is a 4-colouring of $\mathcal{V}(P)$, as illustrated in Figure 1. There is no 3-colouring since $\{(0,0),(1,0),(1,1),(0,1)\}$ is a 4-clique. Therefore $\chi(\mathcal{V}(P)) = 4$. □

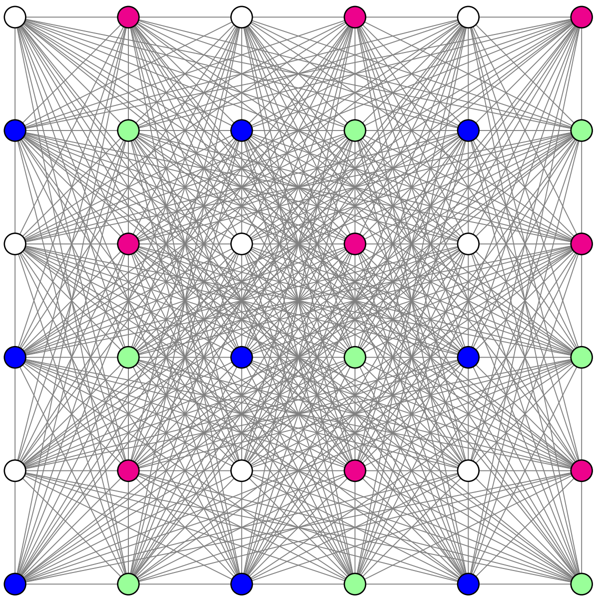

Figure 1: 4-colouring of the visibility graph of the integer lattice.

While the visibility graph of the integer lattice has a quadratic number of edges, Proposition 1 proves that it has small chromatic number. Also note that Proposition 1 generalises to prove that the visibility graph of the d-dimensional[1] integer lattice is 2^d. In this case, the chromatic number and the clique number coincide[2]. Whether there is a similar relationship for all visibility graphs is a fundamental open problem.

Conjecture 1. Visibility graphs are χ-bounded. That is, is there a function f such that $\chi(\mathcal{V}(P)) \leq f(\omega(\mathcal{V}(P)))$ for every finite point set P?

[1]Note that the visibility graph of a set of points in $\mathbb{R}^d$, by a suitable projection, is also a visibility graph of some set of points in $\mathbb{R}^2$.

[2]The visibility graph of the integer lattice is not perfect. For example, $((2,5)(1,3)(5,8)(8,3)(5,1))$ is an induced 5-cycle.

In Section 2 we make some observations about visibility graphs, and give an elementary bound on their chromatic number. In Section 3 we prove that in Conjecture 1, we can take $f(2) = 2$ and $f(3) = 3$. In fact we characterise the finite point sets whose visibility graph has chromatic number 2 or 3. The next interesting case is $\omega(\mathcal{V}(P)) = 4$. Figure 2 shows a visibility graph with $\omega(\mathcal{V}(P)) = 4$, for which it is easily seen that $\chi(\mathcal{V}(P)) = 5$. It is an open problem whether every visibility graph with $\omega(\mathcal{V}(P)) \leq 4$ has $\chi(\mathcal{V}(P)) \leq 5$.

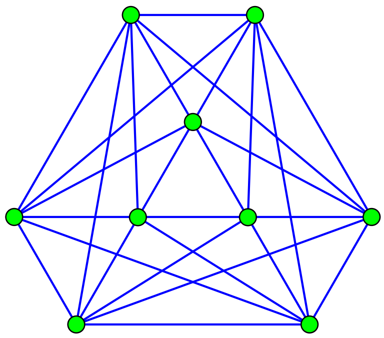

Figure 2: A 5-chromatic visibility graph with maximum clique size 4.

The main result of this paper, presented in Section 4, is a super-polynomial lower bound on the chromatic number in terms of the clique number, for a certain family of visibility graphs.

Note that visibility graphs of polygons are well studied (see [1, 4, 7, 8, 18] for example), but even here, it is an open problem whether the chromatic number is bounded by the clique number. For results and open problems regarding the χ-boundedness of other graph families, especialling those arising in a geometric context, see [10, 11, 12, 13, 15, 16, 17] for example.

2 Observations

The following is a fundamental observation regarding visibility graphs.

Proposition 2. *For every finite point set $P \subset \mathbb{R}^2$, the diameter of the visibility graph $\mathcal{V}(P)$ is*

$$\begin{cases} 1 & \text{if } P \text{ is in general position,} \\ |P| - 1 & \text{if } P \text{ is collinear,} \\ 2 & \text{otherwise.} \end{cases}$$

Proof. The diameter is one if and only if $\mathcal{V}(P)$ is complete, which occurs if and only if P is in general position. If P is collinear, then $\mathcal{V}(P)$ is a path, which has diameter $|P| - 1$.

Thus it suffices to prove that if P is not in general position and not collinear, then the diameter of $\mathcal{V}(P)$ is two. Consider two non-visible points $v, w \in P$. Two such points exist, since P is not in general position. Let L be the line containing v and w. Let x be a point in P not on L, such that the perpendicular distance from x to L is minimised. There is such a point x as P is finite, and not all the points in P are collinear. Then v and x are visible and w and x are visible, as otherwise there is a point in P closer to L than x. Thus the distance from v to w in $\mathcal{V}(P)$ is two. Hence the diameter of $\mathcal{V}(P)$ is two. □

Here is one way to colour $\mathcal{V}(P)$.

Proposition 3. *If a point set $P \subseteq \mathbb{R}^2$ can be covered by k lines, then $\chi(\mathcal{V}(P)) \leq 2k$.*

Proof. Associate each point $v \in P$ with one of the k lines that contain v. The subgraph of $\mathcal{V}(P)$ induced by the set of points assigned to any one line is a collection of disjoint paths, and is thus 2-colourable. Using a different pair of colours for each line we obtain a $2k$-colouring of $\mathcal{V}(P)$. □

Corollary 1. *For every point set $P \subseteq \mathbb{R}^2$, $\chi(\mathcal{V}(P))$ is at most twice the minimum degree of $\mathcal{V}(P)$.*

Proof. The result follows from Proposition 3, since P can clearly be covered by $\deg(v)$ lines for any point $v \in P$. □

We conclude this section with the following Ramsey-type conjecture; see [2, 3, 9] for related research.

Conjecture 2. For all integers $k, \ell \geq 2$ there is an $n = n(k, \ell)$ such that every set P of at least n points in the plane contains ℓ collinear points or k pairwise visible points (that is, $\omega(\mathcal{V}(P)) \geq k$).

Note that $n(k, \ell) > (\ell - 1)^{\log_2(k-1)}$ since the projection of the d-dimensional $(\ell - 1) \times (\ell - 1) \times \cdots \times (\ell - 1)$ integer lattice has no set of ℓ collinear points and no k pairwise visible points for $k = 2^d + 1$.

3 The 2- and 3-Chromatic Visibility Graphs

In what follows we characterise the finite point sets whose visibility graph has chromatic number 2 or 3.

Theorem 1. *Let P be a finite point set. Then the following are equivalent:*
(a) $\chi(\mathcal{V}(P)) \leq 2$*,*
(b) *all the points in P are collinear,*
(c) $\mathcal{V}(P)$ *has no K_3 subgraph.*

Proof. That (a) implies (c) is immediate. If all the points in P are collinear, then $\mathcal{V}(P)$ is a path, which is obviously 2-colourable. Thus (b) implies (a). It remains to prove that (c) implies (b). Suppose that not all the points in P are collinear. Let $\{u, v, w\}$ be a set of three non-collinear points in P such that the triangle uvw has minimum area. If there is a distinct point $x \in P \cap \overline{uv}$, then $\{x, v, w\}$ are non-collinear and the triangle xvw has less area than uvw, which is a contradiction. Thus u and v are visible. Similarly u and w are visible, and v and w are visible. Hence $\{u, v, w\}$ induce K_3 in $\mathcal{V}(P)$. □

Before characterising the 3-colourable visibility graphs, consider when $\mathcal{V}(P)$ is planar. In $\mathcal{V}(P)$ there is a line-segment between every pair of vertices (which may be comprised of many edges). Eppstein [6] characterised those planar graphs in which there is a line-segment between every pair of vertices. He called these the *dilation-free* planar graphs, as illustrated in Figure 3.

Lemma 1 ([6]). *Let P be a point set. Then $\mathcal{V}(P)$ is planar if and only if at least one of the following conditions hold:*
(a) *all the points in P are collinear,*
(b) *all the points in P, except for one, are collinear,*
(c) *all the points in P are collinear, except for two non-visible points,*
(d) *all the points in P are collinear, except for two points $v, w \in P$, such that the line-segment $\overline{vw}$ does not intersect the line-segment that contains $P \setminus \{v, w\}$,*
(e) *$\mathcal{V}(P)$ is the drawing of the octahedron shown in Figure 3(e).*

Theorem 2. *Let P be a finite point set. Then the following are equivalent:*
(*i*) $\chi(\mathcal{V}(P)) \leq 3$*,*
(*ii*) *P satisfies conditions* (a)*,* (b)*,* (c) *or* (e) *in Lemma 1,*
(*iii*) *$\mathcal{V}(P)$ has no K_4 subgraph.*

Proof. That (i) implies (iii) is immediate. It is easy to construct a 3-colouring of a visibility graph that satisfies conditions (a), (b), (c) or (e) in Lemma 1. Thus (ii) implies (i). It remains to prove that (iii) implies (ii). Suppose that $\mathcal{V}(P)$ has no K_4 subgraph. Develin *et al.* [5] proved that a visibility graph is planar or contains K_4. (This result applies to a broad range of visibility graphs that includes visibility graphs of point sets.) Thus $\mathcal{V}(P)$ is planar. Lemma 1 describes all the planar visibility graphs. Of these only those satisfying condition (d) contain K_4. □

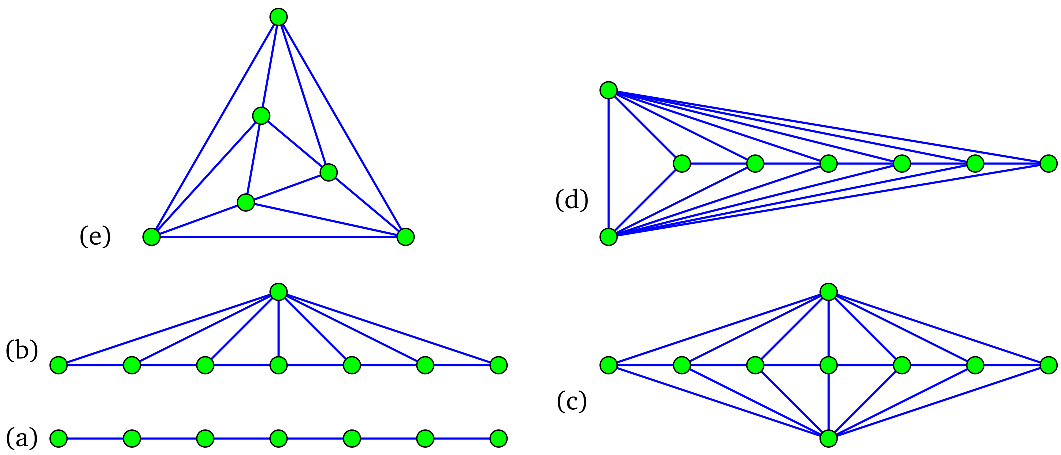

Figure 3: The planar visibility graphs.

4 A Lower Bound

In this section we prove the following super-polynomial lower bound on the chromatic number of a visibility graph.

Theorem 3. *There are constants $c_1, c_2, c_3, c_4 > 0$ and an infinite sequence of visibility graphs $G_0, G_1, G_2, \ldots$, such that $\omega(G_i) \to \infty$ and*

$$\chi(G_i) \;\geq\; (c_1 \log \omega(G_i))^{c_2 \log \omega(G_i)} \;=\; (c_3\, \omega(G_i))^{c_4 + \log\log \omega(G_i)} \enspace .$$

Before proving Theorem 3 we recall two definitions from the literature. Let G and H be graphs. The *lexicographic product* of G by H, denoted by $H[G]$, is the graph with vertex set $V(G) \times V(H)$, where $\{va, wb\}$ is an edge if and only if $ab \in E(H)$, or $a = b$ and $vw \in E(G)$. The *fractional chromatic number* $\chi_f(G)$ of a graph G is the infimum of all fractions a/b such that, to each vertex of G, one can assign a b-element subset of $\{1, 2, \ldots, a\}$ in such a way that adjacent vertices are assigned disjoint subsets. Obviously $\chi_f(G) \leq \chi(G)$. Scheinerman and Ullman [20] proved the following important property about the fractional chromatic number of the lexicographic product.

Lemma 2 ([20]). *For all graphs G and H, $\chi_f(H[G]) = \chi_f(H) \cdot \chi_f(G)$.*

Lemma 3. *For every visibility graph G and for every finite graph H, there is a visibility graph X such that $\omega(H) \cdot \omega(G) \;\leq\; \omega(X) \;\leq\; \omega(H) \cdot \omega(G) + 2|V(H)|$, and $\chi_f(X) \;\geq\; \chi_f(H) \cdot \chi_f(G)$.*

Proof. Suppose $V(H) = \{1, 2, \ldots, n\}$. Let $\{D_1, D_2, \ldots, D_n\}$ be a set of closed unit discs in the plane, whose centres are positioned on the vertices of a sufficiently large regular

n-gon. For all $1 \leq i < j \leq n$, we say a segment with endpoints in D_i and D_j is an *ij-segment*. Here 'sufficiently large' means that for each disc D_i:

(1) the only disc that an ij-segment intersects is D_i and D_j,

(2) there is a line L_i such that every ij-segment crosses L_i, and

(3) whenever an ij-segment crosses an ik-segment ($j \neq k$), the crossing point is on the side of L_i that contains D_i.

Scale G so that its convex hull is enclosed in a unit disc and no vertex is at the centre of the disc. Let $\{G_1, G_2, \dots, G_n\}$ be copies of G, one associated with each vertex of H. Place each G_i in the disc D_i, rotated so that if three points in $\bigcup_i V(G_i)$ are collinear, then they are in a single G_i. This can be achieved by rotating each G_i in turn. At each step, there are only finitely many forbidden rotation angles.

Let X_0 be the visibility graph defined by the point set $\bigcup_i V(G_i)$. By property (1) and the choice of orientations, every point in G_i is visible with every point in G_j for all $i \neq j$. Visibility within each G_i is preserved by scaling and rotating. Thus $X_0 = K_n[G]$.

We now introduce *blocker* points to our set, so that the subgraph of the visibility graph induced by $\bigcup_i V(G_i)$ is $H[G]$. For every non-edge ij of H (that is, an edge of $\overline{H}$), and for all vertices $p \in V(G_i)$ and $q \in V(G_j)$, add one blocker point at the intersection of the segment pq and the line L_i, and add another blocker point at the intersection of the segment pq and the line L_j. If two blocker points coincide, then just use one point. This construction is illustrated in Figure 4.

Let X be the visibility graph of the point set obtained. By property (3) above, for every edge $ij \in E(H)$, every vertex in G_i is visible with every vertex in G_j. Thus the subgraph of X induced by $\bigcup_i V(G_i)$ is $H[G]$.

Obviously $\omega(H[G]) = \omega(H) \cdot \omega(G)$. The blocker vertices on each line L_i can add at most two vertices to a maximum clique. Thus $\omega(H) \cdot \omega(G) \leq \omega(X) \leq \omega(H) \cdot \omega(G) + 2|V(H)|$, as claimed. By Lemma 2 and since $H[G]$ is an induced subgraph of X, $\chi_f(X) \geq \chi_f(H[G]) = \chi_f(H) \cdot \chi_f(G)$, as claimed. □

The following result of Larsen *et al.* [14] is based on the famous construction of Mycielski [19].

Lemma 4 ([14]). *For all $k \geq 0$, there is a triangle-free graph M_k on $3 \cdot 2^k - 1$ vertices such that $\chi_f(M_k) \geq \sqrt{2k}$.*

Proof Construction. Let $M_0 = K_2$. Construct M_{k+1} from M_k as follows. Suppose $V(M_k) = \{v_i : 1 \leq i \leq n_k\}$. Let $V(M_{k+1}) = \{x_i, y_i : 1 \leq i \leq n_k\} \cup \{z\}$. Let $E(M_{k+1}) = \{x_i x_j : v_i v_j \in E(M_k)\} \cup \{x_i y_j : v_i v_j \in E(M_k)\} \cup \{y_i z : 1 \leq i \leq n_k\}$. Note

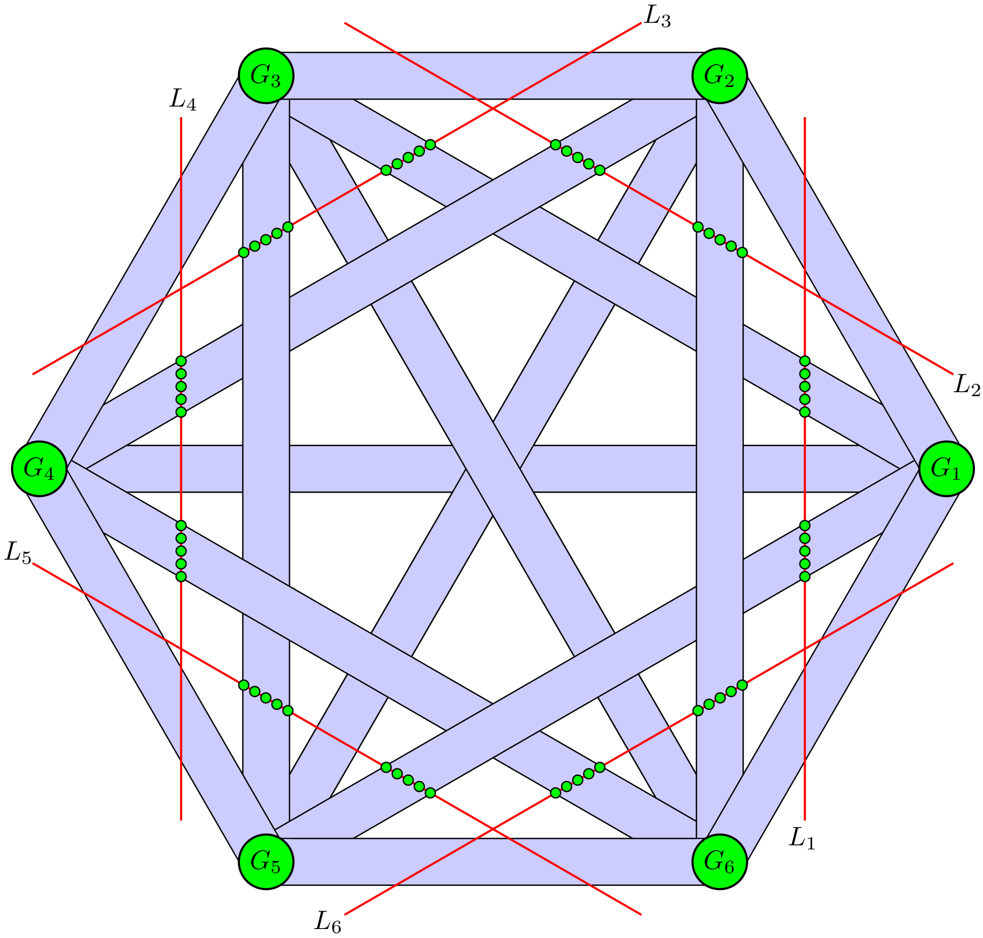

Figure 4: Construction of X from a visibility graph G with $H = K_{3,3}$.

that M_{k+1} has $n_{k+1} = 2n_k + 1$ vertices. Since $n_0 = 2$, it follows that $n_k = 3 \cdot 2^k - 1$. Mycielski [19] proved that M_k is triangle-free and $\chi(M_k) = k + 2$. Larsen *et al.* [14] proved that $\chi_f(M_{k+1}) = \chi_f(M_k) + 1/\chi_f(M_k)$. It follows that $\chi_f(M_k) \geq \sqrt{2k}$ (and this is asymptotically tight; see [14, 20]). □

Proof of Theorem 3. (In what follows we make little effort to optimise the constants c_1 and c_2.) Let $G_0 = K_1$. For all $i \geq 0$, apply Lemma 3 to obtain a visibility graph $G_{i+1}(= X)$ from $G_i(= G)$, where H is the Mycielski graph $M_{k(i)}$, chosen so that

$$3\omega(G_i) \;\leq\; |V(M_{k(i)})| \;=\; 3 \cdot 2^{k(i)} - 1 \;\leq\; 6\,\omega(G_i) \;. \tag{1}$$

By Lemma 3 and since $M_{k(i)}$ is triangle-free,

$$2\omega(G_i) \;\leq\; \omega(G_{i+1}) \;\leq\; 2\omega(G_i) + 2|V(M_{k(i)})| \;\leq\; 14\omega(G_i) \;,$$

where the last inequality follows from (1). Since $\omega(G_0) = 1$,

$$2^i \leq \omega(G_i) \leq 14^i \ . \tag{2}$$

We now prove a lower bound on the chromatic number of G_{i+1}. By Lemmata 3 and 4,

$$\chi_f(G_{i+1}) \geq \chi_f(M_{k(i)}) \cdot \chi_f(G_i) \geq \sqrt{2k(i)}\,\chi_f(G_i) \ .$$

By (1) and the lower bound in (2), $k(i) > \log_2 \omega(G_i) \geq i$. Hence,

$$\chi_f(G_{i+1}) > \sqrt{2i}\,\chi_f(G_i) \ .$$

Since $\chi_f(G_0) = 1$,

$$\chi_f(G_i) \geq \sqrt{2^i i!} \ .$$

By Stirling's Formula,

$$\chi_f(G_i) \geq \sqrt{(2i/\boldsymbol{e})^i} \ .$$

By the upper bound in (2), $i \geq \log_{14} \omega(G_i)$. Hence

$$\chi_f(G_i) \geq \left(\frac{2}{\boldsymbol{e}} \log_{14} \omega(G_i)\right)^{\frac{1}{2}\log_{14}\omega(G_i)} \ .$$

Obviously $\chi(G_i) \geq \chi_f(G_i)$. Thus for an appropriate choice of constants $c_1, c_2 > 0$,

$$\chi(G_i) \geq \chi_f(G_i) \geq (c_1 \log \omega(G_i))^{c_2 \log \omega(G_i)} \ ,$$

as claimed. □

Acknowledgements

Thanks to Ricardo Strausz for stimulating discussions.

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