Channel assignment problem with variable weights

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Abstract

A λ-graph G is a (finite or infinite) graph with k types of edges, x_1-edges, ..., x_k-edges. A labeling c of the vertices of G by non-negative reals is proper with respect to reals $x_1, \ldots, x_k$, if the labels of the end-vertices of an x_i-edge differ by at least x_i. The span of the labeling c is the supremum of the labels used by c. The λ-function $\lambda_G(x_1, \ldots, x_k)$ is the infimum of the spans of all the proper labelings with respect to $x_1, \ldots, x_k$.

We show that the λ-function of any graph G is piecewise linear in $x_1, \ldots, x_k$ with finitely many linear parts. Moreover, we show that for every integers k and Λ, there exist constants $C_{k,\Lambda}$ and $D_{k,\Lambda}$ such that the λ-function of every λ-graph G with k types of edges and chromatic number at most Λ is comprised of at most $C_{k,\Lambda}$ linear parts, and the coefficients of $x_1, \ldots, x_k$ of the linear functions comprising $\lambda_G(x_1, \ldots, x_k)$ are integers between 0 and $D_{k,\Lambda}$. Among others, our results yield proofs of Piecewise Linearity Conjecture, Coefficient Bound Conjecture and Delta Bound Conjecture of Griggs and Jin.

1 Introduction

Radio frequency problems can be expressed as various graph labeling problems [12, 18]. A prominent role among such graph labeling problems plays

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the notion of $L(p_1, \ldots, p_k)$-labelings, graph labelings with distance constraints. Several approaches to study the dependence of the span of optimum labelings on the parameters $p_1, \ldots, p_k$ have recently been proposed: an approach based on real-value relaxation of $L(p_1, \ldots, p_k)$-labelings can be found in the work of Griggs and Jin [8, 9, 10], another approach based on the notion of λ-graphs can be found in [2]. In the present paper, we generalize the notion of λ-graphs introduced in [2] from $k = 2$ to arbitrary k and provide structural results for the general model. The obtained results yield proofs of Piecewise Linearity Conjecture, Coefficient Bound Conjecture and Delta Bound Conjecture of Griggs and Jin stated in [8].

A labeling c of the vertices of a (finite or infinite) graph G by non-negative integers is an *$L(p_1, \ldots, p_k)$-labeling* for positive integers $p_1, \ldots, p_k$, if the labels of any two vertices u and v at distance (exactly) i differ by at least p_i. Let us remark here that all graphs as well as λ-graphs considered in this paper can be finite or infinite unless stated otherwise. The maximum label used by c is said to be the *span* of c and the least span of an $L(p_1, \ldots, p_k)$-labeling of a graph G is denoted by $\lambda_G(p_1, \ldots, p_k)$ (we deviate from the standard notation in order to emphasize the dependence on the parameters $p_1, \ldots, p_k$). There is an enormous amount of literature on algorithms for $L(p_1, \ldots, p_k)$-labelings of graphs [1, 3, 6, 7, 15, 19]. From the structural point of view, the attention of researchers focused mainly on the case of $L(2, 1)$-labelings, partly because of the following conjecture of Griggs and Yeh [11]:

Conjecture 1 (Δ^2 Conjecture) *If G is a finite graph of maximum degree Δ, then $\lambda_G(2, 1) \leq \Delta^2$.*

Conjecture 1 was verified for several special classes of graphs, including graphs of maximum degree two, chordal graphs [20], see also [4, 16], and hamiltonian cubic graphs [13, 14]. In the general case, the original bound $\lambda_G(2, 1) \leq \Delta^2 + 2\Delta$ from [11] has been improved to $\lambda_{2,1}(G) \leq \Delta^2 + \Delta$ in [5] and a recent more general result of the author and Škrekovski [17] yields the present record $\lambda_{2,1}(G) \leq \Delta^2 + \Delta - 1$.

In order to capture the dependence of the optimum spans on the parameters, Griggs and Jin [8] allowed both the parameters $p_1, \ldots, p_k$ and the labels used by a labeling c to be any non-negative reals. Similarly to the original notion, they define the *span* of a labeling c as the supremum of the labels used by c, and $\lambda_G(p_1, \ldots, p_k)$ denotes the span of an optimum labeling of a graph G, i.e., the minimum (that is always attained if $\lambda_G(p_1, \ldots, p_k)$ is finite) of the spans of all $L(p_1, \ldots, p_k)$-labelings of G. In this setting, Griggs and Jin [8] prove that for any reals $p_1, \dots, p_k$, the value of $\lambda_G(p_1, \dots, p_k)$ can be expressed as $\sum_{i=1}^k \alpha_i p_i$ for some non-negative integers α_i. Moreover, if all $p_1, \dots, p_k$ are integers, the values $\lambda_G(p_1, \dots, p_k)$ in the original and the relaxed settings coincide. They also show that the function $\lambda_G(p_1, \dots, p_k)$ is a continuous function piecewise linear in the parameters $p_1, \dots, p_k$, and conjecture the following [8]:

Conjecture 2 (Piecewise Linearity Conjecture) *For any graph G, the graph of the function $\lambda_G(p_1, \dots, p_k)$ is comprised by finitely many linear parts, i.e., there exist finitely many hyperplanes in $\mathbb{R}^k$ through the origin such that the function $\lambda_G(p_1, \dots, p_k)$ is linear in each of the convex polyhedral cones (formed by non-negative reals) that are determined by the hyperplanes.*

Conjecture 3 (Coefficient Bound Conjecture) *For every graph G and every integer k, there exists a constant $D_{k,G}$ such that the following holds for all reals $p_1, \dots, p_k$: the value of the function $\lambda_G(p_1, \dots, p_k)$ is equal to $\sum_{i=1}^k \alpha_i p_i$ for some integer coefficients $\alpha_1, \dots, \alpha_k$ between 0 and $D_{k,G}$ (the integers $\alpha_1, \dots, \alpha_k$ may depend on $p_1, \dots, p_k$). Moreover, there is a labeling c with span $\lambda_G(p_1, \dots, p_k)$ such that $c(v) = \sum_{i=1}^k \alpha_i(v) p_i$ where $\alpha_1(v), \dots, \alpha_k(v)$ are integers between 0 and $D_{k,G}$.*

Conjecture 4 (Delta Bound Conjecture) *For every integers Δ and k, there exists a constant $D_{k,\Delta}$ such that, for every graph G with maximum degree at most Δ and every reals $p_1, \dots, p_k$ the following holds: the value of the function $\lambda_G(p_1, \dots, p_k)$ is equal to $\sum_{i=1}^k \alpha_i p_i$ for some integer coefficients $\alpha_1, \dots, \alpha_k$ between 0 and $D_{k,\Delta}$ (the integers $\alpha_1, \dots, \alpha_k$ may depend on $p_1, \dots, p_k$). Moreover, there is a labeling c with span $\lambda_G(p_1, \dots, p_k)$ such that $c(v) = \sum_{i=1}^k \alpha_i(v) p_i$ where $\alpha_1(v), \dots, \alpha_k(v)$ are integers between 0 and $D_{k,\Delta}$.*

Note that Delta Bound Conjecture implies Coefficient Bound Conjecture. Griggs and Jin [8] proved all the three Conjectures 2, 3 and 4 for $k = 2$ (for $k = 1$, the conjectures are trivial) and Conjecture 2 also for finite graphs G.

In the present paper, we consider the problems posed in [8] in the more general setting of λ-graphs that was introduced for $k = 2$ in [2]. A *λ-graph G with k types of edges* is a graph G whose edges are labeled by variables $x_1, \dots, x_k$. An edge labeled by a variable x_i is called *x_i-edge*. Two vertices of G may be joined by edges of several types. A *proper labeling c* of G with respect to the real numbers $x_1, \dots, x_k$ is a labeling of the vertices of G by non-negative reals such that the labels of the end-vertices of an x_i-edge uv differ by at least x_i, i.e., $|c(u)-c(v)| \geq x_i$. The *span* of the labeling c is the supremum of the labels used by c and $\lambda_G(x_1,\ldots,x_k)$ is defined to be the infimum of the spans of all proper labelings with respect to $x_1,\ldots,x_k$. The results of [2] yield that for every reals $x_1,\ldots,x_k$, if $\lambda_G(x_1,\ldots,x_k)$ is finite, then there exists a proper labeling c with span $\lambda_G(x_1,\ldots,x_k)$ and the span of c is equal to the maximum label used by c, i.e., both the infimum and the supremum in the definitions are attained. The λ-*function* of a λ-graph G is $\lambda_G(x_1,\ldots,x_k)$ viewed as a function of variables $x_1,\ldots,x_k$. The *chromatic number* of a λ-graph G is the chromatic number of the underlying graph, i.e., $\lambda_G(1,\ldots,1)+1$.

$L(p_1,\ldots,p_k)$-labelings of graphs can be modeled as λ-graphs as follows: if G is a graph, form a λ-graph $G^{(k)}$ with the vertex set $V(G)$ such that two vertices u and v are joined by an x_i-edge in $G^{(k)}$, $i=1,\ldots,k$, if their distance in G is exactly i. Clearly, the optimum span $\lambda_G(p_1,\ldots,p_k)$ is equal to the value $\lambda_{G^{(k)}}(x_1,\ldots,x_k)$ of the λ-function of $G^{(k)}$ for $x_i=p_i$, $i=1,\ldots,k$. Because of this close relation, we decided to use the notation $\lambda_G(\ldots)$ both for the spans of optimum $L(p_1,\ldots,p_k)$-labelings and the λ-functions of λ-graphs. Since it is always clear throughout the paper whether G is a graph (in which case, $\lambda_G(p_1,\ldots,p_k)$ stands for the span of an optimum $L(p_1,\ldots,p_k)$-labeling) or a λ-graph (in which case, λ_G stands for the λ-function of G), the confusion of the notations is avoided.

Similarly as in the case of $L(p_1,\ldots,p_k)$-labeling [8], λ-functions of λ-graphs have the *scaling property*, i.e., for every non-negative reals $x_1,\ldots,x_k$ and β, the following holds: $\lambda_G(\beta x_1,\ldots,\beta x_k)=\beta\lambda_G(x_1,\ldots,x_k)$. Therefore, the λ-function of any λ-graph is linear on every ray through the origin in $\mathbb{R}^k$. In Section 4, we show that the λ-function λ_G of any λ-graph G is comprised of finitely many linear parts, i.e., the subset of $\mathbb{R}^k$ formed by non-negative reals can be partitioned into finitely many (infinite) polyhedral cones (with the tips at the origin of $\mathbb{R}^k$) such that λ_G is linear on each of these cones.

Our main result is Theorem 9 that asserts the existence of the constants $C_{k,\Lambda}$ and $D_{k,\Lambda}$ such that the λ-function of any λ-graph G with k types of edges and chromatic number at most Λ is comprised of at most $C_{k,\Lambda}$ linear parts and the coefficients of $x_1,\ldots,x_k$ of the linear functions comprising the λ-function are integers between 0 and $D_{k,\Lambda}$. In this paper, we solely focus on proving the existence of the constants $C_{k,\Lambda}$ and $D_{k,\Lambda}$ without attempting to optimize their growth. Let us remark that the existence of the constants $C_{k,\Lambda}$ and $D_{k,\Lambda}$ for $k=2$ follows from the results of [2]. However, the technique used in [2] does not seem to generalize to $k>2$. As demonstrated in Section 5, our main result yields the proofs of Piecewise Linearity Conjecture, Coefficient Bound Conjecture and Delta Bound Conjecture for $L(p_1, \ldots, p_k)$-labelings (Conjectures 2, 3 and 4).

2 Preliminaries

In [2], an analogue of Gallai-Roy Theorem for infinite graphs with edges of different weights was proved. We will not state the theorem in its full generality but just in the form restricted to λ-graphs. An orientation of an infinite graph G is said to be *finitary* if it does not contain a directed walk of arbitrary length. In particular, a finitary orientation of G is acyclic. A *weight* of a finite directed path P in an orientation of a λ-graph G with respect to $x_1, \ldots, x_k$ is the sum of the variables assigned to its edges, i.e., $\sum_{i=1}^{k} \alpha_i x_i$ if P contains α_i x_i-edges. The *weight of a finitary orientation* $\vec{G}$ of a λ-graph G is the maximum weight of a directed path in $\vec{G}$ (note that the maximum is always attained since the lengths of directed paths in $\vec{G}$ are bounded in a finitary orientation and there are only finitely many different types of edges in G). We now state the version of Gallai-Roy Theorem for λ-graphs:

Theorem 1 *Let G be a λ-graph with k types of edges. For any real numbers $x_1, \ldots, x_k$, $\lambda_G(x_1, \ldots, x_k)$ is equal to the minimum weight of a finitary orientation $\vec{G}$ of G (in particular, there exists a finitary orientation of weight $\lambda_G(x_1, \ldots, x_k)$).*

If $\vec{G}$ is a finitary orientation of a λ-graph G, then the labeling c, where $c(v)$ is the maximum weight of a directed path ending at a vertex v, is a proper labeling of G with respect to $x_1, \ldots, x_k$. We say that the labeling c, defined in this way, *corresponds* to the orientation $\vec{G}$. Clearly, the span of the labeling corresponding to $\vec{G}$ is the weight of $\vec{G}$. On the other hand, for a proper labeling c of G for positive reals $x_1, \ldots, x_k$, whose span is finite, one may define a (finitary) orientation $\vec{G}$ of G such that an edge uv is directed from u to v if $c(u) < c(v)$. Such orientation $\vec{G}$ *corresponds* to the labeling c. Observe that the weight of the orientation corresponding to a proper labeling c is at most the span of c (in general, it can be strictly smaller).

If G is a λ-graph with k types of edges, we say that an edge uv is an $x_{\le \ell}$-edge if uv is an x_i-edge where $i \le \ell$. The set of all $x_{\le \ell}$-edges of G is the set of all x_i-edges with $i = 1, \ldots, \ell$. Similarly, we use the terms $x_{<\ell}$-edges,

$x_{\geq \ell}$-edges, etc. We demonstrate this notation in the next auxiliary lemma that will be used later:

Lemma 2 *Let G be a λ-graph with k types of edges and with chromatic number at most Λ, and let $0 \leq \ell < k$. If there exist an integer D and a finitary orientation $\vec{G}$ of G such that every directed path in $\vec{G}$ contains at most D $x_{\leq \ell}$-edges, then:*

$$\lambda_G(x_1, \ldots, x_k) \leq d_{\max} + (\ell+1)^D \Lambda \cdot \max\{x_{\ell+1}, \ldots, x_k\}$$

where $d_{\max}$ is the maximum sum of weights of $x_{\leq \ell}$-edges on a directed path in $\vec{G}$, i.e., $d_{\max}$ would be the weight of $\vec{G}$ if the parameters $x_{\ell+1}, \ldots, x_k$ were equal to zero.

In particular, it holds that $\lambda_G(x_1, \ldots, x_k) \leq \Lambda \cdot \max\{x_1, \ldots, x_k\}$.

Proof: Fix a finitary orientation $\vec{G}$ that has the properties described in the statement of the lemma (if $\ell = 0$, fix any finitary orientation $\vec{G}$ of G). Let $d(v)$ be the maximum sum of the weights of $x_{\leq \ell}$-edges on a directed path in $\vec{G}$ ending at a vertex v. Clearly, $d_{\max} = \max_{v \in V(G)} d(v)$. Let D be the set of all different values of $d(v)$ and let $\delta(v)$ be the number of the elements of D smaller than $d(v)$. Since every directed path in $\vec{G}$ contains at most D $x_{\leq \ell}$-edges, it holds that $|D| \leq (\ell+1)^D$, and thus $0 \leq \delta(v) < |D| \leq (\ell+1)^D$ for every vertex v of G. Finally, let μ be a coloring of the vertices of G with colors $1, \ldots, \Lambda$.

Let us define a labeling c' of the vertices of G as follows:

$$c'(v) = d(v) + (\delta(v)\Lambda + \mu(v)) \cdot \max\{x_{\ell+1}, \ldots, x_k\} \ .$$

Since $\delta(v) < |D|$ for every vertex v of G, the span of c' does not exceed:

$$d_{\max} + |D|\Lambda \cdot \max\{x_{\ell+1}, \ldots, x_k\} \leq d_{\max} + (\ell+1)^D \Lambda \cdot \max\{x_{\ell+1}, \ldots, x_k\}.$$

In the rest, we show that c' is a proper labeling with respect to $x_1, \ldots, x_k$.

Consider an x_i-edge uv of G. By symmetry, we may assume that the edge uv is directed from u to v in $\vec{G}$. In particular, it holds that $d(u) \leq d(v)$ and $\delta(u) \leq \delta(v)$. We distinguish two major cases: the first one is $i \leq \ell$. In this case, $d(u) + x_i \leq d(v)$ and thus $\delta(u) < \delta(v)$. We can immediately conclude:

$$\begin{aligned} c'(v) - c'(u) &= d(v) - d(u) + ((\delta(v) - \delta(u))\Lambda + \mu(v) - \mu(u)) \cdot \max\{x_{\ell+1}, \ldots, x_k\} \\ &\geq d(v) - d(u) + (\Lambda + \mu(v) - \mu(u)) \cdot \max\{x_{\ell+1}, \ldots, x_k\} \\ &\geq d(v) - d(u) \geq x_i \ . \end{aligned}$$

Therefore, the edge uv is properly colored in the first case.

The other case is that $i > \ell$. If $d(u) = d(v)$, then $\delta(u) = \delta(v)$ and the following holds (similarly to the first case):

$$|c'(u) - c'(v)| = |\mu(u) - \mu(v)| \cdot \max\{x_{\ell+1}, \ldots, x_k\} \geq x_i \ .$$

If $d(u) < d(v)$, then $\delta(u) < \delta(v)$ and we have the following:

$$\begin{aligned} c'(v) - c'(u) &= d(v) - d(u) + ((\delta(v) - \delta(u))\Lambda + \mu(v) - \mu(u)) \cdot \max\{x_{\ell+1}, \ldots, x_k\} \\ &\geq (\Lambda + \mu(v) - \mu(u)) \cdot \max\{x_{\ell+1}, \ldots, x_k\} \geq x_i \ . \end{aligned}$$

Hence, the labels of u and v also differ by at least x_i in the second case. ■

3 Orientations with Minimum Weight

In this section, we construct orientations of λ-graphs with minimum weight such that the maximum length of a directed path in the constructed orientation is bounded. First, let us define numbers $D_{i,\Lambda}$ and $K_{i,\Lambda}$ for integer Λ and i as follows:

$$\begin{aligned} D_{1,\Lambda} &= \Lambda \\ K_{i,\Lambda} &= (i+1)^{D_{i,\Lambda}} \\ D_{i+1,\Lambda} &= (2K_{i,\Lambda})^{K_{i,\Lambda}^2+3} \cdot \Lambda \end{aligned}$$

Next, we state several propositions that can be verified directly from the definitions of $D_{i,\Lambda}$ and $K_{i,\Lambda}$. Their proofs are left to the reader.

Proposition 3 *For integers $\Lambda \geq 2$ and i, the number of multisets that consist of at most $D_{i,\Lambda}$ numbers $1, \ldots, i$ does not exceed $K_{i,\Lambda} - 1$.*

Proposition 4 *The following holds for every integers Λ and i:*

$$D_{i+1,\Lambda} \geq (2K_{i,\Lambda})^{K_{i,\Lambda}^2+2} \cdot \Lambda + K_{i,\Lambda} \cdot \Lambda \ .$$

We now introduce some notations used in the proof the main lemma of this section (Lemma 7). For an integer M and positive reals $x_1, \ldots, x_k$,

$\Gamma_M(x_1,\ldots,x_k)$ denotes the set of all combinations of $x_1,\ldots,x_k$ with non-negative integer coefficients whose sum does not exceed M, i.e.:

$$\Gamma_M(x_1,\ldots,x_k) = \left\{ \sum_{j=1}^{k} \alpha_j x_j \mid 0 \le \alpha_1,\ldots,\alpha_k \;\&\; \sum_{j=1}^{k} \alpha_j \le M \right\} .$$

The set $\Gamma'_M(x_1,\ldots,x_k)$ is then defined to be the set of all non-negative reals that can be expressed as a difference of two numbers from $\Gamma_M(x_1,\ldots,x_k)$, i.e.:

$$\Gamma'_M(x_1,\ldots,x_k) = \{\alpha - \beta | \alpha,\beta \in \Gamma_M(x_1,\ldots,x_k) \;\&\; \alpha - \beta \ge 0\} .$$

Since $0 \in \Gamma_M(x_1,\ldots,x_k)$, the set $\Gamma_M(x_1,\ldots,x_k)$ is contained in the set $\Gamma'_M(x_1,\ldots,x_k)$. The following estimates on the sizes of $\Gamma_M(x_1,\ldots,x_k)$ and $\Gamma'_M(x_1,\ldots,x_k)$ directly follow from Proposition 3:

Proposition 5 *Let $x_1,\ldots,x_k$ be any positive real numbers and let $\Lambda \ge 2$ be a positive integer. The following two estimates hold:*

$$\begin{aligned} |\Gamma_{D_{k,\Lambda}}(x_1,\ldots,x_k)| &< K_{k,\Lambda} \\ |\Gamma'_{D_{k,\Lambda}}(x_1,\ldots,x_k)| &< K_{k,\Lambda}^2 \end{aligned}$$

We now establish an auxiliary lemma that will be extremely useful in the proof of Lemma 7:

Lemma 6 *Let $x_1,\ldots,x_k$, $x_1 \ge \cdots \ge x_k > 0$, be real numbers, $\Lambda \ge 2$ a positive integer, and y another positive real number. There exists a real number t,*

$$K_{k,\Lambda}\Lambda y \le t \le (2K_{k,\Lambda})^{K_{k,\Lambda}^2}\Lambda y$$

such that the set $\Gamma'_{D_{k,\Lambda}}(x_1,\ldots,x_k)$ contains no element strictly between t and $K_{k,\Lambda}(t+\Lambda y)$. In particular, the real t has the following property $(\star)$: if $\gamma \in \Gamma'_{D_{k,\Lambda}}(x_1,\ldots,x_k)$ and $\gamma > t$, then $\gamma \ge K_{k,\Lambda}(t+\Lambda y)$.

Proof: By Proposition 5, the set $\Gamma'_{D_{k,\Lambda}}(x_1,\ldots,x_k)$ contains less than $K_{k,\Lambda}^2$ real numbers. Let us define reals t_j, $j = 1,\ldots,K_{k,\Lambda}^2$, as follows:

$$t_j = (2K_{k,\Lambda})^j \Lambda y .$$

Since $K_{k,\Lambda}(t_j+\Lambda y)\le 2K_{k,\Lambda}t_j=t_{j+1}$ for all $j=1,\ldots,K_{k,\Lambda}^2-1$, all the open intervals I_j,

$$I_j=\left(t_j,K_{k,\Lambda}(t_j+\Lambda y)\right),j=1,\ldots,K_{k,\Lambda}^2,$$

are disjoint. Since all the $K_{k,\Lambda}^2$ intervals I_j are disjoint and $\Gamma'_{D_{k,\Lambda}}(x_1,\ldots,x_k)<K_{k,\Lambda}^2$, there exists j_0 such that no element of $\Gamma'_{D_{k,\Lambda}}(x_1,\ldots,x_k)$ is contained in I_{j_0}. The number t_{j_0} is the desired number t.

■

We state and prove the key lemma of this section:

Lemma 7 *Let G be a (finite or infinite) λ-graph G with k types of edges and with chromatic number at most Λ. Fix real numbers $x_1,\ldots,x_k$, $x_1\ge\cdots\ge x_k>0$. For each $\ell=1,\ldots,k$, there exists a finitary orientation $\vec{G}$ of G of weight $\lambda_G(x_1,\ldots,x_k)$ such that every directed path in $\vec{G}$ contains at most $D_{\ell,\Lambda}$ $x_{\le\ell}$-edges.*

Proof: If $\Lambda=1$, there is nothing to prove since G contains no edges and the statement of the lemma holds vacuously. Therefore, we assume $\Lambda\ge 2$ in the remaining. For the rest of the proof, let us fix a proper coloring μ (in the usual sense) of the vertices of G with colors $1,\ldots,\Lambda$.

The proof of the lemma proceeds by induction on the number ℓ. First, we have to deal with the case $\ell=1$. Consider any finitary orientation $\vec{G}$ of G of weight $\lambda_G(x_1,\ldots,x_k)$. Such an orientation exists by Theorem 1. By Lemma 2, it holds $\lambda_G(x_1,\ldots,x_k)\le\Lambda x_1$. Since the weight of any directed path in $\vec{G}$ does not exceed $\lambda_G(x_1,\ldots,x_k)\le\Lambda x_1$, every directed path in $\vec{G}$ contains at most $D_{1,\Lambda}=\Lambda$ x_1-edges.

We now deal with the case $\ell>1$. By the induction, there exists a finitary orientation $\vec{G}$ of G of weight $\lambda_G(x_1,\ldots,x_k)$ such that any directed path contains at most $D_{\ell-1,\Lambda}$ $x_{\le\ell-1}$-edges. Let $c(v)$ be the labeling corresponding to $\vec{G}$, and let $d(v)$ be the maximum sum of weights of $x_{\le\ell-1}$-edges on a directed path in $\vec{G}$ ending at a vertex v. Clearly, $d(v)\le c(v)$ for every vertex v of G. Finally, let $\delta(v)$ be the number of the elements of $\Gamma_{D_{\ell-1,\Lambda}}$ smaller or equal to $d(v)$. Since $|\Gamma_{D_{\ell-1,\Lambda}}|<K_{\ell-1,\Lambda}$ by Proposition 5, $1\le\delta(v)\le K_{\ell-1,\Lambda}-1$ for every vertex v of G.

By Lemma 6 (applied for $y=x_\ell$ and $k=\ell-1$), there exists a real number t,

$$K_{k,\Lambda}\Lambda x_\ell\le t\le(2K_{k,\Lambda})^{K_{\ell-1,\Lambda}^2}\Lambda x_\ell\ ,$$

such that the set $\Gamma'_{D_{\ell-1,\Lambda}}(x_1,\ldots,x_{\ell-1})$ contains no element strictly between t and $K_{\ell-1,\Lambda}(t+\Lambda x_\ell)$, i.e., t has the property $(\star)$ from Lemma 6. We define a new labeling c' and show that it is is a proper labeling with respect to $x_1,\ldots,x_k$:

1. If $c(v)-d(v)\le(K_{\ell-1,\Lambda}-\delta(v))t$, then $c'(v)=c(v)$.
2. Otherwise, $c'(v)=d(v)+(K_{\ell-1,\Lambda}-1)t+\delta(v)\Lambda x_\ell+\mu(v)x_\ell$.

Fix an x_i-edge uv of G. In order to verify that c' is a proper labeling on the edge uv, we distinguish five major cases:

- **Both the labels $c'(u)$ and $c'(v)$ are defined by the first rule.** Since $c'(u)=c(u)$ and $c'(v)=c(v)$, we have $|c'(u)-c'(v)|=|c(u)-c(v)|\ge x_i$.
- **The label $c'(u)$ is defined by the first rule, the label $c'(v)$ is defined by the second rule and $i<\ell$.**
  We distinguish two cases according to the orientation of the edge uv in $\vec{G}$. If the edge is directed from u to v, we have $d(u)+x_i\le d(v)$. Because the label of u is defined by the first rule, the label $c'(u)=c(u)$ is at most $d(u)+(K_{\ell-1,\Lambda}-1)t$. On the other hand, the label $c'(v)$ is larger than $d(v)+(K_{\ell-1,\Lambda}-1)t$. We infer that $c'(v)-c'(u)\ge d(v)-d(u)\ge x_i$.

  The other case is that the edge uv is directed from v to u. In particular, $d(v)+x_i\le d(u)$, $\delta(v)<\delta(u)$ and $c(v)\le c(u)$. First, we show that $d(u)-d(v)-x_i>t$. Assume for contrary that $d(u)-d(v)-x_i\le t$. Since c is a proper labeling of G, $c(v)\le c(u)-x_i$. Since the label to u was defined by the first rule, we have $c(u)\le d(u)+(K_{\ell-1,\Lambda}-\delta(u))t$. Therefore, the following holds:

$$\begin{aligned}c(v)&\le c(u)-x_i\\&\le d(u)+(K_{\ell-1,\Lambda}-\delta(u))t-x_i\\&\le d(v)+(K_{\ell-1,\Lambda}-\delta(u))t+t\\&\le d(v)+(K_{\ell-1,\Lambda}-\delta(v))t\ .\end{aligned}$$

  However, this yields that the label of u should have been defined by the first rule. We conclude that $d(u)-d(v)-x_i>t$. Moreover, since t has the property $(\star)$ and $d(u)-d(v)-x_i\in\Gamma'_{D_{\ell-1,\Lambda}}(x_1,\ldots,x_{\ell-1})$, it holds that $d(u)-d(v)-x_i\ge K_{\ell-1,\Lambda}(t+\Lambda x_\ell)$.

We now bound the label $c'(v)$ assigned to the vertex v from above (recall that $\mu(v)x_\ell \le \Lambda x_\ell \le t$):

$$\begin{aligned} c'(v) &= d(v) + (K_{\ell-1,\Lambda} - 1)t + \delta(v)\Lambda x_\ell + \mu(v)x_\ell \\ &\le d(v) + K_{\ell-1,\Lambda}t + K_{\ell-1,\Lambda}\Lambda x_\ell \\ &\le d(u) - x_i \;\le\; c(u) - x_i \;=\; c'(u) - x_i \ . \end{aligned}$$

Hence, the labels of the vertices u and v differ by at least x_i as required.

- **The label $c'(u)$ is defined by the first rule, the label $c'(v)$ is defined by the second rule and $i \ge \ell$.**
  If $d(u) \le d(v)$, then $c'(u) \le d(u) + (K_{\ell-1,\Lambda} - 1)t$ and $c'(v) \ge d(v) + (K_{\ell-1,\Lambda} - 1)t + \mu(v)x_\ell \ge d(u) + (K_{\ell-1,\Lambda} - 1)t + x_\ell$. Therefore, $c'(v) - c'(u) \ge x_\ell$ as desired.

  In the rest, we focus on the case $d(u) > d(v)$. This implies that $\delta(u) > \delta(v)$, the edge uv is directed from v to u and $c(u) > c(v)$. First, we exclude the case $d(u) - d(v) \le t$. Since the label of the vertex u was defined by the first rule, we have $c(u) \le d(u) + (K_{\ell-1,\Lambda} - \delta(u))t$. We infer the following upper bound on $c(v)$:

$$\begin{aligned} c(v) &\le c(u) \le d(u) + (K_{\ell-1,\Lambda} - \delta(u))t \\ &\le d(v) + t + (K_{\ell-1,\Lambda} - \delta(u))t \\ &\le d(v) + (K_{\ell-1,\Lambda} - \delta(v))t \ . \end{aligned}$$

  Then the label to v should have been defined by the first rule not by the second one. We may conclude that $d(u) - d(v) > t$. Since t has the property $(\star)$ and $d(u) - d(v) \in \Gamma'_{D_{\ell-1,\Lambda}}(x_1, \ldots, x_{\ell-1})$, it holds that $d(u) - d(v) \ge K_{\ell-1,\Lambda}(t + \Lambda x_\ell)$. The following upper bound on $c'(v)$ readily follows (recall that $\delta(v) \le K_{\ell-1,\Lambda} - 1$):

$$\begin{aligned} c'(v) &= d(v) + (K_{\ell-1,\Lambda} - 1)t + \delta(v)\Lambda x_\ell + \mu(v)x_\ell \\ &\le d(v) + K_{\ell-1,\Lambda}t - t + K_{\ell-1,\Lambda}\Lambda x_\ell \\ &\le d(u) - t \;\le\; c(u) - x_\ell \;=\; c'(u) - x_\ell \;\le\; c'(u) - x_i \ . \end{aligned}$$

  Hence, the labels of the end-vertices of the x_i-edge uv differ by at least x_i as required.

- **Both the labels $c'(u)$ and $c'(v)$ are defined by the second rule and $i < \ell$.**

By symmetry, we may assume that the edge uv is directed from u to v in $\vec{G}$. In such case, $d(u) + x_i \leq d(v)$ and $\delta(u) < \delta(v)$. We now estimate the difference of the labels $c'(u)$ and $c'(v)$:

$$\begin{aligned} c'(v) - c'(u) &= d(v) - d(u) + (\delta(v) - \delta(u))\Lambda x_\ell + (\mu(v) - \mu(u))x_\ell \\ &\geq x_i + \Lambda x_\ell - |\mu(v) - \mu(u)|x_\ell \;\geq\; x_i \ . \end{aligned}$$

We conclude that the edge uv is properly colored.

- **Both the labels $c'(u)$ and $c'(v)$ are defined by the second rule and $i \geq \ell$.**
By symmetry, we may assume that the edge uv is directed from u to v in $\vec{G}$. In such case, $d(u) \leq d(v)$. If $d(u) = d(v)$, then:

$$|c'(v) - c'(u)| = |\mu(v) - \mu(u)|x_\ell \geq x_\ell \geq x_i \ .$$

In the rest, we deal with the case $d(u) < d(v)$. In particular, $\delta(u) < \delta(v)$. We bound the difference between $c'(u)$ and $c'(v)$ as follows (recall that $1 \leq \mu(v), \mu(v) \leq \Lambda$):

$$\begin{aligned} c'(v) - c'(u) &= d(v) - d(u) + (\delta(v) - \delta(u))\Lambda x_\ell + (\mu(v) - \mu(u))x_\ell \\ &\geq \Lambda x_\ell - |\mu(v) - \mu(u)|x_\ell \;\geq\; x_\ell \;\geq\; x_i \ . \end{aligned}$$

Therefore, the difference of the labels $c'(u)$ and $c'(v)$ is at least x_i as desired.

As the next step, we show that the span of c' is equal to $\lambda_G(x_1, \ldots, x_k)$. In order to do so, it is enough to show that $c'(v) \leq \lambda_G(x_1, \ldots, x_k)$ for every vertex v of G. Let $c_{\max}$ and $d_{\max}$ be the maximums of the values $c(v)$ and $d(v)$ taken over all the vertices v of G. Clearly, $c_{\max} = \lambda_G(x_1, \ldots, x_k)$. By Lemma 2, the following holds:

$$c_{\max} \leq d_{\max} + \ell^{D_{\ell-1,\Lambda}}\Lambda \cdot x_\ell = d_{\max} + K_{\ell-1,\Lambda}\Lambda \cdot x_\ell \ .$$

Fix a vertex v of G. In order to show that $c'(v) \leq c_{\max}$, we distinguish three cases according to difference between $d(v)$ and $d_{\max}$:

- $d(v) = d_{\max}$
Since $c(v) \leq c_{\max} \leq d(v) + K_{\ell-1,\Lambda}\Lambda x_\ell \leq d(v) + t \leq d(v) + (K_{\ell-1,\Lambda} - \delta(v))t$, the label of the vertex v was defined by the first rule. Consequently, $c'(v) = c(v) \leq c_{\max}$.

- $d_{\max} - d(v) \leq t$
  First, let us observe that $\delta(v) \leq K_{\ell-1,\Lambda} - 2$. Again, we bound the original label $c(v)$ from above:

  $$c(v) \leq c_{\max} \leq d_{\max} + t \leq d(v) + 2t \leq d(v) + (K_{\ell-1,\Lambda} - \delta(v))t .$$

  Therefore, the label of the vertex v was defined by the first rule, and $c'(v) = c(v) \leq c_{\max}$.

- $d_{\max} - d(v) > t$
  Since t has the property $(\star)$ and $d_{\max} - d(v) \in \Gamma'_{D_{\ell-1,\Lambda}}(x_1, \ldots, x_{\ell-1})$, $d_{\max} - d(v) \geq K_{\ell-1,\Lambda}(t + \Lambda x_\ell)$. If the first rule applies to the vertex v, then $c'(v) = c(v) \leq c_{\max}$. If the second rule applies, then the following estimate on $c'(v)$ holds:

  $$\begin{aligned} c'(v) &= d(v) + (K_{\ell-1,\Lambda} - 1)t + \delta(v)\Lambda x_\ell + \mu(v)x_\ell \\ &\leq d(v) + K_{\ell-1,\Lambda}t + K_{\ell-1,\Lambda}\Lambda x_\ell \\ &\leq d_{\max} \;\leq\; c_{\max} \;. \end{aligned}$$

  Hence, the label $c'(v)$ does not exceed $c_{\max}$.

Let $\vec{G}'$ be the orientation of G corresponding to the labeling c'. Since all $x_1, \ldots, x_k$ are positive, the orientation G is finitary and its weight is at most the span of c'. Since the span of c' is $\lambda_G(x_1, \ldots, x_k)$, the weight of $\vec{G}'$ is exactly $\lambda_G(x_1, \ldots, x_k)$. In order to finish the proof of the lemma, we establish that each directed path in $\vec{G}'$ contains at most $D_{\ell,\Lambda}$ $x_{\leq \ell}$-edges.

All the labels $c'(v)$ defined by the first rule are contained in the following union of intervals:

$$\bigcup_{\gamma \in \Gamma_{D_{\ell-1,\Lambda}}(x_1,\ldots,x_{\ell-1})} \langle \gamma, \gamma + K_{\ell-1,\Lambda}t \rangle$$

$$\subseteq \bigcup_{\gamma \in \Gamma_{D_{\ell-1,\Lambda}}(x_1,\ldots,x_{\ell-1})} \left\langle \gamma, \gamma + (2K_{\ell-1,\Lambda})^{K_{\ell-1,\Lambda}^2+1}\Lambda x_\ell \right\rangle$$

The labels $c'(v)$ assigned by the second rule are from the following set:

$$\bigcup_{\gamma_i \in \Gamma_{D_{\ell-1,\Lambda}}(x_1,\ldots,x_{\ell-1})} \{\gamma_i + (K_{\ell-1,\Lambda} - 1)t + i\Lambda x_\ell + jx_\ell, j = 1, \ldots, \Lambda\}$$

where $\gamma_1, \gamma_2, \ldots$ are all the elements of $\Gamma_{D_{\ell-1,\Lambda}}$ listed in the increasing order. Consider a directed path P in $\vec{G}'$ and let C be the set of labels of the end-vertices of $x_{\le\ell}$-edges on P. Since any two labels in C differ by at least x_ℓ and $|\Gamma_{D_{\ell-1,\Lambda}}(x_1, \ldots, x_{\ell-1})| \le K_{\ell-1,\Lambda}$ by Proposition 5, we have the following upper bound on the number of labels contained in C that were defined by the first rule:

$$|\Gamma_{D_{\ell-1,\Lambda}}(x_1, \ldots, x_{\ell-1})| \frac{(2K_{\ell-1,\Lambda})^{K^2_{\ell-1,\Lambda}+1}\Lambda x_\ell}{x_\ell} \le (2K_{\ell-1,\Lambda})^{K^2_{\ell-1,\Lambda}+2}\Lambda \ .$$

Similarly, the number of the labels defined by the second rule does not exceed:

$$|\Gamma_{D_{\ell-1,\Lambda}}(x_1, \ldots, x_{\ell-1})|\Lambda \le K_{\ell-1,\Lambda}\Lambda \ .$$

Combining both the bounds, we infer from Proposition 4 that the size of C does not exceed $D_{\ell,\Lambda} = (2K_{\ell-1,\Lambda})^{K^2_{\ell-1,\Lambda}+3}\Lambda$. Therefore, every directed path in $\vec{G}'$ contains at most $D_{\ell,\Lambda}$ $x_{\le\ell}$-edges as desired.

■

We modify Lemma 7 to a version used in Section 4:

Lemma 8 *Let G be a (finite or infinite) λ-graph with k types of edges and chromatic number at most Λ. For any k-tuple of non-negative reals $x_1, \ldots, x_k$, there exists a finitary orientation of G with weight $\lambda_G(x_1, \ldots, x_k)$ with maximum length of a directed path at most $D_{k,\Lambda}$.*

Proof: By symmetry, we can assume that $x_1 \ge \cdots \ge x_k$ (otherwise, permute the types of the edges of G). If $x_k > 0$, the statement of the lemma follows directly from Lemma 7. In the rest, we deal with the case when $x_{k'} > 0$ and $x_{k'+1} = \cdots = x_k = 0$.

Fix a coloring μ of G with Λ colors $1, \ldots, \mu$. Let G' be the subgraph of G formed by $x_{\le k'}$-edges. By Lemma 7, there exists a finitary orientation $\vec{G}'$ of G' with weight $\lambda_{G'}(x_1, \ldots, x_{k'}) = \lambda_G(x_1, \ldots, x_k)$ and with maximum path length at most $D_{k',\Lambda}$. Let $c(v)$ be the labeling of G' corresponding to $\vec{G}'$. Observe that $c(v) \in \Gamma_{D_{k',\Lambda}}(x_1, \ldots, x_{k'})$ for every vertex v of G.

We extend $\vec{G}'$ to a finitary orientation $\vec{G}$ of G. An $x_{>k'}$-edge uv is directed from u to v if $c(u) < c(v)$, from v to u if $c(u) > c(v)$. If $c(u) = c(v)$, then the edge uv is directed from u to v if $\mu(u) < \mu(v)$, and from u to v, otherwise. Clearly, the weight of $\vec{G}$ is the same as the weight of $\vec{G}'$.

Let P be a directed path in $\vec{G}$. Clearly, the labels $c(v)$ of vertices v does not decrease along the path P. Moreover, each subpath of P formed by vertices v with the same label has length at most $\Lambda - 1$ as the labels $\mu(v)$ of the vertices comprising the subpath strictly increase. Since all the labels $c(v)$ are from the set $\Gamma_{D_{k',\Lambda}}(x_1, \ldots, x_{k'})$, P contains at most $K_{k',\Lambda}$ such subpaths. Hence, the length of a directed P in $\vec{G}$ does not exceed $K_{k',\Lambda} \cdot \Lambda$. Therefore, the maximum length of a directed path in $\vec{G}$ is at most $K_{k',\Lambda} \cdot \Lambda \leq D_{k,\Lambda}$.

■

4 Main Result

In this section, we prove our main result on the structure of the λ-functions of λ-graphs. Before doing so, we introduce several definitions. $\mathcal{F}_{k,\Lambda}$ denotes the set of all linear functions of k variables with integer coefficients between 0 and $D_{k,\Lambda}$, i.e.,

$$\mathcal{F}_{k,\Lambda} = \left\{ \sum_{i=0}^{k} \alpha_i x_i, 0 \leq \alpha_i \leq D_{k,\Lambda} \right\} .$$

Next, $\mathcal{F}^{\max}_{k,\Lambda}$ is the set of all functions φ that are equal to the maximum of some of the functions from $\mathcal{F}_{k,\Lambda}$, i.e.,

$$\mathcal{F}^{\max}_{k,\Lambda} = \{\varphi(x_1, \ldots, x_p) = \max_{f \in F} f(x_1, \ldots, x_p) \text{ for } F \subseteq \mathcal{F}_{k,\Lambda}, F \neq \emptyset\} .$$

Finally, $\mathcal{F}^{\mathrm{minmax}}_{k,\Lambda}$ is the set of all functions that are equal to the minimum of some of the functions from $\mathcal{F}^{\max}_{k,\Lambda}$, i.e.,

$$\mathcal{F}^{\mathrm{minmax}}_{k,\Lambda} = \{\varphi(x_1, \ldots, x_p) = \max_{f \in F} f(x_1, \ldots, x_p) \text{ for } F \subseteq \mathcal{F}^{\max}_{k,\Lambda}, F \neq \emptyset\} .$$

Observe $\mathcal{F}_{k,\Lambda} \subseteq \mathcal{F}^{\max}_{k,\Lambda} \subseteq \mathcal{F}^{\mathrm{minmax}}_{k,\Lambda}$. Clearly, all the three sets $\mathcal{F}_{k,\Lambda}$, $\mathcal{F}^{\max}_{k,\Lambda}$ and $\mathcal{F}^{\mathrm{minmax}}_{k,\Lambda}$ are finite. Therefore, the subset of $\mathbb{R}^k$ formed by k-tuples of non-negative reals can be partitioned into finitely many closed polyhedral cones (with the tips at the origin of $\mathbb{R}^k$) such that every function contained in $\mathcal{F}^{\mathrm{minmax}}_{k,\Lambda}$ is linear on each of the cones. Let $C_{k,\Lambda}$ be the number of such cones.

We now state and prove the main result of the paper (note that both the numbers $C_{k,\Lambda}$ and $D_{k,\Lambda}$ just depend on k and Λ):

Theorem 9 *For every λ-graph G with k types of edges and chromatic number at most Λ, $\lambda_G(x_1,\ldots,x_k)$ is a piecewise linear function of $x_1,\ldots,x_k$ with at most $C_{k,\Lambda}$ linear parts formed by linear function with integer coefficients between 0 and $D_{k,\Lambda}$. Moreover, the subset of $\mathbb{R}^k$ formed by non-negative integers can be partitioned into at most $C_{k,\Lambda}$ (closed) polyhedral cones such that for each of the cones the following holds: there exist integers $\alpha_i(v)$ between 0 and $D_{k,\Lambda}$ such that the labeling c, $c(v)=\sum_{i=1}^k \alpha_i(v)x_i$, is a proper labeling of G with respect to $x_1,\ldots,x_k$ and the span of c is $\lambda_G(x_1,\ldots,x_k)$.*

Proof: Let $\mathcal{D}$ be the set of all finitary orientations of G with maximum length of a directed path at most $D_{k,\Lambda}$. For an orientation $\vec{G}\in\mathcal{D}$, let $F(\vec{G})$ be the set of all the functions $\sum_{i=1}^k \alpha_i x_i$ such that $\vec{G}$ contains a directed path with precisely α_i x_i-edges. Since the maximum length of a directed path in $\vec{G}$ does not exceed $D_{k,\Lambda}$, the set $F(\vec{G})$ is a subset of $\mathcal{F}_{k,\Lambda}$, i.e., $F(\vec{G})\subseteq\mathcal{F}_{k,\Lambda}$. By the definition, the weight of the orientation $\vec{G}$ with respect to $x_1,\ldots,x_k$ is the following:

$$w_{\vec{G}}(x_1,\ldots,x_k)=\max_{f\in F(\vec{G})} f(x_1,\ldots,x_k)\ .$$

Let W be the set of all the functions $w_{\vec{G}}(x_1,\ldots,x_k)$ where $\vec{G}$ ranges through all the orientations contained in $\mathcal{D}$. Clearly, $W\subseteq\mathcal{F}^{\max}_{k,\Lambda}$. For $w\in W$, let $\vec{G}_w$ be one of the orientations in $\mathcal{D}$ with $w_{\vec{G}}=w$. By Theorem 1 and Lemma 8, the following equality holds:

$$\lambda_G(x_1,\ldots,x_k)=\min_{\vec{G}\in\mathcal{D}} w_{\vec{G}}(x_1,\ldots,x_k)=\min_{w\in W} w(x_1,\ldots,x_k)\ .$$

Similarly as before, we have $\lambda_G(x_1,\ldots,x_k)\in\mathcal{F}^{\mathrm{minmax}}_{k,\Lambda}$.

Consider the partition of k-tuples of non-negative reals into $C_{k,\Lambda}$ polyhedral cones such that every function of $\mathcal{F}^{\mathrm{minmax}}_{k,\Lambda}$ is linear on each of the cones. In particular, $\lambda_G(x_1,\ldots,x_k)\in\mathcal{F}^{\mathrm{minmax}}_{k,\Lambda}$ is linear on each of the cones. Fix one such cone and let $w\in W$ be a function such that $\lambda_G(x_1,\ldots,x_k)=w(x_1,\ldots,x_k)$ on the fixed cone. Let c be the labeling corresponding to the orientation $\vec{G}_w$. Since no directed path of $\vec{G}_w\in\mathcal{D}$ has length more than $D_{k,\Lambda}$, $c(v)$ for a single vertex v when viewed as a function of $x_1,\ldots,x_k$ belongs to $\mathcal{F}^{\max}_{k,\Lambda}$. In particular, every function $c(v)$ is linear on the fixed cone. Therefore, the considered polyhedral cones form a possible partition of $\mathbb{R}^k$ with the properties from the statement of the theorem. The bounds on the integers coefficients of linear functions readily follows from the definitions of $\mathcal{F}_{k,\Lambda}$, $\mathcal{F}_{k,\Lambda}^{\max}$ and $\mathcal{F}_{k,\Lambda}^{\mathrm{minmax}}$.

■

Since only the functions from $\mathcal{F}_{k,\Lambda}^{\mathrm{minmax}}$ could be the λ-function of a λ-graph with k types of edges and with chromatic number at most Λ, we have the following:

Corollary 10 *There exists only finitely many piecewise linear functions that could be the λ-function of a λ-graph with k types of edges and with chromatic number at most Λ.*

Another immediate corollary is the following somewhat surprising statement:

Corollary 11 *Let $x_1, \ldots, x_k$ be a fixed k-tuple of positive reals and let γ be a non-negative real. There exist only finitely many different k-parameter λ-functions λ_G such that $\lambda_G(x_1, \ldots, x_k) = \gamma$.*

Proof: If G is a λ-graph with k types of edges such that $\lambda_G(x_1, \ldots, x_k) = \gamma$, then the chromatic number of G does not exceed $\frac{\lambda_G(x_1,\ldots,x_k)}{\min\{x_1,\ldots,x_k\}} + 1$ by the scaling property. By Corollary 10, λ-graphs with k types of edges with bounded chromatic number have only finitely many different λ-functions.

■

Note that Corollary 11 includes the result of [2] that the number of λ-functions with prescribed boundary values is finite.

5 Labelings with Distance Constraints

In this section, we infer from Theorem 9 Piecewise Linearity Conjecture, Coefficient Bound Conjecture and Delta Bound Conjecture stated in [8]. First, let us state the following simple proposition that can be found in [8] (note that its proof employs the Axiom of Choice):

Proposition 12 *If G is a graph of maximum degree Δ and k is a positive integer, then the chromatic number of the k-th power of G does not exceed $\Delta^k + 1$.*

We can now state the theorem from that the three conjectures mentioned above readily follow:

Theorem 13 *For every integers Δ and k, there exist constants $C'_{k,\Delta}$ and $D'_{k,\Delta}$ with the following property: for any graph G with maximum degree Δ, there exist finitely many hyperplanes in $\mathbb{R}^k$ through the origin such that the function $\lambda_G(p_1, \ldots, p_k)$ is linear in each of at most $C'_{k,\Delta}$ convex polyhedral cones that are determined by the hyperplanes. Moreover, for each of the cones the following is true: there exist integers $\alpha_i(v)$ between 0 and $D_{k,\Lambda}$ such that the labeling c, $c(v) = \sum_{i=1}^{k} \alpha_i(v)p_i$, is a proper $L(p_1, \ldots, p_k)$-labeling of G with respect to $x_1, \ldots, x_k$ and the span of c is $\lambda_G(p_1, \ldots, p_k)$.*

Proof: Set $C'_{k,\Delta} = C_{k,\Delta^k+1}$ and $D'_{k,\Delta} = D_{k,\Delta^k+1}$. Let G be a graph with maximum degree Δ and form the λ-graph $G^{(k)}$ as described in Section 1. By Proposition 12, the chromatic number of $G^{(k)}$ does not exceed $\Delta^k + 1$. Theorem 13 now follows from Theorem 9.

■

An immediate corollary of Theorem 13 (alternatively, of Corollary 10) is the following:

Corollary 14 *For every integers Δ and k, the set $\Lambda_{k,\Delta}$, that consists of all (piecewise linear) functions $\lambda_G(p_1, \ldots, p_k)$ where G is a finite or infinite graph of maximum degree Δ, is finite.*

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