Choosability of Graphs with Infinite Sets of Forbidden Differences

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Abstract

The notion of the list-T-coloring is a common generalization of the T-coloring and the list-coloring. Given a set of non-negative integers T, a graph G and a list-assignment L, the graph G is said to be T-colorable from the list-assignment L if there exists a coloring c such that the color $c(v)$ of each vertex v is contained in its list $L(v)$ and $|c(u) - c(v)| \notin T$ for any two adjacent vertices u and v. The T-choice number of a graph G number is the minimum integer k such that G is T-colorable for any list-assignment L which assigns each vertex of G a list of at least k colors.

We focus on list-T-colorings with infinite sets T. In particular, we show that for any fixed set T of integers, all graphs have finite T-choice number if and only if the T-choice number of K_2 is finite. For the case when the T-choice number of K_2 is finite, two upper bounds on the T-choice number of a graph G are provided: one being polynomial in the maximum degree of the graph G, and one being polynomial in the T-choice number of K_2.

Keywords: graph coloring, list coloring, T-coloring

1 Introduction

Special types of graph colorings attracted attention of researchers in connection with with their applications in wireless networks. Hale [7] formulated

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several frequency assignment problems in the terms of graph theory. Suppose that transmitters are stationed at various locations, and we wish to assign every transmitter a frequency over which it will operate. The frequencies need to be assigned in a way such that the frequencies of nearby standing transmitters do not interfere. If interference occurred only when the transmitters use the same frequency, the problem could be formulated as a graph-coloring problem: every transmitter is represented by a vertex, and frequencies are referred to as colors. Any pair of vertices representing close transmitters is connected by an edge.

In practice, interference occurs even if the frequencies are different, e.g., when the difference of the frequencies equals a certain value. T-colorings of graphs deal with this restriction: given a set of nonnegative integers T, a T-coloring of a graph G is a vertex-coloring (with positive integers) of G such that the absolute value of the difference between any two colors assigned to adjacent vertices does not belong to the set T. Let us remark that the set $T = \{0, 7, 14, 15\}$ is the set of forbidden differences in the model for frequency assignment in UHF television transmitter systems [9]. The concept of T-colorings has been extensively studied as witnessed by the survey of Roberts [10]. The reader is referred to this survey for more detailed introduction.

However, it is not always possible for a given transmitter to operate on all frequencies — instead a transmitter is assigned a set of frequencies which it can operate. This leads us to the concept of *list colorings* introduced independently by Vizing [14], and by Erdős, Rubin and Taylor [4]. Combining list colorings with T-colorings, *list-T-colorings* arise. This notion was first introduced by Tesman [13] and further studied by Alon and Zaks [2], Fiala, Král' and Škrekovski [5], Tesman [11], Waller [15, 16], and others. Given a set $L(v)$ of allowed colors for each vertex of a graph G, a list-T-coloring of G is a proper T-coloring of the graph G such that the color assigned to a vertex v belongs to the set $L(v)$. A graph G is said to be *T-k-choosable* if a list-T-coloring exists for every collection of sets $L(v)$ such that $|L(v)| = k$ for every vertex v. The *T-choice number* $\mathrm{ch}_T(G)$ of a graph G is the minimum number k such that G is T-k-choosable.

So far, researchers mainly focused on the case when the set T is finite. Alon et al. [2] proposed to consider the case when T is infinite, in particular, to characterize those infinite sets T for which the T-choice number of all graphs is finite. In this paper, we attempt to make the first step in this direction. In particular, it is shown that for any set T, the T-choice number is either finite for all graphs or infinite for all (non-trivial) graphs.

We also investigate the behavior of the T-choice number of a given graph G in terms of its maximum degree Δ. In order to do this, we introduce the following function:

Definition 1.1. *If T is a set of integers, then $\mathrm{ch}_T(\Delta)$ is the smallest integer ℓ such that every graph with maximum degree Δ is T-ℓ-choosable.*

In Section 3, we prove that for any integer $\Delta \geq 1$, $\mathrm{ch}_T(\Delta)$ is finite if and only if $\mathrm{ch}_T(1)$ is finite. For the case when the $\mathrm{ch}_T(1)$ is finite, two upper bounds on the T-choice number of a graph G are provided in Sections 3 and 4: one being polynomial in the maximum degree of the graph G and one polynomial in $\mathrm{ch}_T(1) = \mathrm{ch}_T(K_2)$. At the end of the paper, we investigate the connection between $\mathrm{ch}_T(\Delta)$ and the length of the longest arithmetical progression contained in T.

2 Preliminaries

Throughout the paper, the following notation is used: if a is an integer and B is a set of integers, then $a + B$ denotes the set $\{a + b : b \in B\}$. If A and B are two sets of integers, $A + B$ denotes the set $\{a + b : a \in A, b \in B\}$. Similarly, $-A$ stands for the set $\{-a : a \in A\}$. Two colors c_1 and c_2 are said to be *conflicting with respect to a set T* if $|c_1 - c_2| \in T$. When the set T is clear from the context, the colors are said just to be *conflicting*. A *non-trivial* graph is a graph that contains at least one edge.

Next, we establish a proposition which outlines the connection between T-choice number of the graph K_2 (i.e., a single edge) and structure of the set T.

Proposition 2.1. *Let $k \geq 2$ be an integer. The graph K_2 is T-k-choosable if and only if the following inequality holds for every k distinct integers $i_1, \ldots, i_k$:*

$$|(i_1 + (T \cup -T)) \cap \cdots \cap (i_k + (T \cup -T))| < k.$$

Proof. Let u and v be the two vertices of the graph K_2. Firstly, consider the case when there exist k distinct integers $i_1, \ldots, i_k$ such that

$$|(i_1 + (T \cup -T)) \cap \cdots \cap (i_k + (T \cup -T))| \geq k.$$

Let $L(u)$ be $\{i_1, \ldots, i_k\}$ and $L(v)$ be any k-element subset of the set $(i_1 + (T \cup -T)) \cap \cdots \cap (i_k + (T \cup -T))$. Now, it is impossible to color properly both

u and v from their lists because all colors in $L(u)$ conflict with all colors in $L(v)$.

The other implication is also not too difficult: fix a list-assignment L, and let $L(u) = \{i_1, \ldots, i_k\}$. Because

$$|(i_1 + (T \cup -T)) \cap \cdots \cap (i_k + (T \cup -T))| < k,$$

there exist colors $c_1 \in L(u)$ and $c_2 \in L(v)$ such that $|c_1 - c_2| \notin T$. Otherwise, we have that the set $L(v)$ is contained in the above intersection, so the size of the intersection must be at least k, a contradiction. Now, we can use c_1 to color u and c_2 to color v and we obtain a proper coloring. Hence, K_2 can be colored properly from the list assignment L. $\square$

Motivated by the preceeding proposition, we use the following property to ease our arguments:

Definition 2.2. *Let $k \geq 2$ be an integer. A set of integers T is k-*good *if and only if the graph K_2 is T-k-choosable, i.e., it satisfies the condition in Proposition 2.1. Furthermore, T is* good *if it is good for some k.*

3 Upper Bound Polynomial in the Maximum Degree

In this section, we combine methods from the probability theory and the extremal combinatorics to obtain an upper bound on $\mathrm{ch}_T(\Delta)$ which is polynomial in Δ (if a set T is fixed). First, we shortly introduce the concepts we use in our arguments.

3.1 Lovász Local Lemma

The probabilistic method is a remarkable technique based on the probability theory which can be used to prove theorems which have nothing to do with probability and proved its usefulness in many proofs in combinatorics. For examples of such usages and a deeper introduction to the probabilistic method, we refer the reader to the monograph on the subject by Alon and Spencer [1].

In a typical probabilistic proof of a combinatorial result, one usually has to show that the probability of a certain event is positive. If we have mutually independent events and each of them holds with a positive probability, then there is a positive probability that all the events hold simultaneously. This can be generalized to the case when the events are almost independent, as shown in the following theorem proved in [3]:

Theorem 3.1. (Lovász Local Lemma, General Case)
Let $A_1, A_2, \ldots, A_n$ be events in an arbitrary probability space. Let $D(V, E)$ be the dependency digraph for the events $A_1, \ldots, A_n$, i.e., $V = \{1, \ldots, n\}$ and the event A_i is independent of all the events in the set $\{A_j : (i,j) \notin E\}$. Suppose there exist real numbers $x_1, \ldots, x_n$, $0 \leq x_i < 1$, for which the following holds:

$$\mathrm{Prob}[A_i] \leq x_i \prod_{(i,j)\in E} (1 - x_j).$$

Then, the probability $\mathrm{Prob}[\bigwedge_{i=1}^{n} \bar{A}_i]$ that none of the events $A_1, \ldots, A_n$ holds is positive.

In our proof, we use the symmetric version of Theorem 3.1:

Theorem 3.2. (Lovász Local Lemma, Symmetric Case)
Let $A_1, A_2, \ldots, A_n$ be events in an arbitrary probability space. Suppose each event A_i is mutually independent of a set of all the other events A_j but at most d, and that $\mathrm{Prob}[A_i] \leq p$ for all $1 \leq i \leq n$. If $ep(d+1) \leq 1$, then $\mathrm{Prob}[\bigwedge_{i=1}^{n} \bar{A}_i] > 0$.

3.2 The Problem of Zarankiewicz

Extremal combinatorics found a lot of applications in the computer science. We refer the reader to the monograph [8] by Jukna for examples and more background. The problem of Zarankiewicz is an analogue of the well-known theorem of Turán that determines the maximum number of edges in a graph of order n not containing a complete graph of order k as a subgraph [12]. The problem of Zarankiewicz is the following: for given natural numbers m, n, s and t, determine the maximum number of edges in m by n bipartite graph which does not contain a complete s by t bipartite graph. This maximum is denoted by $z(m, n; s, t)$. Zarankiewicz [17] originally asked the question for $s = t = 3$ and $m = n = 4, 5$ and 6. Later, the generalized version of the problem appeared and became known as the problem of Zarankiewicz. Note that this problem can be also reformulated in terms of 0–1 matrices: at most how many 1's can a 0–1 matrix of m rows and n columns contain if it has no s by t submatrix all whose entries are 1's?

Unfortunately, no exact expression for $z(m,n;s,t)$ is known, even the magnitude of $z(n,n;t,t)$ is unknown for fixed (but large) values of t [6]. On the other hand, several upper and lower bounds are known, for example the following one can be found in [6, Theorem 1.3.2]:

Theorem 3.3. *Let m, n, s and t be natural numbers which satisfy $2 \leq s \leq m$ and $2 \leq t \leq n$, the following holds:*

$$z(m,n;s,t) < (s-1)^{1/t}(n-t+1)m^{1-1/t} + (t-1)m.$$

In our proof, it is enough to consider the following specialized version of the preceding theorem for $m = n$ and $s = t$.

Theorem 3.4. *Let n and t be natural numbers such that $2 \leq t \leq n$. The following holds:*

$$z(n,n;t,t) < (t-1)^{1/t}(n-t+1)n^{1-1/t} + (t-1)n.$$

3.3 The Upper Bound

In this subsection, we combine the concepts introduced in the previous two subsections to obtain the desired bound on $\mathrm{ch}_T(\Delta)$. A *conflict graph* of an edge uv is the bipartite graph whose vertices correspond to the colors contained in lists of the vertices u and v and edges join the conflicting colors:

Definition 3.5. *Let G be a graph, L be a list-assignment of G, let T be a set of integers and let $e = uv$ be an edge of G. The* conflict graph *of the edge e is the bipartite graph whose vertex set is $(L(u) \times \{0\}) \cup (L(v) \times \{1\})$ and two vertices $(c_1, 0)$ and $(c_2, 1)$ are joined by an edge if and only if $|c_1 - c_2| \in T$. The conflict graph of an edge e is denoted by CG_e.*

Now, we introduce the notion of *density* for bipartite graphs which relates the number of vertices and the number of edges of a bipartite graph.

Definition 3.6. *The density of a bipartite graph G with parts of orders m and n is the ratio of the number of edges of G to the number of edges in complete m by n bipartite graph, i.e., the density of the graph G is $\frac{|E_G|}{mn}$.*

Before stating the main theorem, let us prove two lemmas. The first lemma shows that if the conflict graphs of all the edges in G have small density, then the graph G can be colored from the given lists:

Lemma 3.7. *Let G be a graph with maximum degree at most Δ, let T be a set of integers and let L be a list-assignment of G. If it is impossible to color G from L, then there exists an edge e such that the density of CG_e is greater than $\frac{1}{2e\Delta}$.*

Proof. We use Theorem 3.2. Color the vertices independently from their lists at random. Let c be the resulting coloring. A_e denotes the event that the colors of the end-vertices of $e = uv$ are conflicting, i.e., $|c(u) - c(v)| \in T$. As the maximum degree of G is at most Δ and each edge has two end-vertices, the event A_i in dependent on at most $2(\Delta - 1)$ other events. Therefore, if $\text{Prob}[A_e] \leq \frac{1}{2e\Delta}$ for all edges, there exists a proper coloring by Theorem 3.2. As no such coloring exists, there must be an edge e, such that $\text{Prob}[A_e] > \frac{1}{2e\Delta}$. Since $\text{Prob}[A_e]$ is exactly the density of CG_e, the statement of the lemma readily follows. $\square$

The following lemma shows that if the density of a bipartite graph is large, then it contains a large complete bipartite subgraph:

Lemma 3.8. *Let $k \geq 2$ be an integer and ρ a real number, $0 < \rho \leq 1$. If G is a n by n bipartite graph where $n \geq (k-1)2^k/\rho^k$, the density of G is at least ρ, then G contains the complete bipartite graph $K_{k,k}$ as a (induced) subgraph.*

Proof. Assuming the contrary, there exists an integer $k \geq 2$, a real number $0 < \rho \leq 1$, and a bipartite graph $G(A \cup B, E)$ with $|A| = |B| \geq (k-1)2^k/\rho^k$ whose density is at least ρ abd which does not contain a copy of $K_{k,k}$. By Theorem 3.4, we have

$$\begin{aligned} |E| &< (k-1)^{1/k}(n-k+1)n^{1-1/k} + (k-1)n \\ &= (k-1)^{1/k}n^{1-1/k}\left(n - (k-1) + (k-1)^{1-1/k}n^{1/k}\right) \end{aligned}$$

Therefore,

$$
\begin{aligned}
\rho &= \frac{|E|}{n^2} \\
&< \left(\frac{k-1}{n}\right)^{1/k}\left(1-\frac{k-1}{n}+\left(\frac{k-1}{n}\right)^{1-1/k}\right) \\
&< \left(\frac{k-1}{n}\right)^{1/k}\left(1+\left(\frac{k-1}{n}\right)^{1-1/k}\right) \\
&\le 2\left(\frac{k-1}{n}\right)^{1/k}
\end{aligned}
$$

Hence, we infer that $n < (k-1)2^k/\rho^k$, a contradiction.

□

Now, we are ready to prove the main theorem of this section:

Theorem 3.9. *Let G be a graph of maximum degree Δ and T a set of integers. If the set T is k-good, then G is T-ℓ-choosable for every integer $\ell \ge (k-1)(4e\Delta)^k$.*

Proof. Fix a list assignment L such that $|L(v)| \ge (k-1)(4e\Delta)^k$ for each vertex v of G. Since the set T is k-good, no conflict graph of CG_e for any edge $e \in E_G$ can contain $K_{k,k}$ (recall the Proposition 2.1). Therefore, by Lemma 3.8, no CG_e has density greater than $\frac{1}{2e\Delta}$. Lemma 3.7 now implies that the vertices of the graph G can be colored from the list assignment L. □

The following simple corollary shows that the finiteness of the T-choice number of a graph G depends only on the set T:

Corollary 3.10. *For every every set T, the following statements are equivalent:*

- $\mathrm{ch}_T(K_2)$ *is finite, i.e., T is good.*
- $\mathrm{ch}_T(G)$ *is finite for every non-trivial graph G.*

Proof. It is easy to see that if $\mathrm{ch}_T(G)$ is finite, $\mathrm{ch}_T(K_2)$ must be finite as well. The other implication directly follows from Theorem 3.9: If K_2 is T-k-choosable, then the graph G must be T-k'-choosable for $k' = \lceil (k-1)(4e\Delta)^k \rceil$ where Δ is the maximum degree of the graph G. □

4 Upper Bound Polynomial in the T-choice number of K_2

In the previous section, we showed that if T is good, then $\mathrm{ch}_T(\Delta)$ is bounded by a polynomial in Δ. The constructed upper bound was, however, exponential in the $\mathrm{ch}_T(K_2)$. In this section, we provide an upper bound polynomial in $\mathrm{ch}_T(K_2)$, but, on the other hand, exponential in the maximum degree Δ.

First, let us state the following lemma:

Lemma 4.1. *Let T, S_1 and S_2 be three sets of integers, where $|S_1| = k$ and $|S_2| \geq k$. If the set T is k-good, then there exists an integer $c \in S_1$ such that the following holds:*

$$|S_2 \cap (c + (T \cup -T))| \leq |S_2| - (|S_2| - k)/k$$

Proof. Set $M = |S_2| - (|S_2| - k)/k$. Consider the set

$$\begin{aligned} S^\star &= \{(c_1, c_2) : c_1 \in S_1, c_2 \in S_2, |c_1 - c_2| \in T\} \\ &= \{(c_1, c_2) : c_1 \in S_1, c_2 \in S_2, \exists t \in T : c_2 = c_1 + t \vee c_2 = c_1 - t\}. \end{aligned}$$

Since the set T is k-good, at most $k-1$ elements of S_2 are allowed to be in the intersection of all the sets $\bigcap_{s \in S_1}(s + (T \cup -T))$. Therefore, we have that at most $k-1$ elements of S_2 may appear k times in the set $S^\star$, the other elements of S_2 may appear at most $k-1$ times. Therefore, $|S^\star| \leq |S_2|(k-1)+k-1$. On the other hand, if no color c with the properties from the statement of the lemma exists, then each color $s \in S_1$ appears in at least $M+1$ pairs in $S^\star$. Hence

$$|S^\star| \geq (M+1)k = |S_2|(k-1) + k.$$

This inequality contradicts the previously established bound $|S^\star| \leq |S_2|(k-1) + k - 1$. □

In the following theorem, we show an upper bound on the T-choice number which is, for a fixed graph G, polynomial in $\mathrm{ch}_T(K_2)$:

Theorem 4.2. *Let G be a graph with maximum degree at most Δ and let T be a set of integers. If the set T is k-good, then the T-choice number of G is at most $(\Delta(k-1)+3)k^\Delta$.*

Proof. Fix a graph G and a list assignment L which assigns each vertex of G a list of at least $(\Delta(k-1)+3)k^{\Delta}$ colors.

We color the vertices of the graph G sequentially. When a vertex v is colored, we remove the conflicting colors from the lists of uncolored neighbors of v. The colors are chosen in a way such that at the time when the vertex v is supposed to be colored, the number of the colors remaining in its list is at least $\Delta(k-1)+1$. The procedure for coloring the vertex v is as follows:

Consider uncolored neighbors of v in G. We show that there exists a color $c \in L(v)$ such that for each uncolored neighbor w of v the number of colors in $L(w)$ which are forbidden by the choice of the color c for the vertex v is not large. More precisely, the following holds for any uncolored neighbor w of v:

$$|L(w) \setminus ((c+T) \cup (c-T))| > (|L(w)| - k)/k \tag{1}$$

Assume the opposite. As $|L(v)| \geq \Delta(k-1)+1$, we have by the pigeon-hole principle, that there exists an uncolored neighbor w of v and a k-element subset B of $L(v)$ such that for each color $c \in B$, the following holds:

$$|L(w) \setminus (c+T \cup c-T)| \leq (|L(w)| - k)/k.$$

However, this is impossible by Lemma 4.1 (consider $S_1 = B$, $S_2 = L(w)$). Therefore, a color $c \in L(v)$ satisfying (1) for all uncolored neighbors w of v exists. Color the vertex v by the color c and remove any conflicting colors from the lists of neighbors of v. Continue with another vertex until all vertices of the graph G are colored.

Since we remove the conflicting colors from the lists of the neighboring vertices at each step, the constructed coloring is a proper list-T-coloring. It remains to prove that $|L(v)| \geq \Delta(k-1)+1$ at the time when the vertex v is to be colored. To show this, we calculate the true number of colors required: If the size of the list $L(v)$ was not changed before the vertex is colored, the required initial size would be $\Delta(k-1)+1$. Each change reduces the size of the list from ℓ to at least $(\ell-k)/k$, i.e., if the size of $L(v)$ changed only once, the required initial size would be $(\Delta(k-1)+1)k+k$. In general, the size of $L(v)$ is changed at most Δ times, hence, we must start with lists of sizes at least

$$
\begin{aligned}
s &= (\dots((\Delta(k-1)+1\underbrace{)k+k)\dots)k+k}_{\Delta\,\text{times}} \\
&= (\Delta(k-1)+1)k^{\Delta} + (\dots((k\underbrace{)k+k)\dots)k+k}_{(\Delta-1)\,\text{times}} \\
&= (\Delta(k-1)+1)k^{\Delta} + k^{\Delta} + k^{\Delta-1} + \dots + k \\
&= (\Delta(k-1)+1)k^{\Delta} + k\frac{k^{\Delta}-1}{k-1} \\
&\leq (\Delta(k-1)+1)k^{\Delta} + 2k^{\Delta} = (\Delta(k-1)+3)k^{\Delta}
\end{aligned}
$$

This completes the proof of the theorem. □

The following corollary follows straightforwardly from Theorem 4.2:

Corollary 4.3. *Let T be a set of integers. If T is k-good, then* $\mathrm{ch}_T(\Delta) \leq (\Delta(k-1)+3)k^{\Delta}$

5 Bad Sets with no Arithmetic Progressions

For a fixed graph G, it is natural to expect, that its T-choice number would depend on the size of the longest arithmetic progression contained in the set T. For the case of T finite, the following result by Waller [15] shows that the choice of the set $T = \{0, d, 2d, \dots, (k-1)d\}$ is the worst possible among all sets T with $|T| = k$:

Theorem 5.1. *A 2-connected graph G with maximum degree Δ is not $(|T|\Delta)$-T-choosable if and only if the set T is arithmetic (i.e., $T = \{0, d, 2d, \dots, (k-1)d\}$ for some integers k and d) and G is either a complete graph or an odd cycle.*

One might think that only the arithmetic progressions contained in the set T of the forbidden differences cause the difficulties. Quite surprisingly, this is not the case:

Theorem 5.2. *There exists an infinite set of integers T such that T contains no three-element arithmetic progression, but the set T is not k-good for any $k \geq 2$.*

Before we prove this theorem, let us remark that for the set T from Theorem 5.2, the T-choice number of every non-trivial graph is infinite (follows easily from Corrolary 3.10). Note also that the length three is the best possible, because any two integers form an arithmetic progression of length two.

Proof of Theorem 5.2: We construct the set T by induction: we create the sets $T_1, T_2, \ldots$, such that $T_i \subset T_{i+1}$. Each T_i does not contain an arithmetic progression of length three and, moreover, the set T_i is not i-good. The desired set T is the union of all the sets T_i: $T = \bigcup_{i=1}^{\infty} T_i$.

Instead of constructing the sets T_i directly, we create auxiliary sets A_i and B_i such that $A_i \subset A_{i+1}$ and $B_i \subset B_{i+1}$. Then, we just set $T_i = A_i + B_i$.

In the first step, set $A_1 = \{0, 1\}$ and $B_1 = \{0, 3\}$, therefore the set T_1 is $\{0, 1, 3, 4\}$.

In the $(i+1)$-th step, we obtain A_{i+1} and B_{i+1} as follows: let m_1 be the maximum value in the set $A_i + B_i$. Set $A_{i+1} = A_i \cup \{2m_1 + 1\}$. Let m_2 be the maximum value in the set $A_{i+1} + B_i$. Set $B_{i+1} = B_i \cup \{2m_2 + 1\}$.

Clearly, the set T_i is a proper subset of T_{i+1} ($A_i \subset A_{i+1}$ and $B_i \subset B_{i+1}$). It is not hard to see that the set T_i is not i-good: consider any i-element subset $\{a_1, \ldots, a_i\}$ of A_i. Then for each a_j, B_i is contained in $(-a_j) + T_i$:

$$(-a_j) + T_i = (-a_j) + (A_i + B_i) \supseteq (-a_j) + (a_j + B_i) = B_i$$

Hence,

$$B_i \subseteq ((-a_1) + T_i) \cap \cdots \cap ((-a_i) + T_i)$$

Since the size of B_i is $i + 1$, the set T_i is not i-good.

It remains to prove that T_i does not contain three-element arithmetic progression. The proof proceeds by induction: the set T_1 clearly contains no three-element arithmetic progression. We show that if the set $A_i + B_i$ contains no three-element arithmetic progression, then the set $A_{i+1} + B_i$ contains no three-element arithmetic progression as well.

Let $a = 2m_1 + 1$ be the newly added element of A_{i+1}. Then,

$$A_{i+1} + B_i = (A_i + B_i) \cup (a + B_i)$$

If the set $A_{i+1} + B_i$ contains a three-element arithmetic progression, two elements of the progression must be contained in $A_i + B_i$ and one in $a + B_i$ or vice-versa. (In the opposite case, we would have three-element arithmetic progression in $A_i + B_i$ or even in B_i itself). As m_1 is the maximum value in the set $A_i + B_i$ and the set B_i contains no negative values, the difference d of the arithmetic progression must be at least $m_1 + 1$. On the other hand, the maximum difference between any two elements contained in the set $A_i + B_i$ (and similarly, in the set $a + B_i$) is at most m_1 - a contradiction.

The case of extending $A_{i+1} + B_i$ to $A_{i+1} + B_{i+1}$ is symmetric. □

6 Future Research

In Sections 3 and 4, we obtained two upper bounds on $\mathrm{ch}_T(\Delta)$. The first is, for a fixed set T, polynomial in Δ. The other one is polynomial in $\mathrm{ch}_T(1)$. It is natural to ask whether there exists an upper bound polynomial in both Δ and $\mathrm{ch}_T(1)$:

Problem 6.1. *Does there exist a polynomial $p(k, \Delta)$ such that for every k-good set of integers T, $\mathrm{ch}_T(\Delta) \leq p(k, \Delta)$?*

The only lower bounds on $\mathrm{ch}_T(\Delta)$ we are aware of are trivial: either linear in $\mathrm{ch}_T(1)$ or in Δ. It is natural to suppose that a better lower bound can be obtained.

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