

Channel assignment problem with variable weights

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Abstract

A λ -graph G is a (finite or infinite) graph with k types of edges, x_1 -edges, \dots , x_k -edges. A labeling c of the vertices of G by non-negative reals is proper with respect to reals x_1, \dots, x_k , if the labels of the end-vertices of an x_i -edge differ by at least x_i . The span of the labeling c is the supremum of the labels used by c . The λ -function $\lambda_G(x_1, \dots, x_k)$ is the infimum of the spans of all the proper labelings with respect to x_1, \dots, x_k .

We show that the λ -function of any graph G is piecewise linear in x_1, \dots, x_k with finitely many linear parts. Moreover, we show that for every integers k and Λ , there exist constants $C_{k,\Lambda}$ and $D_{k,\Lambda}$ such that the λ -function of every λ -graph G with k types of edges and chromatic number at most Λ is comprised of at most $C_{k,\Lambda}$ linear parts, and the coefficients of x_1, \dots, x_k of the linear functions comprising $\lambda_G(x_1, \dots, x_k)$ are integers between 0 and $D_{k,\Lambda}$. Among others, our results yield proofs of Piecewise Linearity Conjecture, Coefficient Bound Conjecture and Delta Bound Conjecture of Griggs and Jin.

1 Introduction

Radio frequency problems can be expressed as various graph labeling problems [12, 18]. A prominent role among such graph labeling problems plays

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the notion of $L(p_1, \dots, p_k)$ -labelings, graph labelings with distance constraints. Several approaches to study the dependence of the span of optimum labelings on the parameters p_1, \dots, p_k have recently been proposed: an approach based on real-value relaxation of $L(p_1, \dots, p_k)$ -labelings can be found in the work of Griggs and Jin [8, 9, 10], another approach based on the notion of λ -graphs can be found in [2]. In the present paper, we generalize the notion of λ -graphs introduced in [2] from $k = 2$ to arbitrary k and provide structural results for the general model. The obtained results yield proofs of Piecewise Linearity Conjecture, Coefficient Bound Conjecture and Delta Bound Conjecture of Griggs and Jin stated in [8].

A labeling c of the vertices of a (finite or infinite) graph G by non-negative integers is an $L(p_1, \dots, p_k)$ -labeling for positive integers p_1, \dots, p_k , if the labels of any two vertices u and v at distance (exactly) i differ by at least p_i . Let us remark here that all graphs as well as λ -graphs considered in this paper can be finite or infinite unless stated otherwise. The maximum label used by c is said to be the *span* of c and the least span of an $L(p_1, \dots, p_k)$ -labeling of a graph G is denoted by $\lambda_G(p_1, \dots, p_k)$ (we deviate from the standard notation in order to emphasize the dependence on the parameters p_1, \dots, p_k). There is an enormous amount of literature on algorithms for $L(p_1, \dots, p_k)$ -labelings of graphs [1, 3, 6, 7, 15, 19]. From the structural point of view, the attention of researchers focused mainly on the case of $L(2, 1)$ -labelings, partly because of the following conjecture of Griggs and Yeh [11]:

Conjecture 1 (Δ^2 Conjecture) *If G is a finite graph of maximum degree Δ , then $\lambda_G(2, 1) \leq \Delta^2$.*

Conjecture 1 was verified for several special classes of graphs, including graphs of maximum degree two, chordal graphs [20], see also [4, 16], and hamiltonian cubic graphs [13, 14]. In the general case, the original bound $\lambda_G(2, 1) \leq \Delta^2 + 2\Delta$ from [11] has been improved to $\lambda_{2,1}(G) \leq \Delta^2 + \Delta$ in [5] and a recent more general result of the author and Škrekovski [17] yields the present record $\lambda_{2,1}(G) \leq \Delta^2 + \Delta - 1$.

In order to capture the dependence of the optimum spans on the parameters, Griggs and Jin [8] allowed both the parameters p_1, \dots, p_k and the labels used by a labeling c to be any non-negative reals. Similarly to the original notion, they define the *span* of a labeling c as the supremum of the labels used by c , and $\lambda_G(p_1, \dots, p_k)$ denotes the span of an optimum labeling of a graph G , i.e., the minimum (that is always attained if $\lambda_G(p_1, \dots, p_k)$ is finite) of the spans of all $L(p_1, \dots, p_k)$ -labelings of G . In this setting, Griggs

and Jin [8] prove that for any reals p_1, \dots, p_k , the value of $\lambda_G(p_1, \dots, p_k)$ can be expressed as $\sum_{i=1}^k \alpha_i p_i$ for some non-negative integers α_i . Moreover, if all p_1, \dots, p_k are integers, the values $\lambda_G(p_1, \dots, p_k)$ in the original and the relaxed settings coincide. They also show that the function $\lambda_G(p_1, \dots, p_k)$ is a continuous function piecewise linear in the parameters p_1, \dots, p_k , and conjecture the following [8]:

Conjecture 2 (Piecewise Linearity Conjecture) *For any graph G , the graph of the function $\lambda_G(p_1, \dots, p_k)$ is comprised by finitely many linear parts, i.e., there exist finitely many hyperplanes in \mathbb{R}^k through the origin such that the function $\lambda_G(p_1, \dots, p_k)$ is linear in each of the convex polyhedral cones (formed by non-negative reals) that are determined by the hyperplanes.*

Conjecture 3 (Coefficient Bound Conjecture) *For every graph G and every integer k , there exists a constant $D_{k,G}$ such that the following holds for all reals p_1, \dots, p_k : the value of the function $\lambda_G(p_1, \dots, p_k)$ is equal to $\sum_{i=1}^k \alpha_i p_i$ for some integer coefficients $\alpha_1, \dots, \alpha_k$ between 0 and $D_{k,G}$ (the integers $\alpha_1, \dots, \alpha_k$ may depend on p_1, \dots, p_k). Moreover, there is a labeling c with span $\lambda_G(p_1, \dots, p_k)$ such that $c(v) = \sum_{i=1}^k \alpha_i(v) p_i$ where $\alpha_1(v), \dots, \alpha_k(v)$ are integers between 0 and $D_{k,G}$.*

Conjecture 4 (Delta Bound Conjecture) *For every integers Δ and k , there exists a constant $D_{k,\Delta}$ such that, for every graph G with maximum degree at most Δ and every reals p_1, \dots, p_k the following holds: the value of the function $\lambda_G(p_1, \dots, p_k)$ is equal to $\sum_{i=1}^k \alpha_i p_i$ for some integer coefficients $\alpha_1, \dots, \alpha_k$ between 0 and $D_{k,\Delta}$ (the integers $\alpha_1, \dots, \alpha_k$ may depend on p_1, \dots, p_k). Moreover, there is a labeling c with span $\lambda_G(p_1, \dots, p_k)$ such that $c(v) = \sum_{i=1}^k \alpha_i(v) p_i$ where $\alpha_1(v), \dots, \alpha_k(v)$ are integers between 0 and $D_{k,\Delta}$.*

Note that Delta Bound Conjecture implies Coefficient Bound Conjecture. Griggs and Jin [8] proved all the three Conjectures 2, 3 and 4 for $k = 2$ (for $k = 1$, the conjectures are trivial) and Conjecture 2 also for finite graphs G .

In the present paper, we consider the problems posed in [8] in the more general setting of λ -graphs that was introduced for $k = 2$ in [2]. A λ -graph G with k types of edges is a graph G whose edges are labeled by variables x_1, \dots, x_k . An edge labeled by a variable x_i is called x_i -edge. Two vertices of G may be joined by edges of several types. A proper labeling c of G with respect to the real numbers x_1, \dots, x_k is a labeling of the vertices of G by

non-negative reals such that the labels of the end-vertices of an x_i -edge uv differ by at least x_i , i.e., $|c(u) - c(v)| \geq x_i$. The *span* of the labeling c is the supremum of the labels used by c and $\lambda_G(x_1, \dots, x_k)$ is defined to be the infimum of the spans of all proper labelings with respect to x_1, \dots, x_k . The results of [2] yield that for every reals x_1, \dots, x_k , if $\lambda_G(x_1, \dots, x_k)$ is finite, then there exists a proper labeling c with span $\lambda_G(x_1, \dots, x_k)$ and the span of c is equal to the maximum label used by c , i.e., both the infimum and the supremum in the definitions are attained. The λ -*function* of a λ -graph G is $\lambda_G(x_1, \dots, x_k)$ viewed as a function of variables x_1, \dots, x_k . The *chromatic number* of a λ -graph G is the chromatic number of the underlying graph, i.e., $\lambda_G(1, \dots, 1) + 1$.

$L(p_1, \dots, p_k)$ -labelings of graphs can be modeled as λ -graphs as follows: if G is a graph, form a λ -graph $G^{(k)}$ with the vertex set $V(G)$ such that two vertices u and v are joined by an x_i -edge in $G^{(k)}$, $i = 1, \dots, k$, if their distance in G is exactly i . Clearly, the optimum span $\lambda_G(p_1, \dots, p_k)$ is equal to the value $\lambda_{G^{(k)}}(x_1, \dots, x_k)$ of the λ -function of $G^{(k)}$ for $x_i = p_i$, $i = 1, \dots, k$. Because of this close relation, we decided to use the notation $\lambda_G(\dots)$ both for the spans of optimum $L(p_1, \dots, p_k)$ -labelings and the λ -functions of λ -graphs. Since it is always clear throughout the paper whether G is a graph (in which case, $\lambda_G(p_1, \dots, p_k)$ stands for the span of an optimum $L(p_1, \dots, p_k)$ -labeling) or a λ -graph (in which case, λ_G stands for the λ -function of G), the confusion of the notations is avoided.

Similarly as in the case of $L(p_1, \dots, p_k)$ -labeling [8], λ -functions of λ -graphs have the *scaling property*, i.e., for every non-negative reals x_1, \dots, x_k and β , the following holds: $\lambda_G(\beta x_1, \dots, \beta x_k) = \beta \lambda_G(x_1, \dots, x_k)$. Therefore, the λ -function of any λ -graph is linear on every ray through the origin in \mathbb{R}^k . In Section 4, we show that the λ -function λ_G of any λ -graph G is comprised of finitely many linear parts, i.e., the subset of \mathbb{R}^k formed by non-negative reals can be partitioned into finitely many (infinite) polyhedral cones (with the tips at the origin of \mathbb{R}^k) such that λ_G is linear on each of these cones.

Our main result is Theorem 9 that asserts the existence of the constants $C_{k,\Lambda}$ and $D_{k,\Lambda}$ such that the λ -function of any λ -graph G with k types of edges and chromatic number at most Λ is comprised of at most $C_{k,\Lambda}$ linear parts and the coefficients of x_1, \dots, x_k of the linear functions comprising the λ -function are integers between 0 and $D_{k,\Lambda}$. In this paper, we solely focus on proving the existence of the constants $C_{k,\Lambda}$ and $D_{k,\Lambda}$ without attempting to optimize their growth. Let us remark that the existence of the constants $C_{k,\Lambda}$ and $D_{k,\Lambda}$ for $k = 2$ follows from the results of [2]. However, the technique used in [2] does not seem to generalize to $k > 2$. As demon-

strated in Section 5, our main result yields the proofs of Piecewise Linearity Conjecture, Coefficient Bound Conjecture and Delta Bound Conjecture for $L(p_1, \dots, p_k)$ -labelings (Conjectures 2, 3 and 4).

2 Preliminaries

In [2], an analogue of Gallai-Roy Theorem for infinite graphs with edges of different weights was proved. We will not state the theorem in its full generality but just in the form restricted to λ -graphs. An orientation of an infinite graph G is said to be *finitary* if it does not contain a directed walk of arbitrary length. In particular, a finitary orientation of G is acyclic. A *weight* of a finite directed path P in an orientation of a λ -graph G with respect to x_1, \dots, x_k is the sum of the variables assigned to its edges, i.e., $\sum_{i=1}^k \alpha_i x_i$ if P contains α_i x_i -edges. The *weight of a finitary orientation* \vec{G} of a λ -graph G is the maximum weight of a directed path in \vec{G} (note that the maximum is always attained since the lengths of directed paths in \vec{G} are bounded in a finitary orientation and there are only finitely many different types of edges in G). We now state the version of Gallai-Roy Theorem for λ -graphs:

Theorem 1 *Let G be a λ -graph with k types of edges. For any real numbers x_1, \dots, x_k , $\lambda_G(x_1, \dots, x_k)$ is equal to the minimum weight of a finitary orientation \vec{G} of G (in particular, there exists a finitary orientation of weight $\lambda_G(x_1, \dots, x_k)$).*

If \vec{G} is a finitary orientation of a λ -graph G , then the labeling c , where $c(v)$ is the maximum weight of a directed path ending at a vertex v , is a proper labeling of G with respect to x_1, \dots, x_k . We say that the labeling c , defined in this way, *corresponds* to the orientation \vec{G} . Clearly, the span of the labeling corresponding to \vec{G} is the weight of \vec{G} . On the other hand, for a proper labeling c of G for positive reals x_1, \dots, x_k , whose span is finite, one may define a (finitary) orientation \vec{G} of G such that an edge uv is directed from u to v if $c(u) < c(v)$. Such orientation \vec{G} *corresponds* to the labeling c . Observe that the weight of the orientation corresponding to a proper labeling c is at most the span of c (in general, it can be strictly smaller).

If G is a λ -graph with k types of edges, we say that an edge uv is an $x_{\leq \ell}$ -edge if uv is an x_i -edge where $i \leq \ell$. The set of all $x_{\leq \ell}$ -edges of G is the set of all x_i -edges with $i = 1, \dots, \ell$. Similarly, we use the terms $x_{< \ell}$ -edges,

$x_{\geq \ell}$ -edges, etc. We demonstrate this notation in the next auxiliary lemma that will be used later:

Lemma 2 *Let G be a λ -graph with k types of edges and with chromatic number at most Λ , and let $0 \leq \ell < k$. If there exist an integer D and a finitary orientation \vec{G} of G such that every directed path in \vec{G} contains at most D $x_{\leq \ell}$ -edges, then:*

$$\lambda_G(x_1, \dots, x_k) \leq d_{\max} + (\ell + 1)^D \Lambda \cdot \max\{x_{\ell+1}, \dots, x_k\}$$

where d_{\max} is the maximum sum of weights of $x_{\leq \ell}$ -edges on a directed path in \vec{G} , i.e., d_{\max} would be the weight of \vec{G} if the parameters $x_{\ell+1}, \dots, x_k$ were equal to zero.

In particular, it holds that $\lambda_G(x_1, \dots, x_k) \leq \Lambda \cdot \max\{x_1, \dots, x_k\}$.

Proof: Fix a finitary orientation \vec{G} that has the properties described in the statement of the lemma (if $\ell = 0$, fix any finitary orientation \vec{G} of G). Let $d(v)$ be the maximum sum of the weights of $x_{\leq \ell}$ -edges on a directed path in \vec{G} ending at a vertex v . Clearly, $d_{\max} = \max_{v \in V(G)} d(v)$. Let D be the set of all different values of $d(v)$ and let $\delta(v)$ be the number of the elements of D smaller than $d(v)$. Since every directed path in \vec{G} contains at most D $x_{\leq \ell}$ -edges, it holds that $|D| \leq (\ell + 1)^D$, and thus $0 \leq \delta(v) < |D| \leq (\ell + 1)^D$ for every vertex v of G . Finally, let μ be a coloring of the vertices of G with colors $1, \dots, \Lambda$.

Let us define a labeling c' of the vertices of G as follows:

$$c'(v) = d(v) + (\delta(v)\Lambda + \mu(v)) \cdot \max\{x_{\ell+1}, \dots, x_k\}.$$

Since $\delta(v) < |D|$ for every vertex v of G , the span of c' does not exceed:

$$d_{\max} + |D|\Lambda \cdot \max\{x_{\ell+1}, \dots, x_k\} \leq d_{\max} + (\ell + 1)^D \Lambda \cdot \max\{x_{\ell+1}, \dots, x_k\}.$$

In the rest, we show that c' is a proper labeling with respect to x_1, \dots, x_k .

Consider an x_i -edge uv of G . By symmetry, we may assume that the edge uv is directed from u to v in \vec{G} . In particular, it holds that $d(u) \leq d(v)$ and $\delta(u) \leq \delta(v)$. We distinguish two major cases: the first one is $i \leq \ell$. In this case, $d(u) + x_i \leq d(v)$ and thus $\delta(u) < \delta(v)$. We can immediately conclude:

$$\begin{aligned} c'(v) - c'(u) &= d(v) - d(u) + ((\delta(v) - \delta(u))\Lambda + \mu(v) - \mu(u)) \cdot \max\{x_{\ell+1}, \dots, x_k\} \\ &\geq d(v) - d(u) + (\Lambda + \mu(v) - \mu(u)) \cdot \max\{x_{\ell+1}, \dots, x_k\} \\ &\geq d(v) - d(u) \geq x_i. \end{aligned}$$

Therefore, the edge uv is properly colored in the first case.

The other case is that $i > \ell$. If $d(u) = d(v)$, then $\delta(u) = \delta(v)$ and the following holds (similarly to the first case):

$$|c'(u) - c'(v)| = |\mu(u) - \mu(v)| \cdot \max\{x_{\ell+1}, \dots, x_k\} \geq x_i .$$

If $d(u) < d(v)$, then $\delta(u) < \delta(v)$ and we have the following:

$$\begin{aligned} c'(v) - c'(u) &= d(v) - d(u) + ((\delta(v) - \delta(u))\Lambda + \mu(v) - \mu(u)) \cdot \max\{x_{\ell+1}, \dots, x_k\} \\ &\geq (\Lambda + \mu(v) - \mu(u)) \cdot \max\{x_{\ell+1}, \dots, x_k\} \geq x_i . \end{aligned}$$

Hence, the labels of u and v also differ by at least x_i in the second case. ■

3 Orientations with Minimum Weight

In this section, we construct orientations of λ -graphs with minimum weight such that the maximum length of a directed path in the constructed orientation is bounded. First, let us define numbers $D_{i,\Lambda}$ and $K_{i,\Lambda}$ for integer Λ and i as follows:

$$\begin{aligned} D_{1,\Lambda} &= \Lambda \\ K_{i,\Lambda} &= (i+1)^{D_{i,\Lambda}} \\ D_{i+1,\Lambda} &= (2K_{i,\Lambda})^{K_{i,\Lambda}^2+3} \cdot \Lambda \end{aligned}$$

Next, we state several propositions that can be verified directly from the definitions of $D_{i,\Lambda}$ and $K_{i,\Lambda}$. Their proofs are left to the reader.

Proposition 3 *For integers $\Lambda \geq 2$ and i , the number of multisets that consist of at most $D_{i,\Lambda}$ numbers $1, \dots, i$ does not exceed $K_{i,\Lambda} - 1$.*

Proposition 4 *The following holds for every integers Λ and i :*

$$D_{i+1,\Lambda} \geq (2K_{i,\Lambda})^{K_{i,\Lambda}^2+2} \cdot \Lambda + K_{i,\Lambda} \cdot \Lambda .$$

We now introduce some notations used in the proof the main lemma of this section (Lemma 7). For an integer M and positive reals x_1, \dots, x_k ,

$\Gamma_M(x_1, \dots, x_k)$ denotes the set of all combinations of x_1, \dots, x_k with non-negative integer coefficients whose sum does not exceed M , i.e.:

$$\Gamma_M(x_1, \dots, x_k) = \left\{ \sum_{j=1}^k \alpha_j x_j \mid 0 \leq \alpha_1, \dots, \alpha_k \text{ \& } \sum_{j=1}^k \alpha_j \leq M \right\} .$$

The set $\Gamma'_M(x_1, \dots, x_k)$ is then defined to be the set of all non-negative reals that can be expressed as a difference of two numbers from $\Gamma_M(x_1, \dots, x_k)$, i.e.:

$$\Gamma'_M(x_1, \dots, x_k) = \{ \alpha - \beta \mid \alpha, \beta \in \Gamma_M(x_1, \dots, x_k) \text{ \& } \alpha - \beta \geq 0 \} .$$

Since $0 \in \Gamma_M(x_1, \dots, x_k)$, the set $\Gamma_M(x_1, \dots, x_k)$ is contained in the set $\Gamma'_M(x_1, \dots, x_k)$. The following estimates on the sizes of $\Gamma_M(x_1, \dots, x_k)$ and $\Gamma'_M(x_1, \dots, x_k)$ directly follow from Proposition 3:

Proposition 5 *Let x_1, \dots, x_k be any positive real numbers and let $\Lambda \geq 2$ be a positive integer. The following two estimates hold:*

$$\begin{aligned} |\Gamma_{D_{k,\Lambda}}(x_1, \dots, x_k)| &< K_{k,\Lambda} \\ |\Gamma'_{D_{k,\Lambda}}(x_1, \dots, x_k)| &< K_{k,\Lambda}^2 \end{aligned}$$

We now establish an auxiliary lemma that will be extremely useful in the proof of Lemma 7:

Lemma 6 *Let x_1, \dots, x_k , $x_1 \geq \dots \geq x_k > 0$, be real numbers, $\Lambda \geq 2$ a positive integer, and y another positive real number. There exists a real number t ,*

$$K_{k,\Lambda} \Lambda y \leq t \leq (2K_{k,\Lambda})^{K_{k,\Lambda}^2} \Lambda y$$

such that the set $\Gamma'_{D_{k,\Lambda}}(x_1, \dots, x_k)$ contains no element strictly between t and $K_{k,\Lambda}(t + \Lambda y)$. In particular, the real t has the following property (\star): if $\gamma \in \Gamma'_{D_{k,\Lambda}}(x_1, \dots, x_k)$ and $\gamma > t$, then $\gamma \geq K_{k,\Lambda}(t + \Lambda y)$.

Proof: By Proposition 5, the set $\Gamma'_{D_{k,\Lambda}}(x_1, \dots, x_k)$ contains less than $K_{k,\Lambda}^2$ real numbers. Let us define reals t_j , $j = 1, \dots, K_{k,\Lambda}^2$, as follows:

$$t_j = (2K_{k,\Lambda})^j \Lambda y .$$

Since $K_{k,\Lambda}(t_j + \Lambda y) \leq 2K_{k,\Lambda}t_j = t_{j+1}$ for all $j = 1, \dots, K_{k,\Lambda}^2 - 1$, all the open intervals I_j ,

$$I_j = (t_j, K_{k,\Lambda}(t_j + \Lambda y)), j = 1, \dots, K_{k,\Lambda}^2,$$

are disjoint. Since all the $K_{k,\Lambda}^2$ intervals I_j are disjoint and $\Gamma'_{D_{k,\Lambda}}(x_1, \dots, x_k) < K_{k,\Lambda}^2$, there exists j_0 such that no element of $\Gamma'_{D_{k,\Lambda}}(x_1, \dots, x_k)$ is contained in I_{j_0} . The number t_{j_0} is the desired number t . ■

We state and prove the key lemma of this section:

Lemma 7 *Let G be a (finite or infinite) λ -graph G with k types of edges and with chromatic number at most Λ . Fix real numbers x_1, \dots, x_k , $x_1 \geq \dots \geq x_k > 0$. For each $\ell = 1, \dots, k$, there exists a finitary orientation \vec{G} of G of weight $\lambda_G(x_1, \dots, x_k)$ such that every directed path in \vec{G} contains at most $D_{\ell,\Lambda} x_{\leq \ell}$ -edges.*

Proof: If $\Lambda = 1$, there is nothing to prove since G contains no edges and the statement of the lemma holds vacuously. Therefore, we assume $\Lambda \geq 2$ in the remaining. For the rest of the proof, let us fix a proper coloring μ (in the usual sense) of the vertices of G with colors $1, \dots, \Lambda$.

The proof of the lemma proceeds by induction on the number ℓ . First, we have to deal with the case $\ell = 1$. Consider any finitary orientation \vec{G} of G of weight $\lambda_G(x_1, \dots, x_k)$. Such an orientation exists by Theorem 1. By Lemma 2, it holds $\lambda_G(x_1, \dots, x_k) \leq \Lambda x_1$. Since the weight of any directed path in \vec{G} does not exceed $\lambda_G(x_1, \dots, x_k) \leq \Lambda x_1$, every directed path in \vec{G} contains at most $D_{1,\Lambda} = \Lambda$ x_1 -edges.

We now deal with the case $\ell > 1$. By the induction, there exists a finitary orientation \vec{G} of G of weight $\lambda_G(x_1, \dots, x_k)$ such that any directed path contains at most $D_{\ell-1,\Lambda} x_{\leq \ell-1}$ -edges. Let $c(v)$ be the labeling corresponding to \vec{G} , and let $d(v)$ be the maximum sum of weights of $x_{\leq \ell-1}$ -edges on a directed path in \vec{G} ending at a vertex v . Clearly, $d(v) \leq c(v)$ for every vertex v of G . Finally, let $\delta(v)$ be the number of the elements of $\Gamma_{D_{\ell-1,\Lambda}}$ smaller or equal to $d(v)$. Since $|\Gamma_{D_{\ell-1,\Lambda}}| < K_{\ell-1,\Lambda}$ by Proposition 5, $1 \leq \delta(v) \leq K_{\ell-1,\Lambda} - 1$ for every vertex v of G .

By Lemma 6 (applied for $y = x_\ell$ and $k = \ell - 1$), there exists a real number t ,

$$K_{k,\Lambda} \Lambda x_\ell \leq t \leq (2K_{k,\Lambda})^{K_{\ell-1,\Lambda}^2} \Lambda x_\ell,$$

such that the set $\Gamma'_{D_{\ell-1,\Lambda}}(x_1, \dots, x_{\ell-1})$ contains no element strictly between t and $K_{\ell-1,\Lambda}(t + \Lambda x_\ell)$, i.e., t has the property (\star) from Lemma 6. We define a new labeling c' and show that it is a proper labeling with respect to x_1, \dots, x_k :

1. If $c(v) - d(v) \leq (K_{\ell-1,\Lambda} - \delta(v))t$, then $c'(v) = c(v)$.
2. Otherwise, $c'(v) = d(v) + (K_{\ell-1,\Lambda} - 1)t + \delta(v)\Lambda x_\ell + \mu(v)x_\ell$.

Fix an x_i -edge uv of G . In order to verify that c' is a proper labeling on the edge uv , we distinguish five major cases:

- **Both the labels $c'(u)$ and $c'(v)$ are defined by the first rule.**
Since $c'(u) = c(u)$ and $c'(v) = c(v)$, we have $|c'(u) - c'(v)| = |c(u) - c(v)| \geq x_i$.
- **The label $c'(u)$ is defined by the first rule, the label $c'(v)$ is defined by the second rule and $i < \ell$.**

We distinguish two cases according to the orientation of the edge uv in \vec{G} . If the edge is directed from u to v , we have $d(u) + x_i \leq d(v)$. Because the label of u is defined by the first rule, the label $c'(u) = c(u)$ is at most $d(u) + (K_{\ell-1,\Lambda} - 1)t$. On the other hand, the label $c'(v)$ is larger than $d(v) + (K_{\ell-1,\Lambda} - 1)t$. We infer that $c'(v) - c'(u) \geq d(v) - d(u) \geq x_i$.

The other case is that the edge uv is directed from v to u . In particular, $d(v) + x_i \leq d(u)$, $\delta(v) < \delta(u)$ and $c(v) \leq c(u)$. First, we show that $d(u) - d(v) - x_i > t$. Assume for contrary that $d(u) - d(v) - x_i \leq t$. Since c is a proper labeling of G , $c(v) \leq c(u) - x_i$. Since the label to u was defined by the first rule, we have $c(u) \leq d(u) + (K_{\ell-1,\Lambda} - \delta(u))t$. Therefore, the following holds:

$$\begin{aligned}
c(v) &\leq c(u) - x_i \\
&\leq d(u) + (K_{\ell-1,\Lambda} - \delta(u))t - x_i \\
&\leq d(v) + (K_{\ell-1,\Lambda} - \delta(u))t + t \\
&\leq d(v) + (K_{\ell-1,\Lambda} - \delta(v))t.
\end{aligned}$$

However, this yields that the label of u should have been defined by the first rule. We conclude that $d(u) - d(v) - x_i > t$. Moreover, since t has the property (\star) and $d(u) - d(v) - x_i \in \Gamma'_{D_{\ell-1,\Lambda}}(x_1, \dots, x_{\ell-1})$, it holds that $d(u) - d(v) - x_i \geq K_{\ell-1,\Lambda}(t + \Lambda x_\ell)$.

We now bound the label $c'(v)$ assigned to the vertex v from above (recall that $\mu(v)x_\ell \leq \Lambda x_\ell \leq t$):

$$\begin{aligned} c'(v) &= d(v) + (K_{\ell-1,\Lambda} - 1)t + \delta(v)\Lambda x_\ell + \mu(v)x_\ell \\ &\leq d(v) + K_{\ell-1,\Lambda}t + K_{\ell-1,\Lambda}\Lambda x_\ell \\ &\leq d(u) - x_i \leq c(u) - x_i = c'(u) - x_i . \end{aligned}$$

Hence, the labels of the vertices u and v differ by at least x_i as required.

- **The label $c'(u)$ is defined by the first rule, the label $c'(v)$ is defined by the second rule and $i \geq \ell$.**

If $d(u) \leq d(v)$, then $c'(u) \leq d(u) + (K_{\ell-1,\Lambda} - 1)t$ and $c'(v) \geq d(v) + (K_{\ell-1,\Lambda} - 1)t + \mu(v)x_\ell \geq d(u) + (K_{\ell-1,\Lambda} - 1)t + x_\ell$. Therefore, $c'(v) - c'(u) \geq x_\ell$ as desired.

In the rest, we focus on the case $d(u) > d(v)$. This implies that $\delta(u) > \delta(v)$, the edge uv is directed from v to u and $c(u) > c(v)$. First, we exclude the case $d(u) - d(v) \leq t$. Since the label of the vertex u was defined by the first rule, we have $c(u) \leq d(u) + (K_{\ell-1,\Lambda} - \delta(u))t$. We infer the following upper bound on $c(v)$:

$$\begin{aligned} c(v) &\leq c(u) \leq d(u) + (K_{\ell-1,\Lambda} - \delta(u))t \\ &\leq d(v) + t + (K_{\ell-1,\Lambda} - \delta(u))t \\ &\leq d(v) + (K_{\ell-1,\Lambda} - \delta(v))t . \end{aligned}$$

Then the label to v should have been defined by the first rule not by the second one. We may conclude that $d(u) - d(v) > t$. Since t has the property (\star) and $d(u) - d(v) \in \Gamma'_{D_{\ell-1,\Lambda}}(x_1, \dots, x_{\ell-1})$, it holds that $d(u) - d(v) \geq K_{\ell-1,\Lambda}(t + \Lambda x_\ell)$. The following upper bound on $c'(v)$ readily follows (recall that $\delta(v) \leq K_{\ell-1,\Lambda} - 1$):

$$\begin{aligned} c'(v) &= d(v) + (K_{\ell-1,\Lambda} - 1)t + \delta(v)\Lambda x_\ell + \mu(v)x_\ell \\ &\leq d(v) + K_{\ell-1,\Lambda}t - t + K_{\ell-1,\Lambda}\Lambda x_\ell \\ &\leq d(u) - t \leq c(u) - x_\ell = c'(u) - x_\ell \leq c'(u) - x_i . \end{aligned}$$

Hence, the labels of the end-vertices of the x_i -edge uv differ by at least x_i as required.

- **Both the labels $c'(u)$ and $c'(v)$ are defined by the second rule and $i < \ell$.**

By symmetry, we may assume that the edge uv is directed from u to v in \vec{G} . In such case, $d(u) + x_i \leq d(v)$ and $\delta(u) < \delta(v)$. We now estimate the difference of the labels $c'(u)$ and $c'(v)$:

$$\begin{aligned} c'(v) - c'(u) &= d(v) - d(u) + (\delta(v) - \delta(u))\Lambda x_\ell + (\mu(v) - \mu(u))x_\ell \\ &\geq x_i + \Lambda x_\ell - |\mu(v) - \mu(u)|x_\ell \geq x_i. \end{aligned}$$

We conclude that the edge uv is properly colored.

- **Both the labels $c'(u)$ and $c'(v)$ are defined by the second rule and $i \geq \ell$.**

By symmetry, we may assume that the edge uv is directed from u to v in \vec{G} . In such case, $d(u) \leq d(v)$. If $d(u) = d(v)$, then:

$$|c'(v) - c'(u)| = |\mu(v) - \mu(u)|x_\ell \geq x_\ell \geq x_i.$$

In the rest, we deal with the case $d(u) < d(v)$. In particular, $\delta(u) < \delta(v)$. We bound the difference between $c'(u)$ and $c'(v)$ as follows (recall that $1 \leq \mu(v), \mu(v) \leq \Lambda$):

$$\begin{aligned} c'(v) - c'(u) &= d(v) - d(u) + (\delta(v) - \delta(u))\Lambda x_\ell + (\mu(v) - \mu(u))x_\ell \\ &\geq \Lambda x_\ell - |\mu(v) - \mu(u)|x_\ell \geq x_\ell \geq x_i. \end{aligned}$$

Therefore, the difference of the labels $c'(u)$ and $c'(v)$ is at least x_i as desired.

As the next step, we show that the span of c' is equal to $\lambda_G(x_1, \dots, x_k)$. In order to do so, it is enough to show that $c'(v) \leq \lambda_G(x_1, \dots, x_k)$ for every vertex v of G . Let c_{\max} and d_{\max} be the maximums of the values $c(v)$ and $d(v)$ taken over all the vertices v of G . Clearly, $c_{\max} = \lambda_G(x_1, \dots, x_k)$. By Lemma 2, the following holds:

$$c_{\max} \leq d_{\max} + \ell^{D_{\ell-1, \Lambda}} \Lambda \cdot x_\ell = d_{\max} + K_{\ell-1, \Lambda} \Lambda \cdot x_\ell.$$

Fix a vertex v of G . In order to show that $c'(v) \leq c_{\max}$, we distinguish three cases according to difference between $d(v)$ and d_{\max} :

- $d(v) = d_{\max}$
Since $c(v) \leq c_{\max} \leq d(v) + K_{\ell-1, \Lambda} \Lambda x_\ell \leq d(v) + t \leq d(v) + (K_{\ell-1, \Lambda} - \delta(v))t$, the label of the vertex v was defined by the first rule. Consequently, $c'(v) = c(v) \leq c_{\max}$.

- $d_{\max} - d(v) \leq t$

First, let us observe that $\delta(v) \leq K_{\ell-1, \Lambda} - 2$. Again, we bound the original label $c(v)$ from above:

$$c(v) \leq c_{\max} \leq d_{\max} + t \leq d(v) + 2t \leq d(v) + (K_{\ell-1, \Lambda} - \delta(v))t .$$

Therefore, the label of the vertex v was defined by the first rule, and $c'(v) = c(v) \leq c_{\max}$.

- $d_{\max} - d(v) > t$

Since t has the property (\star) and $d_{\max} - d(v) \in \Gamma'_{D_{\ell-1, \Lambda}}(x_1, \dots, x_{\ell-1})$, $d_{\max} - d(v) \geq K_{\ell-1, \Lambda}(t + \Lambda x_{\ell})$. If the first rule applies to the vertex v , then $c'(v) = c(v) \leq c_{\max}$. If the second rule applies, then the following estimate on $c'(v)$ holds:

$$\begin{aligned} c'(v) &= d(v) + (K_{\ell-1, \Lambda} - 1)t + \delta(v)\Lambda x_{\ell} + \mu(v)x_{\ell} \\ &\leq d(v) + K_{\ell-1, \Lambda}t + K_{\ell-1, \Lambda}\Lambda x_{\ell} \\ &\leq d_{\max} \leq c_{\max} . \end{aligned}$$

Hence, the label $c'(v)$ does not exceed c_{\max} .

Let \vec{G}' be the orientation of G corresponding to the labeling c' . Since all x_1, \dots, x_k are positive, the orientation G is finitary and its weight is at most the span of c' . Since the span of c' is $\lambda_G(x_1, \dots, x_k)$, the weight of \vec{G}' is exactly $\lambda_G(x_1, \dots, x_k)$. In order to finish the proof of the lemma, we establish that each directed path in \vec{G}' contains at most $D_{\ell, \Lambda}$ $x_{\leq \ell}$ -edges.

All the labels $c'(v)$ defined by the first rule are contained in the following union of intervals:

$$\begin{aligned} &\bigcup_{\gamma \in \Gamma_{D_{\ell-1, \Lambda}}(x_1, \dots, x_{\ell-1})} \langle \gamma, \gamma + K_{\ell-1, \Lambda}t \rangle \\ &\subseteq \bigcup_{\gamma \in \Gamma_{D_{\ell-1, \Lambda}}(x_1, \dots, x_{\ell-1})} \langle \gamma, \gamma + (2K_{\ell-1, \Lambda})^{K_{\ell-1, \Lambda}^2 + 1} \Lambda x_{\ell} \rangle \end{aligned}$$

The labels $c'(v)$ assigned by the second rule are from the following set:

$$\bigcup_{\gamma_i \in \Gamma_{D_{\ell-1, \Lambda}}(x_1, \dots, x_{\ell-1})} \{ \gamma_i + (K_{\ell-1, \Lambda} - 1)t + i\Lambda x_{\ell} + jx_{\ell}, j = 1, \dots, \Lambda \}$$

where $\gamma_1, \gamma_2, \dots$ are all the elements of $\Gamma_{D_{\ell-1, \Lambda}}$ listed in the increasing order. Consider a directed path P in \vec{G}' and let C be the set of labels of the end-vertices of $x_{\leq \ell}$ -edges on P . Since any two labels in C differ by at least x_ℓ and $|\Gamma_{D_{\ell-1, \Lambda}}(x_1, \dots, x_{\ell-1})| \leq K_{\ell-1, \Lambda}$ by Proposition 5, we have the following upper bound on the number of labels contained in C that were defined by the first rule:

$$|\Gamma_{D_{\ell-1, \Lambda}}(x_1, \dots, x_{\ell-1})| \frac{(2K_{\ell-1, \Lambda})^{K_{\ell-1, \Lambda}^2 + 1} \Lambda x_\ell}{x_\ell} \leq (2K_{\ell-1, \Lambda})^{K_{\ell-1, \Lambda}^2 + 2} \Lambda.$$

Similarly, the number of the labels defined by the second rule does not exceed:

$$|\Gamma_{D_{\ell-1, \Lambda}}(x_1, \dots, x_{\ell-1})| \Lambda \leq K_{\ell-1, \Lambda} \Lambda.$$

Combining both the bounds, we infer from Proposition 4 that the size of C does not exceed $D_{\ell, \Lambda} = (2K_{\ell-1, \Lambda})^{K_{\ell-1, \Lambda}^2 + 3} \Lambda$. Therefore, every directed path in \vec{G}' contains at most $D_{\ell, \Lambda}$ $x_{\leq \ell}$ -edges as desired. ■

We modify Lemma 7 to a version used in Section 4:

Lemma 8 *Let G be a (finite or infinite) λ -graph with k types of edges and chromatic number at most Λ . For any k -tuple of non-negative reals x_1, \dots, x_k , there exists a finitary orientation of G with weight $\lambda_G(x_1, \dots, x_k)$ with maximum length of a directed path at most $D_{k, \Lambda}$.*

Proof: By symmetry, we can assume that $x_1 \geq \dots \geq x_k$ (otherwise, permute the types of the edges of G). If $x_k > 0$, the statement of the lemma follows directly from Lemma 7. In the rest, we deal with the case when $x_{k'} > 0$ and $x_{k'+1} = \dots = x_k = 0$.

Fix a coloring μ of G with Λ colors $1, \dots, \mu$. Let G' be the subgraph of G formed by $x_{\leq k'}$ -edges. By Lemma 7, there exists a finitary orientation \vec{G}' of G' with weight $\lambda_{G'}(x_1, \dots, x_{k'}) = \lambda_G(x_1, \dots, x_k)$ and with maximum path length at most $D_{k', \Lambda}$. Let $c(v)$ be the labeling of G' corresponding to \vec{G}' . Observe that $c(v) \in \Gamma_{D_{k', \Lambda}}(x_1, \dots, x_{k'})$ for every vertex v of G .

We extend \vec{G}' to a finitary orientation \vec{G} of G . An $x_{> k'}$ -edge uv is directed from u to v if $c(u) < c(v)$, from v to u if $c(u) > c(v)$. If $c(u) = c(v)$, then the edge uv is directed from u to v if $\mu(u) < \mu(v)$, and from u to v , otherwise. Clearly, the weight of \vec{G} is the same as the weight of \vec{G}' .

Let P be a directed path in \vec{G} . Clearly, the labels $c(v)$ of vertices v does not decrease along the path P . Moreover, each subpath of P formed by vertices v with the same label has length at most $\Lambda - 1$ as the labels $\mu(v)$ of the vertices comprising the subpath strictly increase. Since all the labels $c(v)$ are from the set $\Gamma_{D_{k',\Lambda}}(x_1, \dots, x_{k'})$, P contains at most $K_{k',\Lambda}$ such subpaths. Hence, the length of a directed P in \vec{G} does not exceed $K_{k',\Lambda} \cdot \Lambda$. Therefore, the maximum length of a directed path in \vec{G} is at most $K_{k',\Lambda} \cdot \Lambda \leq D_{k,\Lambda}$. ■

4 Main Result

In this section, we prove our main result on the structure of the λ -functions of λ -graphs. Before doing so, we introduce several definitions. $\mathcal{F}_{k,\Lambda}$ denotes the set of all linear functions of k variables with integer coefficients between 0 and $D_{k,\Lambda}$, i.e.,

$$\mathcal{F}_{k,\Lambda} = \left\{ \sum_{i=1}^k \alpha_i x_i, 0 \leq \alpha_i \leq D_{k,\Lambda} \right\}.$$

Next, $\mathcal{F}_{k,\Lambda}^{\max}$ is the set of all functions φ that are equal to the maximum of some of the functions from $\mathcal{F}_{k,\Lambda}$, i.e.,

$$\mathcal{F}_{k,\Lambda}^{\max} = \{ \varphi(x_1, \dots, x_p) = \max_{f \in F} f(x_1, \dots, x_p) \text{ for } F \subseteq \mathcal{F}_{k,\Lambda}, F \neq \emptyset \}.$$

Finally, $\mathcal{F}_{k,\Lambda}^{\min\max}$ is the set of all functions that are equal to the minimum of some of the functions from $\mathcal{F}_{k,\Lambda}^{\max}$, i.e.,

$$\mathcal{F}_{k,\Lambda}^{\min\max} = \{ \varphi(x_1, \dots, x_p) = \max_{f \in F} f(x_1, \dots, x_p) \text{ for } F \subseteq \mathcal{F}_{k,\Lambda}^{\max}, F \neq \emptyset \}.$$

Observe $\mathcal{F}_{k,\Lambda} \subseteq \mathcal{F}_{k,\Lambda}^{\max} \subseteq \mathcal{F}_{k,\Lambda}^{\min\max}$. Clearly, all the three sets $\mathcal{F}_{k,\Lambda}$, $\mathcal{F}_{k,\Lambda}^{\max}$ and $\mathcal{F}_{k,\Lambda}^{\min\max}$ are finite. Therefore, the subset of \mathbb{R}^k formed by k -tuples of non-negative reals can be partitioned into finitely many closed polyhedral cones (with the tips at the origin of \mathbb{R}^k) such that every function contained in $\mathcal{F}_{k,\Lambda}^{\min\max}$ is linear on each of the cones. Let $C_{k,\Lambda}$ be the number of such cones.

We now state and prove the main result of the paper (note that both the numbers $C_{k,\Lambda}$ and $D_{k,\Lambda}$ just depend on k and Λ):

Theorem 9 For every λ -graph G with k types of edges and chromatic number at most Λ , $\lambda_G(x_1, \dots, x_k)$ is a piecewise linear function of x_1, \dots, x_k with at most $C_{k,\Lambda}$ linear parts formed by linear function with integer coefficients between 0 and $D_{k,\Lambda}$. Moreover, the subset of \mathbb{R}^k formed by non-negative integers can be partitioned into at most $C_{k,\Lambda}$ (closed) polyhedral cones such that for each of the cones the following holds: there exist integers $\alpha_i(v)$ between 0 and $D_{k,\Lambda}$ such that the labeling c , $c(v) = \sum_{i=1}^k \alpha_i(v)x_i$, is a proper labeling of G with respect to x_1, \dots, x_k and the span of c is $\lambda_G(x_1, \dots, x_k)$.

Proof: Let \mathcal{D} be the set of all finitary orientations of G with maximum length of a directed path at most $D_{k,\Lambda}$. For an orientation $\vec{G} \in \mathcal{D}$, let $F(\vec{G})$ be the set of all the functions $\sum_{i=1}^k \alpha_i x_i$ such that \vec{G} contains a directed path with precisely α_i x_i -edges. Since the maximum length of a directed path in \vec{G} does not exceed $D_{k,\Lambda}$, the set $F(\vec{G})$ is a subset of $\mathcal{F}_{k,\Lambda}$, i.e., $F(\vec{G}) \subseteq \mathcal{F}_{k,\Lambda}$. By the definition, the weight of the orientation \vec{G} with respect to x_1, \dots, x_k is the following:

$$w_{\vec{G}}(x_1, \dots, x_k) = \max_{f \in F(\vec{G})} f(x_1, \dots, x_k).$$

Let W be the set of all the functions $w_{\vec{G}}(x_1, \dots, x_k)$ where \vec{G} ranges through all the orientations contained in \mathcal{D} . Clearly, $W \subseteq \mathcal{F}_{k,\Lambda}^{\max}$. For $w \in W$, let \vec{G}_w be one of the orientations in \mathcal{D} with $w_{\vec{G}_w} = w$. By Theorem 1 and Lemma 8, the following equality holds:

$$\lambda_G(x_1, \dots, x_k) = \min_{\vec{G} \in \mathcal{D}} w_{\vec{G}}(x_1, \dots, x_k) = \min_{w \in W} w(x_1, \dots, x_k).$$

Similarly as before, we have $\lambda_G(x_1, \dots, x_k) \in \mathcal{F}_{k,\Lambda}^{\min\max}$.

Consider the partition of k -tuples of non-negative reals into $C_{k,\Lambda}$ polyhedral cones such that every function of $\mathcal{F}_{k,\Lambda}^{\min\max}$ is linear on each of the cones. In particular, $\lambda_G(x_1, \dots, x_k) \in \mathcal{F}_{k,\Lambda}^{\min\max}$ is linear on each of the cones. Fix one such cone and let $w \in W$ be a function such that $\lambda_G(x_1, \dots, x_k) = w(x_1, \dots, x_k)$ on the fixed cone. Let c be the labeling corresponding to the orientation \vec{G}_w . Since no directed path of $\vec{G}_w \in \mathcal{D}$ has length more than $D_{k,\Lambda}$, $c(v)$ for a single vertex v when viewed as a function of x_1, \dots, x_k belongs to $\mathcal{F}_{k,\Lambda}^{\max}$. In particular, every function $c(v)$ is linear on the fixed cone. Therefore, the considered polyhedral cones form a possible partition of \mathbb{R}^k with the properties from the statement of the theorem. The bounds on the

integers coefficients of linear functions readily follows from the definitions of $\mathcal{F}_{k,\Lambda}$, $\mathcal{F}_{k,\Lambda}^{\max}$ and $\mathcal{F}_{k,\Lambda}^{\min\max}$. ■

Since only the functions from $\mathcal{F}_{k,\Lambda}^{\min\max}$ could be the λ -function of a λ -graph with k types of edges and with chromatic number at most Λ , we have the following:

Corollary 10 *There exists only finitely many piecewise linear functions that could be the λ -function of a λ -graph with k types of edges and with chromatic number at most Λ .*

Another immediate corollary is the following somewhat surprising statement:

Corollary 11 *Let x_1, \dots, x_k be a fixed k -tuple of positive reals and let γ be a non-negative real. There exist only finitely many different k -parameter λ -functions λ_G such that $\lambda_G(x_1, \dots, x_k) = \gamma$.*

Proof: If G is a λ -graph with k types of edges such that $\lambda_G(x_1, \dots, x_k) = \gamma$, then the chromatic number of G does not exceed $\frac{\lambda_G(x_1, \dots, x_k)}{\min\{x_1, \dots, x_k\}} + 1$ by the scaling property. By Corollary 10, λ -graphs with k types of edges with bounded chromatic number have only finitely many different λ -functions. ■

Note that Corollary 11 includes the result of [2] that the number of λ -functions with prescribed boundary values is finite.

5 Labelings with Distance Constraints

In this section, we infer from Theorem 9 Piecewise Linearity Conjecture, Coefficient Bound Conjecture and Delta Bound Conjecture stated in [8]. First, let us state the following simple proposition that can be found in [8] (note that its proof employs the Axiom of Choice):

Proposition 12 *If G is a graph of maximum degree Δ and k is a positive integer, then the chromatic number of the k -th power of G does not exceed $\Delta^k + 1$.*

We can now state the theorem from that the three conjectures mentioned above readily follow:

Theorem 13 *For every integers Δ and k , there exist constants $C'_{k,\Delta}$ and $D'_{k,\Delta}$ with the following property: for any graph G with maximum degree Δ , there exist finitely many hyperplanes in \mathbb{R}^k through the origin such that the function $\lambda_G(p_1, \dots, p_k)$ is linear in each of at most $C'_{k,\Delta}$ convex polyhedral cones that are determined by the hyperplanes. Moreover, for each of the cones the following is true: there exist integers $\alpha_i(v)$ between 0 and $D_{k,\Delta}$ such that the labeling c , $c(v) = \sum_{i=1}^k \alpha_i(v)p_i$, is a proper $L(p_1, \dots, p_k)$ -labeling of G with respect to x_1, \dots, x_k and the span of c is $\lambda_G(p_1, \dots, p_k)$.*

Proof: Set $C'_{k,\Delta} = C_{k,\Delta^k+1}$ and $D'_{k,\Delta} = D_{k,\Delta^k+1}$. Let G be a graph with maximum degree Δ and form the λ -graph $G^{(k)}$ as described in Section 1. By Proposition 12, the chromatic number of $G^{(k)}$ does not exceed $\Delta^k + 1$. Theorem 13 now follows from Theorem 9. ■

An immediate corollary of Theorem 13 (alternatively, of Corollary 10) is the following:

Corollary 14 *For every integers Δ and k , the set $\Lambda_{k,\Delta}$, that consists of all (piecewise linear) functions $\lambda_G(p_1, \dots, p_k)$ where G is a finite or infinite graph of maximum degree Δ , is finite.*

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