Labelings of graphs with fixed and variable edge-weights*

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Abstract

Motivated by $L(p,q)$-labelings of graphs, we introduce a notion of λ-graphs: a λ-graph G is a graph with two types of edges: 1-edges and x-edges. For a parameter $x \in [0,1]$, a proper labeling of G is a labeling of vertices of G by non-negative reals such that the labels of the end-vertices of a 1-edge differ by at least 1 and the labels of the end-vertices of an x-edge differ by at least x; $\lambda_G(x)$ is the smallest real such that G has a proper labeling by labels from the interval $[0, \lambda_G(x)]$.

We study properties of the function $\lambda_G(x)$ for finite and infinite λ-graphs and establish the following results: if the function $\lambda_G(x)$ is well-defined, then it is a piecewise linear function of x with finitely many linear parts. Surprisingly, the set $\Lambda(\alpha, \beta)$ of all functions λ_G with $\lambda_G(0) = \alpha$ and $\lambda_G(1) = \beta$ is finite for any $\alpha \leq \beta$. We also prove a tight upper bound on the number of segments for finite λ-graphs G with convex functions $\lambda_G(x)$.

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1 Introduction

Several graph theory models for radio frequency assignment were suggested by Hale [14]. One of the most important models is $L(p,q)$-labeling of graphs, which can be traced back to the paper by Griggs and Yeh [13]. An $L(p,q)$-*labeling* of a graph G for $1 \leq q \leq p$ is a labeling of the vertices by non-negative integers such that the labels of adjacent vertices differ by at least p and the labels of vertices at distance two differ by at least q. The least integer K such that there is a proper labeling using integers between 0 and K is called the *span* and is denoted by $\lambda_{p,q}(G)$.

The case of $L(2,1)$-labelings attracted a special attention of researchers, in particular with the connection to the conjecture of Griggs and Yeh [13] that $\lambda_{2,1}(G) \leq \Delta^2$ for every graph G with maximum degree Δ. Bounds on the span in terms of the maximum degree have been proved in a series of papers [13, 5, 21], and the currently best upper bound is $\lambda_{2,1}(G) \leq \Delta^2 + \Delta - 1$. The conjecture itself has been verified for several classes of graphs, including graphs of maximum degree two, chordal graphs [25], see also [4, 20], and hamiltonian cubic graphs [17, 18]. However, even the case of general cubic graphs remains open. Because of practical motivation of the problem, $L(p,q)$-labelings are also widely studied from the algorithmic point of view [1, 3, 7, 8, 19, 24].

In this paper, we study how the span $\lambda_{p,q}(G)$ depends on the parameters p and q. This is well-motivated from practical point of view since in applications, the parameters p and q are not fixed in advance but rather adjusted ad hoc depending on to the level of interference experienced for their different combinations. Our approach is similar to that of [22], but we focus on the original notion of $L(p,q)$-labeling rather than its circular coloring version, and we do not determine the behavior for some particular graphs, but we rather prove general results. Also, we do not restrict our attention to finite graphs. The inclusion of infinite graphs is motivated by applications, e.g., $L(p,q)$-labelings of infinite triangular, square and hexagonal planar lattices naturally arise in practice, and have been addressed from the theoretical point of view as well [16].

$L(p,q)$-labelings are closely related to channel assignment problems. Our definition of channel assignment problem is slightly more general than usual: both the weights of edges and the labels of vertices are real numbers rather than just integers. A *channel assignment problem* is determined by a pair (G,w) consisting of a (finite or infinite) graph G and a function $w : E(G) \to \mathbb{R}^+$. A labeling $c : V(G) \to \mathbb{R}_0^+$ of the vertices of G by non-negative reals is *proper* if $|c(v)-c(v')| \geq w(vv')$ for each edge vv' of G. The *span of a labeling* c is the supremum of the labels used by c and the *span* $\lambda_w(G)$ *of a channel assignment problem* (G,w) is the infimum of the spans of proper labelings for (G,w). An $L(p,q)$-labeling of a graph G can be viewed as the channel assignment problem for the square of G (the second distance power): the edges of G have weights p, and the edges of G^2 not belonging to G have weights q. The reader is also welcome to see the survey [23] on the channel assignment problem.

The alternative view of $L(p,q)$-labelings presented above is a starting point for our work. A *λ-graph* G is a graph with two types of edges: *1-edges* and *x-edges*. For a parameter $x \in [0,1]$, one forms a channel assignment problem on G by assigning the weight 1 to every 1-edge and the weight x to every x-edge. The span of this channel assignment problem is denoted by $\lambda_G(x)$; the function $\lambda_G(x)$ is called the *λ-function* of G. For a graph H, let G_H be the λ-graph on the same set of vertices as H such that the vertices adjacent in H are joined by 1-edges in G_H, and the vertices at distance two in H are joined by x-edges in G_H. Clearly, the following holds:

$$\lambda_{G_H}\left(\frac{q}{p}\right) = \frac{\lambda_{p,q}(H)}{p}.$$

Therefore, the λ-function of G_H can be viewed as normalized one-dimensional function describing the behavior of the two-parameter function $\lambda_{p,q}(H)$. Note also that $\lambda_G(0) = \chi(G^{(1)}) - 1$ and $\lambda_G(1) = \chi(G) - 1$, where $G^{(1)}$ is the spanning subgraph of G formed by the 1-edges. This approach reflects the practical application of radio frequency assignment: the 1-edges represent the pairs of close transmitters where huge interference occurs, and the x-edges correspond to more distant transmitters where smaller interference may appear. The value of the parameter x is then proportional to the interference experienced, and is adjusted according to its level. To get acquainted with the principal concepts of this paper, the reader may consult the Appendix, where we provide the complete list of λ-graphs with four vertices together with their λ-functions, as well as examples of other interesting λ-graphs.

A similar approach to the study of the span of $L(p,q)$-labeling was developed independently of us (and before us) by Griggs and Jin [10, 11, 12]. They presented their results, e.g., during the SIAM Conference on Discrete Mathematics in Nashville, TN, in June 2004. In particular, they proved (using a different terminology) that if H is a (finite or infinite) graph with bounded maximum degree, then λ_{G_H} is a piecewise linear function of x for

$x \in [0, \infty)$ with finitely many linear parts. Moreover, the coefficients of the linear functions forming λ_{G_H} are bounded by a constant that depends solely on the maximum degree of H. The former statement can be derived from our Theorem 4 (see Corollary 5). Our Theorem 12 yields that there are only finitely many different λ-functions for λ-graphs of the form G_H where H is a graph of bounded maximum degree. Hence, Theorem 12 also implies that the coefficients of linear functions forming λ_{G_H} are bounded by a constant depending only on the maximum degree of H.

Our method is different from that in [10]: the arguments in [10] are based on the structure of optimum labelings for a graph H obeying the given distance constraints, whereas we use a close correspondence between orientations of graphs and their labelings, developed in Section 2. Still, some of our results, e.g., Lemma 3, have their counterparts in the work [10]. Since we prove our results in a more general setting, we decided, for the sake of completeness, to include full arguments even in such cases.

1.1 Our results

We study general λ-graphs without restricting our attention to those equal to G_H for some H. In Section 3, we show that the function $\lambda_G(x)$ is a piecewise linear function with finitely many linear parts, under the assumption that it is well-defined for some $x > 0$. The proof of this statement is quite straightforward if G is finite, but it becomes more complex for infinite λ-graphs. In Section 4, we study λ-functions with prescribed values for $x = 0, 1$. Let $\Lambda(\alpha, \beta)$ be the set of all λ-functions $\lambda_G(x)$ of finite and infinite λ-graphs G with $\lambda_G(0) = \alpha$ and $\lambda_G(1) = \beta$. One could expect that the set $\Lambda(\alpha, \beta)$ is infinite for $\alpha < \beta$, but the opposite is true: in fact, the set $\Lambda(\alpha, \beta)$ is finite for any integers $\alpha \leq \beta$. In Theorem 12, we present the bound $2^{2^{\frac{(2\alpha\beta^2+\alpha\beta+\beta^2+2)^2}{2}}}$ on the size of the set $\Lambda(\alpha, \beta)$. At the end of the paper, we focus on finite λ-graphs whose λ-function is convex, and prove an asymptotically tight upper bound on the number of the linear parts of the λ-functions in terms of the order of a λ-graph: if G is a finite λ-graph of order n and the function $\lambda_G(x)$ is convex, then $\lambda_G(x)$ consists of at most $O(n^{2/3})$ linear parts.

2 Gallai-Roy Theorem

We establish an analogue of the Gallai-Roy Theorem for channel assignment problems with (finite and) infinite underlying graphs. The Gallai-Roy theorem in its original form relates colorings and lengths of paths in acyclic orientations of a graph. Our proof follows the lines of a similar theorem for channel assignment problems with finite graphs by McDiarmid [24], but we include the proof for the sake of completeness.

First, we introduce some additional definitions necessary for stating and proving the theorem. An orientation of a graph is *finitary* if there is a constant $K \geq 0$ such that every oriented walk has length at most K. The *weight* of a path is the sum of the weights of the edges on the path. The channel assignment problem (G, w) is said to be *finitary* if the image set of the function w is finite. If (G, w) is finitary, then there exists a proper labeling c whose span is equal to the span of (G, w), and the span of the optimum labeling c is equal to the maximum label used by c (these claims will be established in the proof of Theorem 1).

We now state and prove the announced analogue of the Gallai-Roy Theorem:

Theorem 1. *Let (G, w) be a finitary channel assignment problem. The span of (G, w) is finite if and only if G has a finitary orientation. In this case, the span of (G, w) is equal to the minimum of the maximum weight of a path in a finitary orientation of G, where the minimum is taken over all finitary orientations of G.*

Proof. Consider a finitary orientation of G and let w_0 be the maximum weight of a path in the orientation. Label a vertex v of G with the maximum weight of an oriented path which ends at v. Clearly, the span of this labeling does not exceed w_0. Moreover, the labeling is proper: consider two vertices v and v' joined by an edge of G. Assume that the edge between v and v' is oriented from v to v'. Since each path leading to the vertex v can be prolonged to v', the label of v' is greater than the label of v and they differ by at least $w(vv')$. Since there is a finite number of edge weights (recall that both the channel assignment problem and the orientation are finitary), we conclude that the span of (G, w) is at most the minimum of the maximum weight of a path taken over all finitary orientations of G.

On the other hand, if c is a proper labeling of (G, w), then there is a finitary orientation of G such that the maximum weight of a path in the orientation is at most the span of (G, w). Consider the following orientation:

an edge between two vertices v and v' is oriented from v to v' if $c(v) < c(v')$, otherwise it is oriented from v' to v. Since the labels of the vertices on an oriented path increase on each edge at least by its weight, the maximum weight of the path in the orientation is bounded by the maximum label assigned to a vertex of G. The statement of the theorem now readily follows. $\square$

The next corollary of Theorem 1 on the λ-functions of finite λ-graphs immediately follows:

Corollary 2. *If G is a finite λ-graph of order n, then for each $x \in [0,1]$, there exist non-negative integers a and b with $a + b \leq n - 1$ such that $\lambda_G(x) = a + b \cdot x$.*

Proof. Consider the channel assignment problem (G', w') where G' is the underlying graph of G, the weight $w'(e)$ of a 1-edge e is one and the weight $w'(e)$ of an x-edge e is x. Since the channel assignment problem (G', w') is finitary, its span is equal to the maximum weight of a finite path of a finitary orientation of G'. Therefore, $\lambda_G(x) = a + b \cdot x$ for some non-negative integers $a + b \leq n - 1$. $\square$

3 Piecewise linearity

In this section, we show that the function $\lambda_G(x)$ of every λ-graph is a piecewise linear function of x. As the first step, we show that the function $\lambda_G(x)$ is a linear function of x on some neighborhood of 0:

Lemma 3. *Let G be a (finite or infinite) λ-graph. If the function $\lambda_G(x)$ is finite for some $x > 0$, then the function $\lambda_G(x)$ is a linear function of x on the interval $[0, \varepsilon]$ for some $\varepsilon > 0$.*

Proof. Since $\lambda_G(x)$ is finite for some $x > 0$, there is a finitary orientation $\vec{D_0}$ of G. In particular, the chromatic number $\chi(G^{(1)})$ is finite (recall that $G^{(1)}$ is the spanning subgraph of G whose edges are exactly the 1-edges of G), and $\lambda_G(0) = \chi(G^{(1)}) - 1$.

Next, we construct a finitary orientation of G that does not contain any oriented path with more than $\lambda_G(0)$ 1-edges. Let c be any proper coloring of $G^{(1)}$ with $\chi(G^{(1)})$ colors $0, \ldots, \lambda_G(0)$. Consider the orientation $\vec{D}$ of G such that an edge vv' of G is

- oriented from v to v', if $c(v) < c(v')$,

- oriented from v' to v, if $c(v) > c(v')$, and
- oriented as in the orientation $\vec{D_0}$, otherwise.

Since on each oriented path, the colors of the vertices form a non-decreasing sequence that strictly increases on each 1-edge, there is no oriented path with more than $\lambda_G(0)$ 1-edges. It remains to show that the orientation $\vec{D}$ is finitary. Let k be the maximum length of a path in $\vec{D_0}$. As we have observed, the colors assigned by c to the vertices of an oriented path of $\vec{D}$ form a non-decreasing sequence. A subpath formed by the vertices of the same color is also an oriented path in $\vec{D_0}$. Hence, its length is at most k. We conclude that each oriented path in $\vec{D}$ has length at most $\chi(G^{(1)})(k+1)$. In particular, the orientation $\vec{D}$ is finitary.

Choose $\vec{D}$ to be a finitary orientation of G such that:

1. $\vec{D}$ does not contain any oriented path with more than $\lambda_G(0)$ 1-edges, and
2. the maximum length of an oriented path with exactly $\lambda_G(0)$ 1-edges is minimal.

Since the orientation of G constructed in the previous paragraph has the first property, the orientation $\vec{D}$ exists and is well-defined.

Let k_D be the maximum length of a path in $\vec{D}$, and let k_x be the maximum number of x-edges on a path of $\vec{D}$ that has $\lambda_G(0)$ 1-edges. We show the following:

$$\lambda_G(x) = \lambda_G(0) + k_x x \text{ for every } x \in [0, \varepsilon],$$

where $\varepsilon = \frac{1}{k_D}$.

Assume, for contradiction, that there is $x \in (0, \varepsilon]$ such that $\lambda_G(x) < \lambda_G(0) + k_x x$. Note that this inequality implies that $\lambda_G(x) < \lambda_G(0) + 1$ since $k_x < k_D$. By Theorem 1, G has a finitary orientation $\vec{D}'$ that does not contain any oriented path with more than $\lambda_G(0)$ 1-edges, and in addition, at least one of the following holds:

- $\vec{D}'$ has no oriented path with $\lambda_G(0)$ 1-edges, or
- any oriented path of $\vec{D}'$ with $\lambda_G(0)$ 1-edges has less than k_x x-edges.

The former is impossible because every finitary orientation of $G^{(1)}$, and therefore of G, has a path with $\lambda_G(0)$ edges by Theorem 1. The latter contradicts the choice of the orientation $\vec{D}$. We infer that $\lambda_G(x) \geq \lambda_G(0) + k_x x$ for all $x \in [0, \varepsilon]$.

It remains to establish the opposite inequality, i.e., $\lambda_G(x) \leq \lambda_G(0) + k_x x$ for $x \in [0, \varepsilon]$. Consider an oriented path P in $\vec{D}$. If P contains $\lambda_G(0)$ 1-edges, then it contains at most k_x x-edges, and consequently its weight is at most $\lambda_G(0) + k_x x$. On the other hand, if P contains less than $\lambda_G(0)$ 1-edges, then its weight is at most $\lambda_G(0) - 1 + k_D x \leq \lambda_G(0)$. We conclude that the maximum weight of an oriented path in $\vec{D}$ is at most $\lambda_G(0) + k_x x$. Therefore, $\lambda_G(x) \leq \lambda_G(0) + k_x x$ by Theorem 1. □

We are ready to establish the main result of this section. Note that the statement of Theorem 4 for finite λ-graphs can be easily derived from Corollary 2.

Theorem 4. *Let G be a (finite or infinite) λ-graph. If the function $\lambda_G(x)$ is finite for some $x > 0$, then the function $\lambda_G(x)$ is a piecewise linear function of x on the interval $[0, 1]$ with finitely many linear parts.*

Proof. Since the function $\lambda_G(x)$ is finite for some $x > 0$, G has a finitary orientation and the function $\lambda_G(x)$ is finite for all $x \in [0, 1]$ by Theorem 1. Let $\varepsilon > 0$ be a real such that the function $\lambda_G(x)$ is linear for $x \in [0, \varepsilon]$. Such ε exists by Lemma 3. We may assume that $\varepsilon \leq 1/4$. Moreover, if $\lambda_G(1) = 0$, then λ_G is identically equal to 0 and the theorem holds. Therefore, we only need to consider the case $\lambda_G(1) \geq 1$. Let $K = \lfloor \lambda_G(1)/\varepsilon \rfloor$. By the previous assumptions, $K \geq 4$. Consider the set $\mathcal{D}$ of finitary orientations $\vec{D}$ of G such that the maximum length of an oriented path in $\vec{D}$ is at most K. Note that the set $\mathcal{D}$ is non-empty since G has a finitary orientation with maximum path length $\lambda_G(1)$ by Theorem 1 applied to the graph G with all edge weights equal to one.

For an orientation $\vec{D} \in \mathcal{D}$, let $\mathcal{F}(\vec{D})$ be the set of all the functions $a + bx$ such that $\vec{D}$ contains an oriented path with a 1-edges and b x-edges. Since the maximum length of an oriented path of $\vec{D}$ is at most K, the sum $a + b$ is bounded by K. Therefore, the set $\mathcal{F}(\vec{D})$ is finite for every orientation $\vec{D} \in \mathcal{D}$. Let $f_{\vec{D}}(x) = \max_{f \in \mathcal{F}(\vec{D})} f(x)$. Since the set $\mathcal{F}(\vec{D})$ is finite, the function $f_{\vec{D}}(x)$ is the maximum of a finite number of linear functions. In particular, the function $f_{\vec{D}}(x)$ is piecewise linear and has finitely many linear parts. Let us define:

$$f_0(x) := \min_{\vec{D} \in \mathcal{D}} f_{\vec{D}}(x) = \min_{\vec{D} \in \mathcal{D}} \max_{f \in \mathcal{F}(\vec{D})} f(x).$$

Since there are at most $\binom{K+2}{2} \leq K^2$ functions $a + bx$ with $0 \leq a, b$ and $a + b \leq K$, there are at most 2^{K^2} distinct sets $\mathcal{F}(\vec{D})$, and the minimum in the definition of $f_0(x)$ is always attained. Moreover, the function $f_0(x)$ is the minimum of at most 2^{K^2} distinct piecewise linear functions, and thus $f_0(x)$ is also a piecewise linear function. In the rest of this proof, we show that $\lambda_G(x) = f_0(x)$ for all $x \in [\varepsilon, 1]$.

Fix $x \in [\varepsilon, 1]$. Let $\vec{D}$ be an orientation of G such that $f_{\vec{D}}(x) = f_0(x)$. In the orientation $\vec{D}$, the maximum weight of an oriented path is $f_{\vec{D}}(x)$ and $\lambda_G(x) \leq f_0(x)$ by Theorem 1. Assume for the sake of contradiction that $\lambda_G(x) < f_0(x)$ for some $x \in [\varepsilon, 1]$. By Theorem 1, there exists a finitary orientation $\vec{D}$ of G with the maximum weight of an oriented path equal to $\lambda_G(x)$. If $\vec{D}$ contains a path with more than K edges, then the weight of this path is at least $(K+1)x > \frac{\lambda_G(1)}{\varepsilon} x \geq \lambda_G(1)$. This is impossible, because $\lambda_G(x) \leq \lambda_G(1)$. Therefore, the length of each oriented path in $\vec{D}$ is at most K and $\vec{D} \in \mathcal{D}$. Since the maximum weight of an oriented path in $\vec{D}$ is $f_{\vec{D}}(x)$, we have $f_0(x) \leq f_{\vec{D}}(x) = \lambda_G(x) < f_0(x)$ — contradiction.

We have shown that $\lambda_G(x) = f_0(x)$ for all $x \in [\varepsilon, 1]$. Since the function $\lambda_G(x)$ is piecewise linear on both the intervals $[0, \varepsilon]$ and $[\varepsilon, 1]$ and it has finitely many linear parts on each of the two intervals, it is a piecewise linear function with finitely many linear parts on the whole interval $[0, 1]$. $\square$

Let us now show how Theorem 4 implies the results of Griggs and Jin on λ-functions of λ-graphs of the form G_H:

Corollary 5. *Let H be a (finite or infinite) graph with a bounded maximum degree, and let $\ell_H(x) := \frac{1}{p}\lambda_{p,q}(H)$ for $x = q/p$. The function $\ell_H(x)$ is a piecewise linear function for $x \in [0, \infty)$ with finitely many linear parts.*

Proof. For $x \in [0, 1]$, the statement follows from Theorem 4 applied to the graph G_H whose definition can be found in Section 1. Next, consider the graph G' obtained from G_H by replacing 1-edges by x-edges and x-edges by 1-edges. Observe that $\ell_H(x) = x \cdot \lambda_{G'}(1/x)$. Again, Theorem 4 yields that $\lambda_{G'}(x')$ is a piecewise linear function with finitely many linear parts for $x' \in [0, 1]$. Hence, $\ell_H(x)$ is a piecewise linear function with finitely many linear parts for $x \in [1, \infty)$, too. $\square$

Note that if H has bounded maximum degree, then G_H has bounded maximum degree as well, and in particular, G_H has bounded chromatic number. Our results from Section 4, namely Theorem 12, imply that for every finite bound K there is a finite set $\mathcal{L}_K$ of piecewise linear functions defined on $[0,\infty)$, with finitely many linear parts, such that for any (finite or infinite) graph H with maximum degree at most K we have $\ell_H \in \mathcal{L}_K$.

Another immediate corollary of Theorem 4 is the following:

Corollary 6. *If G is a finite λ-graph of order n, then there exist an integer k, $1 \le k \le n^2$, real numbers $x_0,\dots,x_k$, $0 = x_0 < x_1 < \dots < x_k = 1$, and non-negative integers $a_1,\dots,a_k$ and $b_1,\dots,b_k$ with $a_i + b_i \le n-1$, such that $\lambda_G(x) = a_i + b_i x$ for every $x \in [x_{i-1}, x_i]$. Moreover, $x_i = \frac{c_i}{d_i}$ for some integers c_i, d_i, with $0 \le c_i \le d_i \le n-1$.*

Proof. Since the function $\lambda_G(x)$ is piecewise linear by Theorem 4, there exist real numbers $x_0,\dots,x_k$, $0 = x_0 < x_1 < \dots < x_k = 1$, such that the function $\lambda_G(x)$ is linear on each interval $[x_{i-1}, x_i]$, $i = 1,\dots,k$, for some integer k. By Corollary 2, the coefficients of these linear functions are non-negative integers whose sum does not exceed $n-1$.

Furthermore, each of the reals $x_1,\dots,x_{k-1}$ can be expressed as a fraction with both the numerator and denominator between 1 and $n-1$: clearly, x_i is the (unique) solution of the equation $a_i + b_i x_i = a_{i+1} + b_{i+1} x$. Hence, $x_i = \frac{a_i - a_{i+1}}{b_{i+1} - b_i} = \frac{|a_{i+1} - a_i|}{|b_{i+1} - b_i|}$ (the latter equality follows from the fact that x_i is positive). Since there are at most $(n-1)^2$ fractions with both the numerator and denominator between 1 and $n-1$, the bound on the number k follows. □

Let us remark that the bound on the number of linear parts in Corollary 6 can be improved to $\frac{3n^2}{\pi^2} + o(n^2)$ using the results on the Farey fractions discussed in Section 5. However, we think that the order of the bound from Corollary 6 can be asymptotically improved and conjecture the following:

Conjecture 7. *If G is a finite λ-graph of order n, then the function $\lambda_G(x)$ consists of at most n linear parts.*

4 λ-functions with boundary constraints

As the first step towards the proof of Theorem 12, we establish two bounds on the growth of a λ-function:

Lemma 8. *Let G be a (finite or infinite) λ-graph whose λ-function is finite, and let $\lambda_G(x) = a + bx$ for all $x \in [0,\gamma]$ and some $\gamma > 0$. The following* *inequality holds:*

$$\lambda_G(x) \geq a + bx$$

for all $x \in [0, 1/b]$*, if* $b > 0$*, and for all* $x \in [0, 1]$*, otherwise.*

Proof. If $b = 0$, the lemma holds trivially, because $\lambda_G(x) \geq \lambda_G(0)$ for all $x \in [0, 1]$. In the rest of the proof, we consider the case $b > 0$. Assume for the sake of contradiction that there exists $x_0 \in [0, 1/b]$ such that $\lambda_G(x_0) < a + bx_0 \leq a + 1$. Note that $x_0 > \gamma$ because $\lambda_G(x)$ is equal to $a + bx$ for $x \in [0, \gamma]$. By Theorem 1, there exists an orientation $\vec{D}$ of G with the following property: for every oriented path P in $\vec{D}$ it holds that $a' + b'x_0 \leq \lambda_G(x_0) < a + bx_0$, where a' and b' are the numbers of 1-edges and x-edges of P. Since $a + bx_0 \leq a + 1$, we have $a' \leq a$. Therefore, $a' + b'\gamma < a + b\gamma$ for each such path P. We infer from Theorem 1 that $\lambda_G(\gamma) < a + b\gamma$. This contradicts the assumptions of the lemma. □

Lemma 9. *Let G be a (finite or infinite) λ-graph whose λ-function is finite. The following inequality holds:*

$$\lambda_G(x) \leq \alpha + (\alpha + 1)\beta x$$

where $\alpha = \lambda_G(0)$ *and* $\beta = \lambda_G(1)$*.*

Proof. Fix vertex colorings $c^{(1)}$ and c of the graphs $G^{(1)}$ and G with colors $0, \ldots, \alpha$ and $0, \ldots, \beta$. Let $\vec{D}$ be the following orientation of G: an edge $e = uv$ of G with $c^{(1)}(u) < c^{(1)}(v)$ is oriented from u to v. An edge $e = uv$ with $c^{(1)}(u) = c^{(1)}(v)$ is oriented from u to v if $c(u) < c(v)$, and from v to u, otherwise.

We now bound the maximum weight of a path in $\vec{D}$. Consider an oriented path P in $\vec{D}$. The function $c^{(1)}$ is non-decreasing along the path P. Since the value of $c^{(1)}$ increases on each 1-edge of P, the path P contains at most α 1-edges. There are also at most $\alpha + 1$ subpaths of P formed by the vertices with the same color assigned by $c^{(1)}$. On each such subpath, the function c is strictly increasing, and consequently such a subpath can consist of at most β x-edges. We conclude that each oriented path in $\vec{D}$ contains at most α 1-edges and at most $(\alpha + 1)\beta$ x-edges. By Theorem 1, $\lambda_G(x) \leq \alpha + (\alpha + 1)\beta x$. □

A key essence of the proof that the set $\Lambda(\alpha, \beta)$ is finite is the following lower bound on the length of the initial linear part of a λ-function in terms of $\lambda_G(0)$ and $\lambda_G(1)$:

Lemma 10. *Let G be a (finite or infinite) λ-graph whose λ-function is finite. The length of the initial linear part of $\lambda_G(x)$ is at least*

$$\frac{1}{2\alpha\beta+\alpha+\beta+1}$$

where $\alpha = \lambda_G(0)$ and $\beta = \lambda_G(1)$.

Proof. Let a and b be the non-negative integers such that $\lambda_G(x) = a + bx$ for all $x \in [0, \gamma]$ for some $\gamma > 0$. Note that $a = \alpha$. By Lemma 9, we have $0 \le b \le (\alpha+1)\beta$. We show that $\lambda_G(x) = a + bx$ for all $x \in [0, \frac{1}{2\alpha\beta+\alpha+\beta+1}]$. The inequality $\lambda_G(x) \ge a+bx$ follows from Lemma 8. In the remaining, we focus on establishing the opposite inequality $\lambda_G(x) \le a + bx$.

By Theorem 1 applied to the channel assignment problem derived from G for $x = \min\{1/(b+1), \gamma\}$, there exists a finitary orientation $\vec{D}$ of G with the following properties:

1. $\vec{D}$ contains no oriented path with $a+1$ or more 1-edges, and
2. each oriented path of $\vec{D}$ with a 1-edges contains at most b x-edges.

Let a_v, $v \in V(G)$, be the maximum number of 1-edges on an oriented path of $\vec{D}$ which ends at v, and let b_v be the maximum number of x-edges on an oriented path with a_v 1-edges which ends at v. In addition, let c_β be a coloring of G with colors $0, \ldots, \beta$. For $x \in [0, \frac{1}{2\alpha\beta+\alpha+\beta+1}]$, we define a labeling c of G as follows:

$$c(v) = \begin{cases} a_v + b_v x & \text{if } b_v \le b, \\ a_v + a_v(\beta+1)x + (c_\beta(v)+b+1)x & \text{otherwise.} \end{cases}$$

We now prove that c is a proper labeling of G for every $x \in [0, \frac{1}{2\alpha\beta+\alpha+\beta+1}]$.

As the first step towards this goal, we show that if $b_v > b$, then the label $c(v)$ is at most $a_v + 1 - x$ (note that $b_v > b$ implies $a_v < \alpha$):

$$\begin{aligned} c(v) &= a_v + a_v(\beta+1)x + (c_\beta(v)+b+1)x \\ &\le a_v + (\alpha-1)(\beta+1)x + (\beta+(\alpha+1)\beta+1)x \\ &= a_v + (2\alpha\beta+\alpha+\beta+1)x - x \le a_v + 1 - x \end{aligned} \tag{1}$$

Next, we show that the labeling is proper on each edge of G. Consider an edge uv, oriented from u to v in $\vec{D}$. We distinguish two major cases according to the type of the edge uv:

- uv **is an** x**-edge.**
  Clearly, $a_u \leq a_v$, and if $a_u = a_v$, then $b_u < b_v$. We verify that $|c(u) - c(v)| \geq x$ by considering the following four subcases:

  - $a_u < a_v$
    By (1), $c(u) \leq a_u + 1 - x$. Since $a_u + 1 \leq a_v \leq c(v)$, we have $c(v) - c(u) \geq x$ as desired.
  - $a_u = a_v$ and $b_u < b_v \leq b$
    The inequality $c(v) - c(u) \geq x$ follows from the definition of c.
  - $a_u = a_v$ and $b_u \leq b < b_v$
    We have $c(v) - c(u) \geq (c_\beta(v) + b + 1)x - b_u x \geq x$.
  - $a_u = a_v$ and $b < b_u < b_v$
    By the definition of c, we have $|c(v) - c(u)| = |c_\beta(v) - c_\beta(u)|x \geq x$.

- uv **is a** 1**-edge.**
  Clearly, $a_u < a_v$. If $a_u = a_v - 1$, then $b_u \leq b_v$. We establish that $|c(u) - c(v)| \geq 1$ by considering the next four subcases:

  - $a_u \leq a_v - 2$
    Observe that $c(u) \leq a_u + 1$ and $a_v \leq c(v)$. Since $a_u \leq a_v - 2$, we can immediately conclude that $c(v) - c(u) \geq 1$.
  - $a_u = a_v - 1$ and $b_u \leq b_v \leq b$
    The definition of c immediately yields that $c(v) - c(u) \geq 1$.
  - $a_u = a_v - 1$ and $b_u \leq b < b_v$
    By the definition of c, we have $c(v) - c(u) \geq 1 + (c_\beta(v) + b + 1)x - b_u x \geq 1$.
  - $a_u = a_v - 1$ and $b < b_u \leq b_v$
    We again inspect the definition of c: $c(v) - c(u) \geq 1 + (\beta + 1)x + [(c_\beta(v) + b + 1) - (c_\beta(u) + b + 1)]\, x \geq 1$.

We have shown that c is a proper labeling of G. Note that the maximum label assigned by c does not exceed $a + bx$. The inequality $\lambda_G(x) \leq a + bx$ for $x \in [0, \frac{1}{2\alpha\beta+\alpha+\beta+1}]$ readily follows. □

Before we prove Theorem 12, we observe the following proposition. Its statement can be verified by inspection of the proof of Theorem 4.

Proposition 11. *Let G be a (finite or infinite) λ-graph whose λ-function is finite. Furthermore, let $\mathcal{F}$ be the set of all linear functions $ax + b$ with* *integral non-negative coefficients a and b such that $a + b \leq \beta/\gamma$, where $\beta = \lambda_G(1)$ and γ is a real such that $\lambda_G(x)$ is linear on the interval $[0, \gamma]$. There exist sets $\mathcal{F}_1, \dots, \mathcal{F}_k \subseteq \mathcal{F}$ such that the following equality holds for all $x \in [\gamma, 1]$:*

$$\lambda_G(x) = \min_{i=1,\dots,k} \max_{f \in \mathcal{F}_i} f(x) .$$

Finally, we are ready to prove the main result of this section:

Theorem 12. *Let $\alpha \leq \beta$ be any two non-negative integers. The following estimate on the size of $\Lambda(\alpha, \beta)$ holds:*

$$|\Lambda(\alpha, \beta)| \leq 2^{2^{\frac{(2\alpha\beta^2+\alpha\beta+\beta^2+2)^2}{2}}} .$$

In particular, the set $\Lambda(\alpha, \beta)$ is finite.

Proof. Let $f_0 \in \Lambda(\alpha, \beta)$, i.e., there exists a λ-graph G with $\lambda_G(x) = f_0(x)$ and $f_0(0) = \alpha$ and $f_0(1) = \beta$. By Lemma 10, the function f_0 is a linear function of x on the interval $[0, \gamma]$ where $\gamma = \frac{1}{2\alpha\beta+\alpha+\beta+1}$. In particular, the values of f_0 on the interval $[0, \gamma]$ are uniquely determined by the value of $f_0(\gamma)$ (recall that $f_0(0) = \alpha$).

As in Proposition 11, let $\mathcal{F}$ be the set of all linear functions $ax + b$ with integral non-negative coefficients a and b such that $a + b \leq \beta/\gamma$. Let us estimate the cardinality of the set $\mathcal{F}$:

$$|\mathcal{F}| = \sum_{i=0}^{\lfloor \beta/\gamma \rfloor} (i + 1) \leq \frac{(\beta/\gamma + 2)^2}{2} = \frac{(2\alpha\beta^2 + \alpha\beta + \beta^2 + 2)^2}{2} \qquad (2)$$

By Proposition 11, there exist subsets $\mathcal{F}_1, \dots, \mathcal{F}_k \subseteq \mathcal{F}$ such that $f_0(x) = \min_{i=1,\dots,k} \max_{f \in \mathcal{F}_i} f(x)$ for all $x \in [\gamma, 1]$. Once the sets $\mathcal{F}_1, \dots, \mathcal{F}_k$ are fixed, the value $f_0(\gamma)$ is uniquely determined and thus the function f_0 is uniquely determined by $\mathcal{F}_1, \dots, \mathcal{F}_k$ on the entire interval $[0, 1]$. Since $\mathcal{F}$ contains $2^{|\mathcal{F}|}$ subsets, there are $2^{2^{|\mathcal{F}|}}$ choices of the subsets $\mathcal{F}_1, \dots, \mathcal{F}_k$. The statement of the theorem now follows from the estimate (2). $\square$

5 Convex λ-functions

In this section, we focus on λ-graphs with convex λ-functions. First, we show a simple upper bound on the number of linear parts of convex λ-functions of finite λ-graphs:

Theorem 13. *Let G be a finite λ-graph of order n. If the function $\lambda_G(x)$ is convex, then it consists of at most $3n^{2/3}+1$ linear parts.*

Proof. Let k be the number of linear parts of $\lambda_G(x)$ and let the reals $x_0,\ldots,x_k$ and the integers $a_1,\ldots,a_k$ and $b_1,\ldots,b_k$ be as in Corollary 6. Since the function $\lambda_G(x)$ is convex, $a_i > a_j$ and $b_i < b_j$ for every $1 \le i < j \le k$.

Let $\alpha_i = a_i - a_{i+1} > 0$ and $\beta_i = b_{i+1} - b_i > 0$ for $i = 1,\ldots,k-1$. In particular, $a_1 = a_k + \alpha_1 + \ldots + \alpha_{k-1}$ and $b_k = b_1 + \beta_1 + \ldots + \beta_{k-1}$. Note that $x_i = \alpha_i/\beta_i$ for all $i = 1,\ldots,k-1$. Let I_A be the set of the indices $i = 1,\ldots,k-1$ such that $\alpha_i \ge n^{1/3}$, and let I_B be the set of the indices $i = 1,\ldots,k-1$ such that $\beta_i \ge n^{1/3}$. Since $a_1 < n$ by Corollary 6, $|I_A| \le n^{2/3}$. Similarly, $|I_B| \le n^{2/3}$.

Let $I = \{1,\ldots,k\}\setminus(I_A \cup I_B)$. Each number $x_i = \alpha_i/\beta_i$ is a fraction with both the numerator and denominator between 1 and $n^{1/3}$. Since there are at most $n^{2/3}$ such distinct fractions, we infer that $|I| \le n^{2/3}$. Consequently, $k \le |I|+|I_A|+|I_B| \le 3n^{2/3}$. The statement of the theorem now follows. □

In the rest of this section, we construct λ-graphs whose convex λ-functions have $\Omega(n^{2/3})$ linear parts. The first step towards our construction is the next proposition. We leave its straightforward proof to the reader.

Proposition 14. *Let G be the λ-graph which is the disjoint union of a clique of order k_1 with 1-edges and a clique of order k_x with x-edges. If $k_x > k_1$, then the function $\lambda_G(x)$ consists of two linear parts meeting at the point k_1/k_x.*

The second tool is the next lemma on joins of λ-graphs:

Lemma 15. *Let G_1 and G_2 be two disjoint λ-graphs, and let $G = G_1 \oplus G_2$ be the λ-graph obtained from G_1 and G_2 by adding 1-edges v_1v_2 between any pair of vertices $v_1 \in V(G_1)$ and $v_2 \in V(G_2)$. The following holds:*

$$\lambda_G(x) = \lambda_{G_1}(x) + \lambda_{G_2}(x) + 1 \ .$$

Proof. Fix the number $x \in [0,1]$. By Theorem 1, G_1 and G_2 have finitary orientations $\vec{D_1}$ and $\vec{D_2}$ with the maximum weights of an oriented path equal to $\lambda_{G_1}(x)$ and $\lambda_{G_2}(x)$. Let $\vec{D}$ be the orientation of G obtained from $\vec{D_1}$ and $\vec{D_2}$ by orienting all the edges between G_1 and G_2 from G_1 to G_2. Clearly, the maximum weight of an oriented path in $\vec{D}$ is $\lambda_{G_1}(x) + \lambda_{G_2}(x) + 1$. By

Theorem 1, $\lambda_G(x) \leq \lambda_{G_1}(x) + \lambda_{G_2}(x) + 1$. In the next paragraph, we finish the proof of the lemma by establishing the opposite inequality.

Assume for contradiction that $\lambda_G(x) < \lambda_{G_1}(x) + \lambda_{G_2}(x) + 1$. By Theorem 1, G has a finitary orientation $\vec{D}$ with the maximum weight of an oriented path strictly less than $\lambda_{G_1}(x) + \lambda_{G_2}(x) + 1$. On the other hand, the orientation $\vec{D}$ restricted to G_1 contains an oriented path P_1 with weight at least $\lambda_{G_1}(x)$, and $\vec{D}$ restricted to G_2 contains a path P_2 with weight at least $\lambda_{G_2}(x)$. Let G' be the subgraph of G induced by the vertices of P_1 and P_2, and let $p = |V(G')|$. Note that the orientation $\vec{D}$ is acyclic and any two vertices of G' are connected by an oriented path, which implies that there is a unique way to order the vertices of G' into a sequence $S = (v_1, v_2, v_3, \ldots, v_p)$ which is topologically sorted with respect to $\vec{D}$, i.e., if $\vec{D}$ contains an edge oriented from v_i to v_j, then $i < j$. The uniqueness of S implies that for each $i < p$ the vertices v_i and v_{i+1} are connected by an oriented edge $v_i v_{i+1}$. Therefore, G' contains an oriented Hamilton path $P = v_1 v_2 \cdots v_p$. Furthermore, every x-edge of P is also an edge of P_1 or P_2, and thus the weight of P is at least $\lambda_{G_1}(x) + \lambda_{G_2}(x) + 1$. This contradicts our assumption that the weight of every oriented path in $\vec{D}$ is strictly smaller than $\lambda_{G_1}(x) + \lambda_{G_2}(x) + 1$. □

Finally, we recall some results on the Farey fractions. The Farey sequence is formed by sets F_n of rationals, where F_n is the set of all irreducible fractions a/b with $0 \leq a \leq b \leq n$, e.g., $F_4 = \{0, 1/4, 1/3, 1/2, 2/3, 3/4, 1\}$ (note that $1/2 = 2/4$). The Farey fractions appear, e.g., in [2, 6, 15]. For our considerations, the following result [9, 26, 27] on the Farey fractions is of interest:

$$\lim_{n\to\infty} \frac{|F_n|}{n^2} = \frac{3}{\pi^2} \tag{3}$$

We can now construct a λ-graph whose λ-function consists of $\Omega(n^{2/3})$ linear parts:

Theorem 16. *For every positive integer n, there is a λ-graph G of order n whose λ-function consists of $\frac{\sqrt[3]{3}}{(2\pi)^{2/3}} n^{2/3} - o(n^{2/3}) \approx 0.42 n^{2/3}$ linear parts.*

Proof. Fix a positive integer k. We construct a graph G of order at most $2k|F_k|$ whose λ-function consists of $|F_k| - 1$ linear parts. The statement of the theorem will then follow from the limit (3).

Let F_k° be the set of the Farey fractions from F_k strictly between 0 and 1. For each fraction $\frac{a}{b} \in F_k^\circ$, consider the graph $G_{a/b}$ from Proposition 14 with $k_1 = a$ and $k_x = b$. Note that there are $|F_k| - 2$ choices of a and b (we exclude the fractions 0 and 1). The λ-function of $G_{a/b}$ consists of two linear parts meeting at the point $\frac{a}{b}$.

Let G be the λ-graph obtained from vertex-disjoint copies of $G_{a/b}$, $\frac{a}{b} \in F_k^\circ$, by adding 1-edges between all pairs of vertices from distinct copies, i.e., $G = \bigoplus_{\frac{a}{b} \in F_k^\circ} G_{a/b}$. By Lemma 15, the λ-function of G is equal to the following:

$$\lambda_G(x) = |F_k^\circ| - 1 + \sum_{\frac{a}{b} \in F_k^\circ} \lambda_{G_{a/b}}(x).$$

Therefore, the function $\lambda_G(x)$ consists of $|F_k| - 1$ linear parts.

It remains to estimate the order of the λ-graph G. The order of every graph $G_{a/b}$ is at most $2k$. Hence, the order of G does not exceed $2k|F_k|$ as claimed in the beginning. $\square$

We remark that the multiplicative factors both in Theorems 13 and 16 can be improved by a finer analysis of the estimates used in the proofs. We decided not to do so in order to keep our arguments simple.

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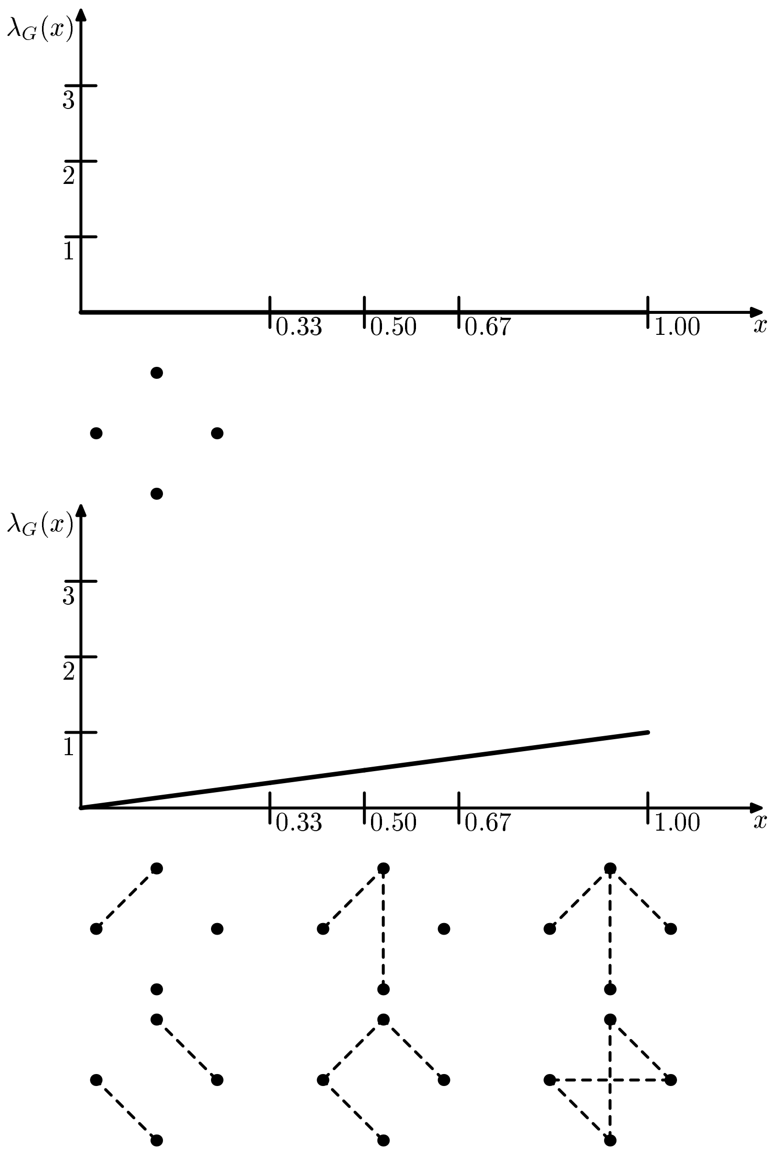
Appendix

All λ-graphs on four vertices

First, we list all non-isomorphic λ-graphs on four vertices together with their λ-functions. The λ-graphs corresponding to the depicted λ-function can be found under the graph of the function. The 1-edges are depicted as solid segments, while the x-edges are represented by dashed segments.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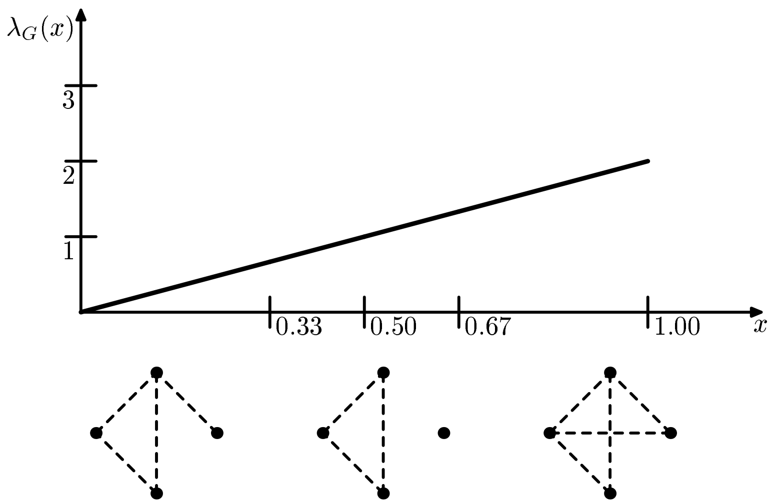
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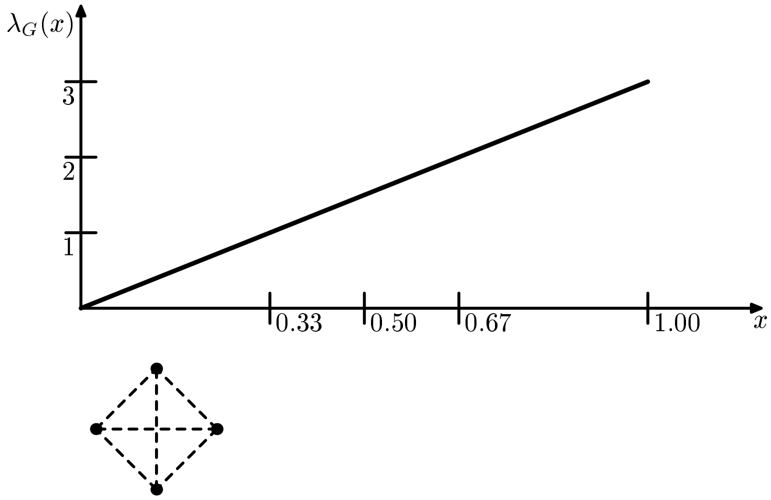
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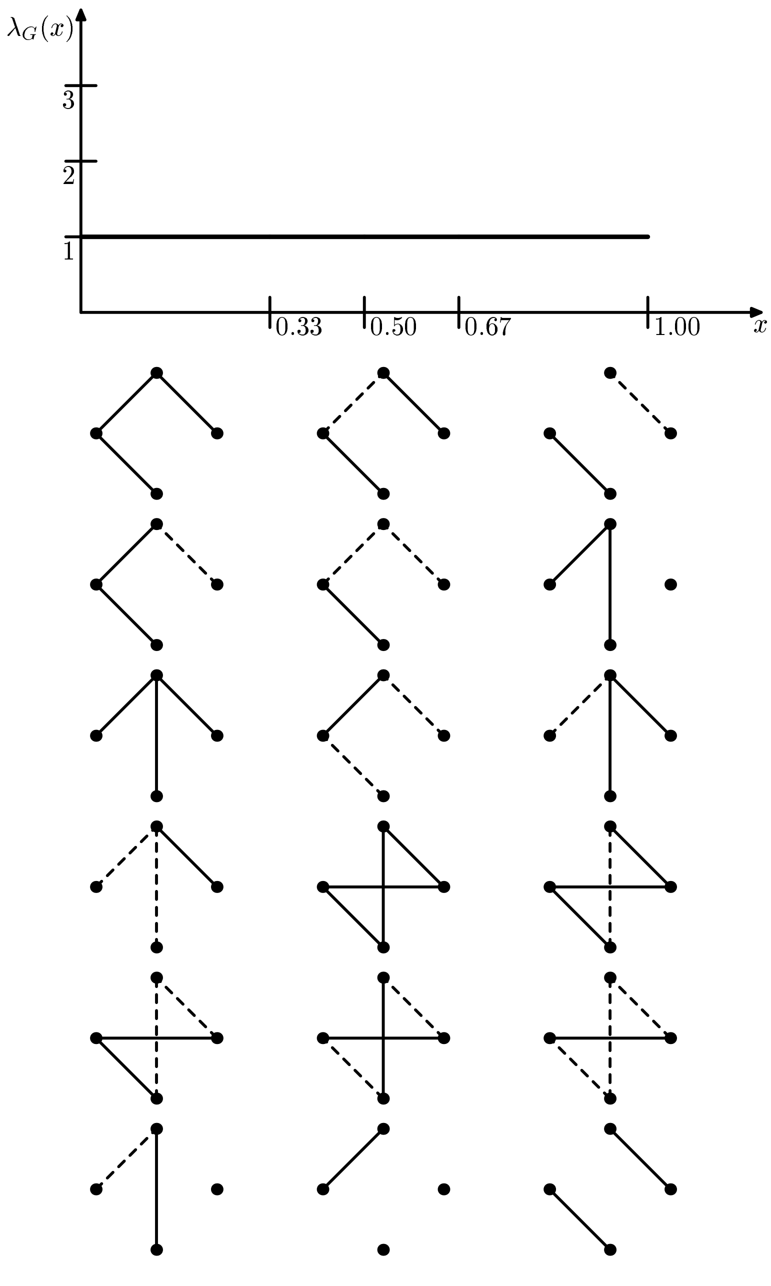
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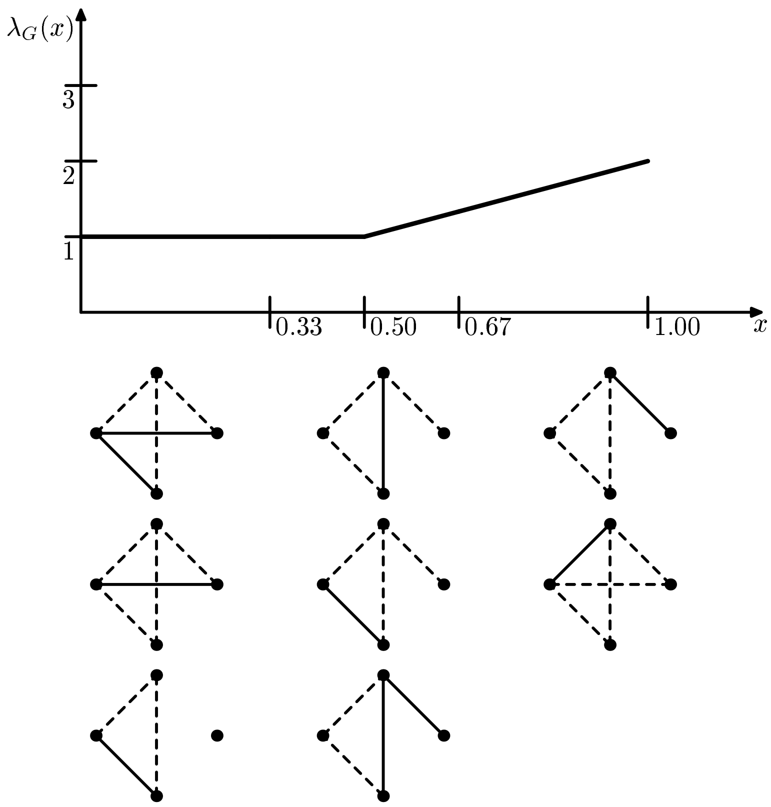
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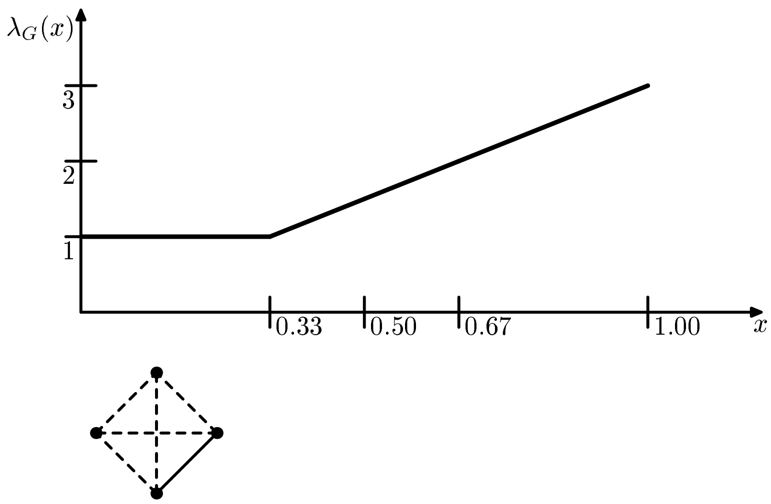

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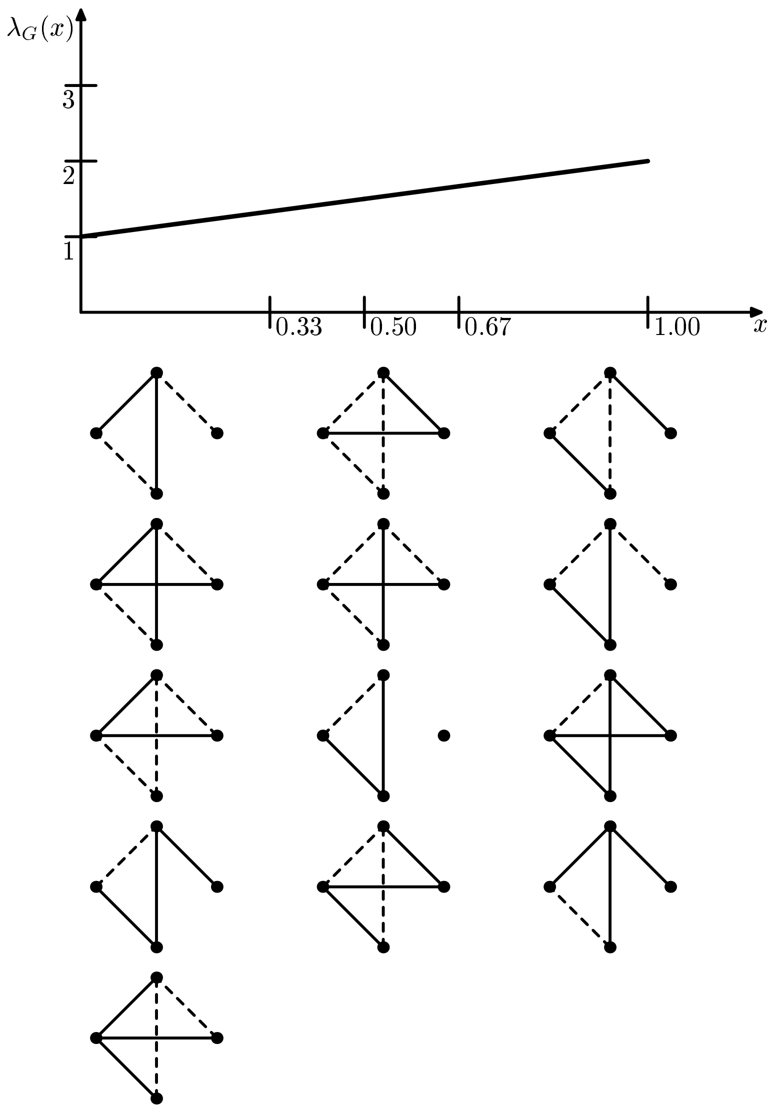

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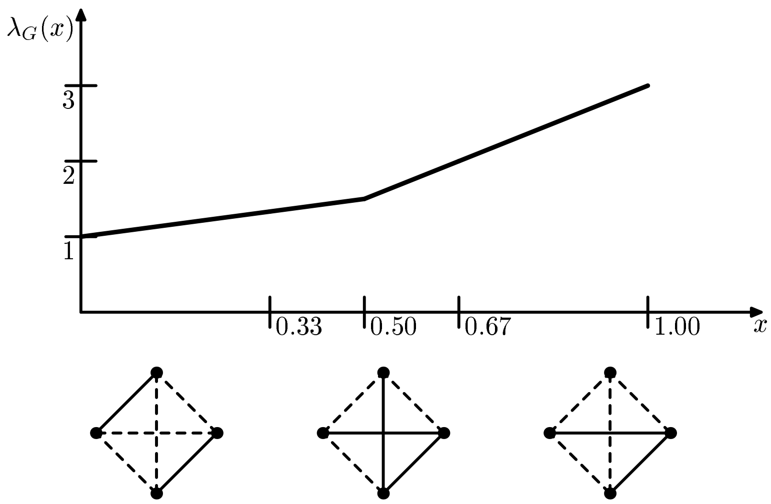
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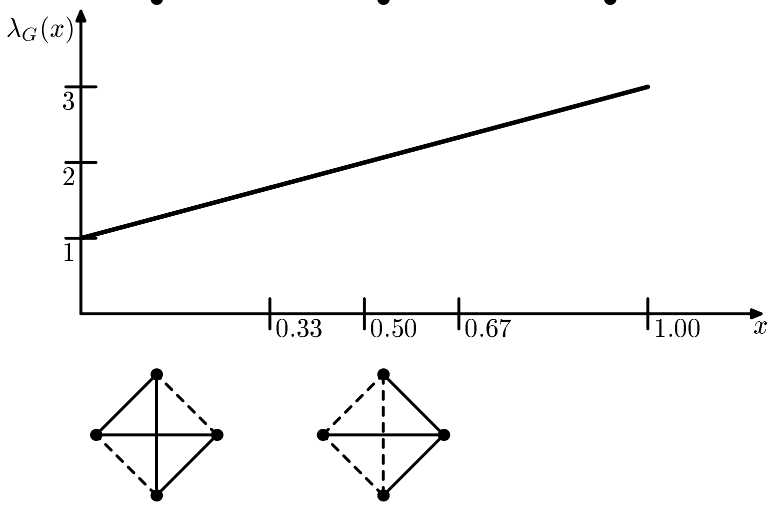
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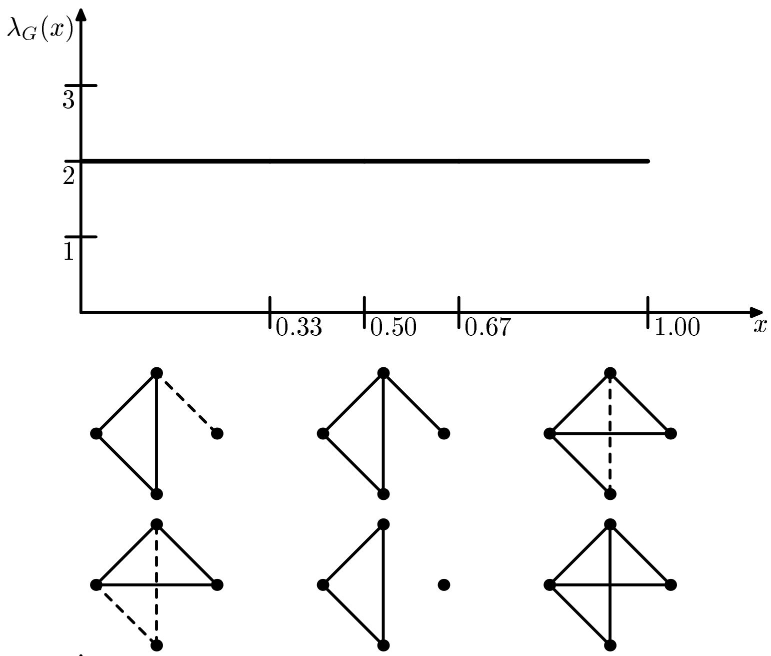
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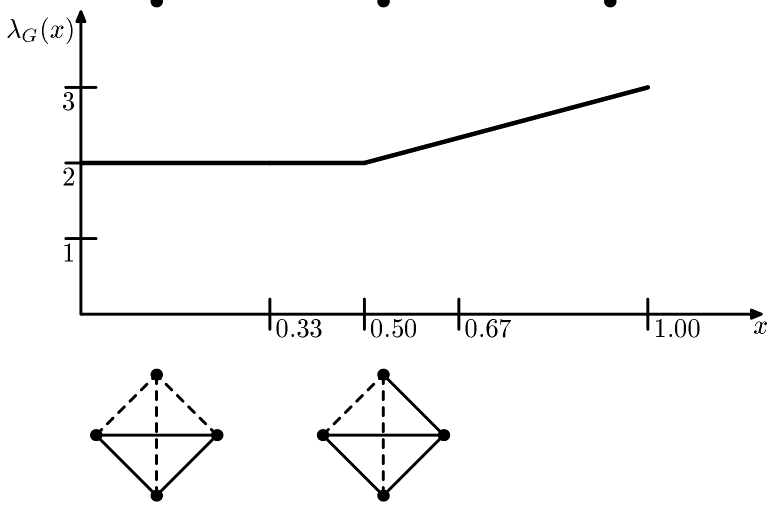
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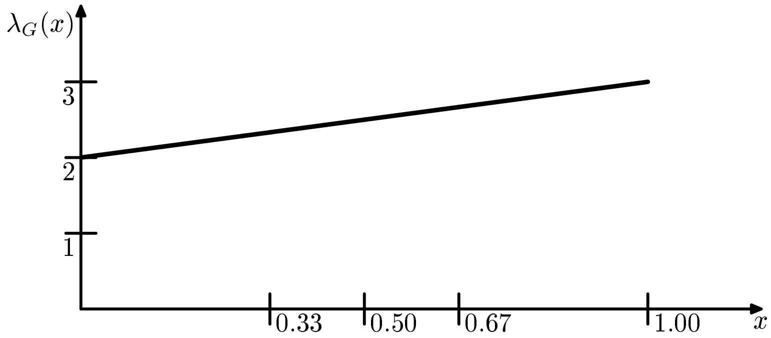
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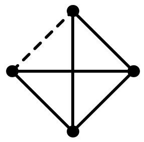

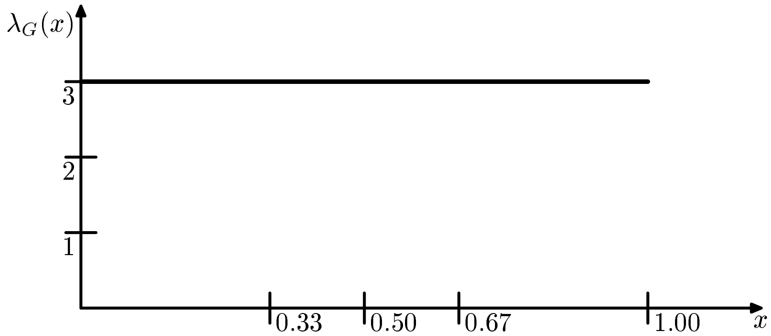
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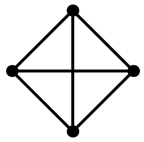
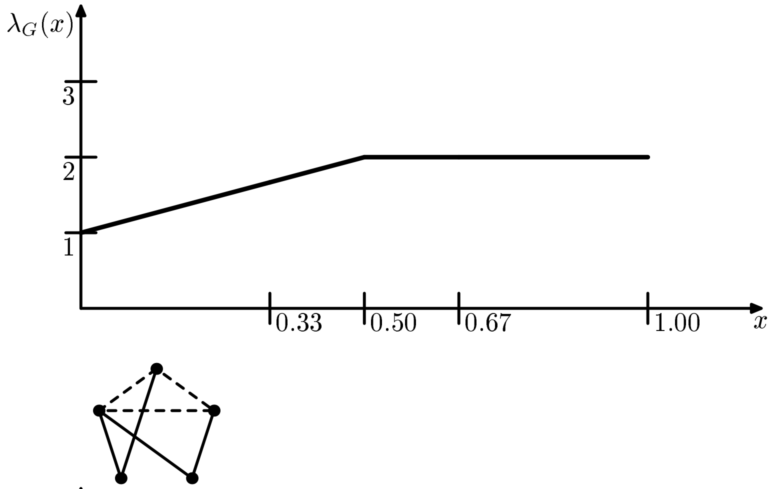
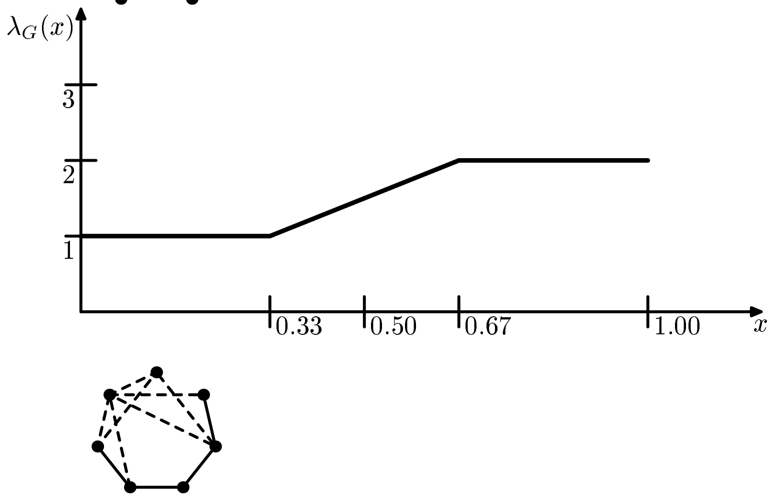
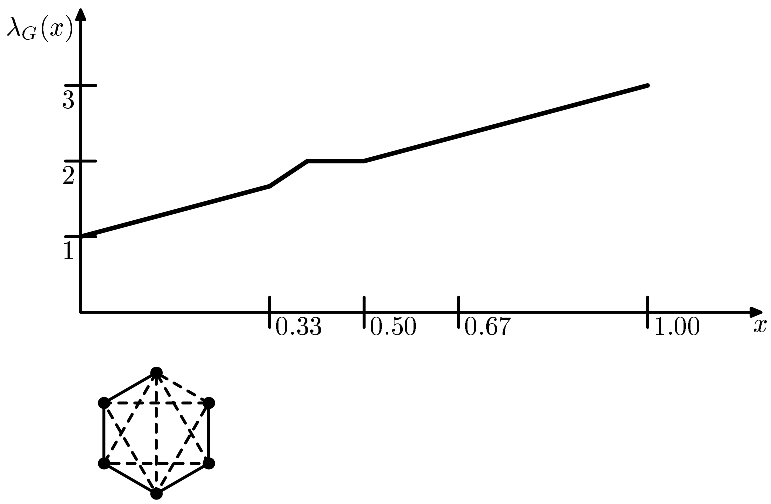


Other selected λ-graphs

We also list some other small λ-graphs with interesting λ-functions: the first one is an example of a λ-graph with a concave λ-function, the second one is an example of a λ-graph whose λ-function is neither convex nor concave (note that it even contains two different constant parts), and the third one is an example of a λ-graph such that two different linear parts of its λ-function correspond to the same linear function.






$\lambda_G(x)$
3
2
1
0.33
0.50
0.67
1.00
x