Cycles intersecting edge-cuts of prescribed sizes

Tomáš Kaiser*

Riste Škrekovski*†

Department of Mathematics
University of West Bohemia
Univerzitní 8
306 14 Plzeň
Czech Republic
kaisert@kma.zcu.cz

Department of Mathematics
University of Ljubljana
Jadranska 19
1111 Ljubljana
Slovenia
skreko@fmf.uni-lj.si

Institute for Theoretical
Computer Science (ITI)
Charles University
Praha, Czech Republic

Abstract

We prove that every cubic bridgeless graph G contains a 2-factor which intersects all (minimal) edge-cuts of size 3 or 4. This generalizes an earlier result of the authors, namely that such a 2-factor exists provided that G is planar. As a further extension, we show that every graph contains a cycle (a union of edge-disjoint circuits) that intersects all edge-cuts of size 3 or 4. Motivated by this result, we introduce the concept of a coverable set of integers and discuss a number of questions, some of which are related to classical problems of graph theory such as Tutte's 4-flow conjecture or the Dominating cycle conjecture.

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1 Introduction

We study the existence of cycles intersecting all edge-cuts of prescribed sizes in a graph. Throughout this paper, a *cycle* in a graph G is a union of edge-disjoint circuits and an *edge-cut* (in short, a *cut*) is an inclusionwise minimal set of edges whose removal increases the number of components of G. Our graphs are undirected and contain no loops, but they may contain parallel edges.

Our starting point is the main result of [7]:

Theorem 1.1 *For any planar graph G, there exists a (not necessarily proper) 2-coloring of $V(G)$ such that there is no monochromatic circuit of length 3 or 4.*

In an equivalent dual form, Theorem 1.1 states that every bridgeless planar cubic graph has a 2-factor intersecting all cuts of size 3 or 4. (A graph is *bridgeless* if it is connected and has no bridges.) In the present paper, we extend the latter result to all bridgeless cubic graphs. Furthermore, we lift the regularity assumption, proving the following:

Theorem 1.2 *Every graph G has a cycle intersecting all cuts of size 3 or 4.*

Motivated by this, we introduce the following concept. Let $\mathbb{N}$ be the set of positive integers and $A \subseteq \mathbb{N}$. We say that a cycle C in a graph G is *A-covering* if it intersects all cuts T with $|T| \in A$. If $\mathcal{Q}$ is a class of graphs, then A is *coverable in $\mathcal{Q}$* if every graph from $\mathcal{Q}$ contains an A-covering cycle. A set that is coverable in the class of all graphs is just said to be *coverable*.

Thus, an equivalent version of Theorem 1.2 is that the set $\{3, 4\}$ is coverable. Which other sets are coverable? $\mathbb{N}$ itself is not; clearly, a graph has an $\mathbb{N}$-covering cycle if and only if it has a spanning Eulerian subgraph (spanning closed trail), which is not the case, for instance, for the graph $K_{2,3}$ (or for any graph with a bridge). In fact, $K_{2,3}$ shows that even the set $\{2\}$ is not coverable.

For a less trivial example of a non-coverable set, consider $A = \{3, 5\}$ and the Petersen graph P_{10}. For any vertex v of P_{10}, the edges incident with v constitute a cut as P_{10} is 3-edge-connected. Since $3 \in A$, any A-covering cycle is a 2-factor. Every 2-factor F of P_{10} is formed by two circuits of length 5. The complement of F is a cut of size 5 that is not intersected by F. It follows that P_{10} has no A-covering cycle.

On the other hand, it may well be that the presence of P_{10} in a graph G (as a minor) is the only obstruction to the existence of a $\{3,5\}$-covering cycle in G. Recall that a graph H is a *minor* of a graph G if H can be obtained from G by a sequence of edge contractions and edge deletions. The graph G is *Petersen-minor-free* (or P_{10}*-free*) if P_{10} is not a minor of G. Petersen-minor-free graphs are the subject of the famous 4*-flow conjecture* of Tutte [13]. Since we will not need to go into the details of integer flows (which can be found in [14]), let us state Tutte's conjecture in a form that does not refer to 4-flows:

Conjecture 1.1 *The edges of any P_{10}-free bridgeless graph can be covered by two cycles.*

Observe that if $E(G)$ is covered by two cycles C_1 and C_2, then C_1 is $(2\mathbb{N}+1)$-covering, where $2\mathbb{N}+1 = \{3,5,7,\ldots\}$. Indeed, any odd cut not intersected by C_1 cannot be covered by C_2, for the intersection of a cycle and a cut has even size. Conversely, it is not difficult to prove that if G has a $(2\mathbb{N}+1)$-covering cycle, then $E(G)$ can be covered by two cycles. Thus, Conjecture 1.1 can equivalently be stated in terms of coverability:

Conjecture 1.2 *The set $2\mathbb{N}+1 = \{3,5,7,\ldots\}$ is coverable in the class of P_{10}-free graphs.*

Observe that Conjecture 1.2 is not restricted to bridgeless graphs. The reason is that any set $A \subseteq \mathbb{N}$ with $1 \notin A$ is coverable in the class of bridgeless graphs if and only if it is coverable in the class of all graphs.

Conjecture 1.1 is well known to be true for planar graphs. Indeed, this special case is equivalent to the Four Color Theorem (see, e.g., [14]). It follows that $2\mathbb{N}+1$ is coverable in the class of planar graphs.

To conclude this section, we point out a relation to another long-standing conjecture. A *dominating cycle* in G is a circuit C such that each edge of G is incident with a vertex of C. (Note that in our terminology, 'dominating circuit' would be a more appropriate term.) The *Dominating cycle conjecture* has several equivalent forms [3, 8, 11]; we state the one due to Fleischner and Jackson [5] (see Section 2 for a definition of cyclically k-connected graphs):

Conjecture 1.3 *Every cyclically 4-edge-connected cubic graph has a dominating cycle.*

By Tutte's theorem [12], Conjecture 1.3 is true for planar graphs. Note that if G is a cyclically 4-edge-connected cubic graph, then a circuit is a dominating cycle in G if and only if it is $(\mathbb{N}+3)$-covering, where $\mathbb{N}+3 = \{4, 5, 6, \ldots\}$. Thus, the following is a generalization of the Dominating cycle conjecture:

Conjecture 1.4 *The set* $\mathbb{N}+3$ *is coverable.*

A result of Thomassen [10, Theorem 4.1] implies that $\mathbb{N}+3$ is coverable in the class of planar graphs. Further questions related to coverable sets are asked in Section 6.

2 Notation and definitions

Let us review a few definitions. As mentioned above, all the graphs we consider are loopless multigraphs. The vertex set and the edge set of a graph G are denoted by $V(G)$ and $E(G)$, respectively. If $E = E(G)$, we write $E(v)$ for the set of edges incident with the vertex v. For a subset $X \subseteq V(G)$, let $G[X]$ denote the subgraph of G induced by the vertices of X. If V_1, V_2 are vertex-disjoint subsets of $V(G)$, let $[V_1, V_2]$ denote the set of edges of G with one endvertex in V_1 and the other endvertex in V_2. Recall that a *cut* in a connected graph G is a subset $C \subseteq E(G)$ such that $G - C$ is disconnected and C is minimal with this property. Note that $G - C$ has two components, say, G_1 and G_2, and that $C = [V(G_1), V(G_2)]$. If any of the graphs G_i consists of a single vertex, then C is a *trivial* cut; otherwise C is called *non-trivial*. Similarly, if any of the graphs G_i is a tree, then C is an *acyclic* cut; otherwise C is *cyclic*.

We write $G_1(C)$ (G_2(C), respectively) for the graph obtained from G by contracting all of X_2 (X_1, respectively) into a new vertex and removing any loops which arise. Note that in both graphs thus constructed, C corresponds to a trivial cut.

Let v be a vertex of degree 2 in a graph. *Suppressing* v consists in removing v along with the incident edges, and joining the former neighbors of v by an edge if they are distinct. Note that if v is incident with two parallel edges, then suppressing v is the same as removing it. To *contract* an edge means to identify its endvertices and remove all the resulting loops. By definition, the contraction of a subgraph is the contraction of all of its edges.

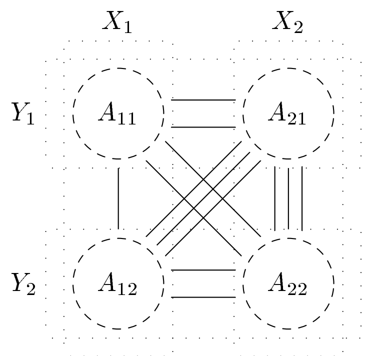

Figure 1: The regions A_{ij} corresponding to cuts $C_1 = [X_1, X_2]$ and $C_2 = [Y_1, Y_2]$.

A graph G is *cyclically k-edge-connected* if $|E(G)| > k$ and G has no cyclic cut of size at most $k-1$. A cycle H is *spanning* in G if each vertex of G is incident with an edge of H.

We refer to edge cuts of size k as *k-cuts*. Similarly, a set of size n is referred to as an *n-set*. A set of even size is an *even set*. The terms *n-subset* and *even subset* are defined in an analogous way.

3 Interlaced cuts

Before proving Theorem 1.2 (in Sections 4 and 5), we need to obtain some information on the possible configurations of cuts of size 3 and 4. Throughout this section, $C_1 = [X_1, X_2]$ and $C_2 = [Y_1, Y_2]$ are two cuts in a connected graph G. For $i, j \in \{1, 2\}$, set $A_{ij} = X_i \cap Y_j$. (See Figure 1 for an illustration.) The sets A_{ij} are called the *regions* corresponding to C_1 and C_2. We say that C_1 *interlaces* C_2 if each of $G[X_1]$ and $G[X_2]$ contains an edge of C_2. In this section, we study the interlacement relation for small cuts.

Proposition 3.1 *With the above definitions,*

$$C_1 \cap C_2 = [A_{11}, A_{22}] \cup [A_{12}, A_{21}].$$

Proof. An edge of C_1 joins a vertex of X_1 to a vertex of X_2, and an edge of C_2 joins a vertex of Y_1 to a vertex of Y_2. Thus, if $e \in C_1 \cap C_2$, then either e has one endvertex in $X_1 \cap Y_1 = A_{11}$ and the other one in $X_2 \cap Y_2 = A_{22}$, or else e has one endvertex in $X_1 \cap Y_2 = A_{12}$ and the other one in $X_2 \cap Y_1 = A_{21}$. $\square$

By the following proposition, C_1 interlaces C_2 if and only if C_2 interlaces C_1. If these equivalent conditions hold, then we say that C_1 and C_2 are *interlaced* or that they form an *interlacing pair*.

Proposition 3.2 *The following three claims are equivalent:*

(a) C_1 *interlaces* C_2*,*

(b) C_2 *interlaces* C_1*,*

(c) all the sets A_{ij} $(i, j \in \{1, 2\})$ *are non-empty.*

Proof. By symmetry, it is enough to prove that (a) $\Rightarrow$ (c) $\Rightarrow$ (b).

We first show that (a) $\Rightarrow$ (c). Since C_1 interlaces C_2, there exists an edge of C_2 with both endvertices in X_1. Notice that one of these two endvertices is in $X_1 \cap Y_1$ and the other one is in $X_1 \cap Y_2$. Thus, $A_{11} \neq \emptyset$ and $A_{12} \neq \emptyset$. Similarly, one can show that $A_{21} \neq \emptyset$ and $A_{22} \neq \emptyset$.

It remains to show that (c) $\Rightarrow$ (b). The graph G has an edge e_1 with one endvertex in A_{11} and the other one in A_{21}. Otherwise, the assumptions $A_{11} \neq \emptyset$ and $A_{21} \neq \emptyset$ imply that the set $C^* = [A_{11}, V(G) \backslash A_{11}]$ is non-empty and that it is a proper subset of C_2. Thus, C^* contradicts the minimality of C_2. Since $e_1 \in [A_{11}, A_{21}]$ belongs to Y_1, it follows that $e_1 \in C_1$ and $e_1 \in G[Y_1]$. Similarly, one can show that C_1 has an edge e_2 in $G[Y_2]$. Thus, C_2 interlaces C_1. $\square$

¿From the above two propositions, we easily obtain the following:

Proposition 3.3 *Suppose that the cuts* C_1 *and* C_2 *are interlaced. Then there exist edges* $e_1, e_2 \in C_1 \setminus C_2$ *and* $f_1, f_2 \in C_2 \setminus C_1$ *such that for each* $i \in \{1, 2\}$*, it holds that* $e_i \in [A_{1i}, A_{2i}]$ *and* $f_i \in [A_{i1}, A_{i2}]$. $\square$

Proposition 3.4 *Suppose that* G *is a bridgeless graph,* C_1 *is a 2-cut,* C_2 *is a cut of size* 3 *or* 4*, and the cuts* C_1 *and* C_2 *are interlaced. Then* $C_1 \cap C_2 = \emptyset$.

Proof. Apply Propositions 3.1 and 3.3. $\square$

Observe that by Propositions 3.1–3.3, all the possible mutual positions of the cuts C_1, C_2 from Proposition 3.4 are as shown in Figure 2.

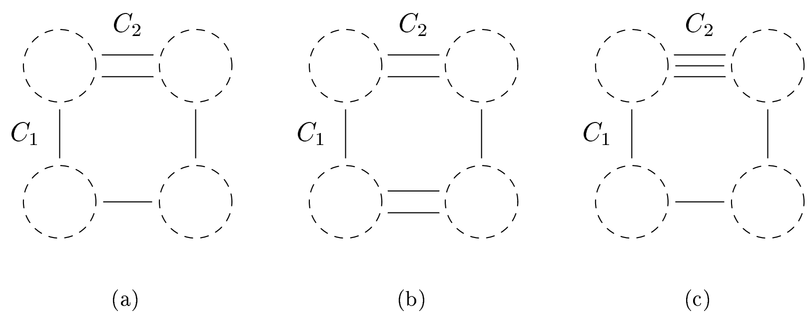

Figure 2: A 2-cut C_1 (vertical edges) interlaced with a cut C_2 of size 3 or 4 (horizontal edges).

Proposition 3.5 *If G is a 3-edge-connected graph, then it has no interlacing pair of 3-cuts.*

Proof. Suppose that C_1 and C_2 are two such cuts. Propositions 3.1–3.3 imply that one of the sets $A_{11}, A_{12}, A_{21}, A_{22}$ is connected to the rest of G by at most 2 edges. This contradicts the 3-edge-connectedness of G. □

Propositions 3.1 and 3.3 imply the following:

Proposition 3.6 *Suppose that G is a 3-edge-connected graph, C_1 is a 3-cut, C_2 is a 4-cut, and the cuts C_1 and C_2 are interlaced. Then $C_1 \cap C_2 = \emptyset$.* □

The structure of G with respect to the cuts C_1, C_2 from Proposition 3.6 is necessarily the one shown in Figure 3.

Proposition 3.7 *Suppose that G is a cyclically 4-edge-connected graph and C_1, C_2 are interlaced cyclic 4-cuts. Then $C_1 \cap C_2 = \emptyset$.*

Proof. Suppose that $C_1 \cap C_2 \neq \emptyset$. If the set $[A_{11}, A_{22}]$ contains at least two edges, then Proposition 3.3 implies that one of the sets A_{12}, A_{21} is connected to the rest of G by at most 2 edges; this contradicts the edge-connectivity assumption of G. Thus, $[A_{11}, A_{22}]$ and (by a symmetric argument) $[A_{12}, A_{21}]$ contain at most one edge each.

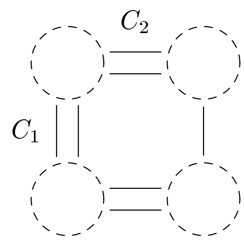

Figure 3: A 3-cut C_1 interlaced with a 4-cut C_2.

Suppose now that the set $[A_{12}, A_{21}]$ is empty. Then $[A_{11}, A_{22}]$ contains precisely one edge. Using the assumption that all 3-cuts are trivial, one can easily show that either one of the sets A_{ij} $(i, j \in \{1, 2\})$ is connected to the rest of the graph by at most 2 edges, or one of the subgraphs $G[X_i]$, $G[Y_i]$ $(i \in \{1, 2\})$, is isomorphic to K_2. The former possibility contradicts the edge-connectivity assumption on G, while the latter one contradicts the assumption that the cuts C_1, C_2 are cyclic.

By the above, each of the sets $[A_{11}, A_{22}]$ and $[A_{12}, A_{21}]$ contains precisely one edge. Proposition 3.3 implies that each A_{ij} is connected to the rest of the graph by exactly 3 edges. Since G is cyclically 4-edge-connected, it follows that each A_{ij} consists of a single vertex, and so G is isomorphic to K_4. Hence, the cuts C_1, C_2 are acyclic, a contradiction. □

The structure of G with respect to the cuts C_1, C_2 from Proposition 3.7 is as depicted in Figure 4.

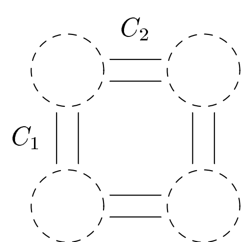

Figure 4: Two interlaced 4-cuts.

Proposition 3.8 *If C_1 and C_2 are distinct non-interlaced cuts with $C_1 \cap C_2 \neq \emptyset$, then precisely one of the sets A_{11}, A_{12}, A_{21}, A_{22} is empty.*

Proof. Note that by Proposition 3.2, at least one of the sets A_{ij} is empty. Since the cuts are distinct, no more than one of these sets can be empty. □

The structure of two distinct non-interlaced cuts with non-empty intersection is shown in Figure 5. Note that between any two of the 3 parts, there must be at least one edge.

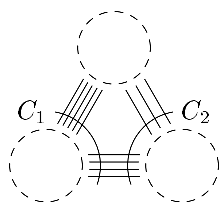

Figure 5: Non-interlaced cuts C_1 and C_2 with common edges.

4 Graphs with small degrees

By a well-known theorem of Petersen, every bridgeless cubic graph G has a 2-factor. In this section, we prove a result which implies that in fact, G has a 2-factor which is a $\{3,4\}$-covering cycle. To this end, we shall make use of the following extension of the Petersen theorem, due to Schönberger [9]:

Theorem 4.1 *Let G be a cubic bridgeless multigraph and $e, f \in E(G)$. Then G has a 2-factor containing both e and f.*

Let v be a vertex of degree 4 in a graph $G = (V, E)$. Let $Y \subset E(v)$ be a 2-set and let $X \subseteq E(v)$ be an even set. Note that if $|X| \neq 2$, then $X = E(v)$. We say that X *crosses* Y if $X \cap Y \neq \emptyset$ and $Y \neq X$. Thus, $E(v)$ crosses each of its 2-subsets. For example, if $E(v) = \{a, b, c, d\}$ and $Y = \{a, b\}$, then the even sets which cross Y are $\{a, c\}$, $\{a, d\}$, $\{b, c\}$, $\{b, d\}$ and $\{a, b, c, d\}$.

Let $w \in V(G)$. We say that a subgraph $H \subseteq G$ *extends* a set $X \subseteq E(w)$ if $E(H) \cap E(w) = X$.

For a partition E_1, E_2 of $E(w)$, let $G(w, E_1, E_2)$ denote the graph constructed from G by splitting w into two new adjacent vertices w_1 and w_2 so that $E(w_1) = E_1 \cup \{w_1 w_2\}$ and $E(w_2) = E_2 \cup \{w_1 w_2\}$. Notice that G can be obtained from $G(w, E_1, E_2)$ by contracting the edge $w_1 w_2$. Furthermore, any cut in G is a cut in $G(w, E_1, E_2)$. Thus, a good cycle in $G(w, E_1, E_2)$ gives rise to a good cycle in G.

If C is a cut in a graph G, then we identify C with the corresponding edge-sets in $G_1(C)$ and $G_2(C)$. We say that a subgraph F_1 of $G_1(C)$ and a subgraph F_2 of $G_2(C)$ *agree on* C if they contain precisely the same edges of C (with respect to this identification). For such a pair of subgraphs, $F_1 \cup F_2$ denotes the subgraph of G consisting of all edges corresponding to those in F_1 or in F_2. For brevity, we call a cycle *good* if it is $\{3, 4\}$-covering.

Lemma 4.2 *Let C be a cut of size at most 3 in a bridgeless graph G. If each graph $G_i(C)$ $(i = 1, 2)$ has a good cycle F_i such that F_1 and F_2 agree on C, then $F_1 \cup F_2$ is a good cycle of G.*

Proof. Clearly, $F_1 \cup F_2$ is a spanning cycle of G. We need to verify that it intersects each cut of size 3 or 4 in G. Let D be such a cut. Let A_{ij} $(i, j \in \{1, 2\})$ be the regions corresponding to the cuts $C_1 = C$ and $C_2 = D$ in G as defined at the beginning of this section. If C and D are not interlaced, then D corresponds to a cut D' of the same size in some $G_i(C)$ (in $G_1(C)$, say). Since F_1 is a good cycle, it intersects D', which implies that $F_1 \cup F_2$ intersects D.

We can thus assume that C and D are interlaced. The structure of G with respect to C and D is shown in Figure 2 or 3. Observe that in each of the possible cases, there is a region (say, A_{11}) incident with a single edge of C and 2 or 3 edges of D. Let R be the set of edges of $G_1(C)$ with exactly one endvertex in A_{11}. Since R is a cut of size 3 or 4 in $G_1(C)$, it is intersected by F_1. As $|R \cap C| = 1$, F_1 must use at least one edge of D. It follows that $F_1 \cup F_2$ intersects D. □

The following theorem deals with $\{3, 4\}$-covering cycles in graphs with maximum degree at most 4, the focus being on cubic graphs where any such cycle is a 2-factor.

Theorem 4.3 *Let G be a 2-connected graph and let v be a vertex of G. Assume that v is of degree at most 4 and all the other vertices of G are of degree at most 3.*

(a) *If v is of degree at most 3, then each 2-set $Y \subset E(v)$ can be extended to a good cycle of G.*

(b) *If v is of degree 4, then there exists a 2-set $X \subset E(v)$ such that every even set $Y \subseteq E(v)$ which crosses X can be extended to a good cycle of G.*

Before proving Theorem 4.3, let us consider an example. Let v be a vertex of degree 4 and write $E(v) = \{a, b, c, d\}$ and $Y = \{a, b\}$. Part (b) of the theorem claims that each of the sets $\{a, c\}$, $\{a, d\}$, $\{b, c\}$, $\{b, d\}$, and $\{a, b, c, d\}$ can be extended to a $\{3, 4\}$-covering cycle of G.

Proof of Theorem 4.3. Suppose that the theorem is false and G is a counterexample with the minimum number of vertices. Since G is 2-connected, $|V(G)| > 2$ and G has no vertices of degree 1. In a series of claims, we show that G is cyclically 5-edge-connected.

Whenever we consider a cut $C = [X_1, X_2]$ and the graphs $G_1(C)$ and $G_2(C)$, we write v_1 for the vertex of $G_2(C)$ obtained by contracting X_1. Similarly, we let v_2 denote the vertex of $G_1(C)$ obtained by contracting X_2.

Claim 1 *The graph G contains no vertex of degree 2.*

If $w \neq v$ is a vertex of degree 2, then use the induction hypothesis to find a good cycle in the graph obtained by suppressing w and note that this yields a suitable good cycle in G. We may thus assume that the only vertex of degree 2 is v. Let z be a neighbor of v. Since G is 2-connected, v and z are joined by a single edge. Consider the graph G' obtained by suppressing v and let e be the newly created edge. The degree of z in G' is 3. Let $Y \subset E(G')$ be a set consisting of e and one other edge incident with z. Use the induction hypothesis to find a good cycle of G' extending Y. This yields a suitable good cycle of G.

Claim 2 *The degree of v is 4.*

Assume that v has degree 3. Let $Y = \{a_1, a_2\} \subset E(v)$ be the given 2-set, and let b be the edge in $E(v) \setminus Y$. Let w be the endvertex of a_1 distinct from v. If the edges a_1, a_2 are parallel, then since G is bridgeless, $|V(G)| > 2$ and the degree of w is 3, there is a unique edge $c \in E(w) \setminus Y$. This implies that $\{b, c\}$ is a 2-cut. Let G' be the graph obtained from $G \setminus \{v, w\}$ by adding an edge e' joining the endvertex $v' \neq v$ of b to the endvertex $w' \neq w$ of c. The edge e' is not a loop as otherwise G would contain a bridge incident with v'.

Note that the degree of v' in G' is 3. Let Y' be the set of edges incident with v' distinct from e'. By the induction hypothesis (in which v' plays the role of v), G' has a good cycle F' that extends Y' and, therefore, does not contain e'. Since any cut of size 3 or 4 in G corresponds to a cut of size 3 or 4 in G' (although not necessarily a cut of the same size), it follows that adding the edges a_1 and a_2 to F', we obtain a good cycle of G extending Y.

It remains to consider the case that a_1 and a_2 are not parallel. Let G'' be the graph obtained by contracting a_1. Note that G'' contains a unique vertex v'' of degree 4. Write E'' for the set of edges incident with v''. By the induction hypothesis (applied to v''), find a 2-set $X'' \subset E''$ with the property stated in the theorem. Since a subset of E'' crosses X'' if and only if it crosses its complement in E'', we may assume that $a_2 \in X''$. Observe that there are exactly two 2-subsets of E'' containing a_2 and crossing X''. It follows that one of them, call it Y'', is different from $\{a_2, b\}$. By the assumption, Y'' can be extended to a good cycle F'' of G''. The corresponding edges in G, together with a_1, comprise a good cycle of G.

Claim 3 *The graph G is cyclically 4-edge-connected (hence, 3-edge-connected).*

Assume the claim false. Since G is bridgeless, there is a cut C that is either a 2-cut or a cyclic 3-cut. Consider the graphs $G_1(C)$ and $G_2(C)$. We may assume that $v \in V(G_1(C))$. Let E_1 be the set of edges of $G_1(C)$ incident with v. By the minimality of G, there is a 2-set $X \subset E_1$ such that any even set $Y \subset E_1$ crossing X can be extended to a good cycle of $G_1(C)$. We assert that X, as a subset of $E(G)$, has the property stated in the theorem.

Indeed, let $Y \subset E(v) \subset E(G)$ be an even set crossing X. In $G_1(C)$, Y can be extended to a good cycle F_1. Let Y_2 be the set of edges of $F_1 \subset G_1(C)$ containing the vertex v_2 of $G_1(C)$ (recall that this is the vertex representing a contracted component of $G-C$). Since $|Y_2| = 2$, we can use the minimality of G to extend Y_2 to a good cycle F_2 of $G_2(C)$. The cycles F_1 and F_2 agree on C. By Lemma 4.2, $F_1 \cup F_2$ is a good cycle of G.

Claim 4 *Every cyclic 4-cut is interlaced with some other 4-cut.*

Suppose that the claim is false. We may thus assume that a cyclic 4-cut $C = \{a, b, c, d\}$ is not interlaced with any other 4-cut. It follows that every cyclic 4-cut corresponds to a 4-cut in either $G_1(C)$ or $G_2(C)$. Clearly, a trivial cut in G corresponds to a cut of the same size in G_1 or G_2. Since, by Claim 3, G has no nontrivial 3-cuts, we conclude that

> (*) *Each cut in G of size 3 or 4 corresponds to a cut of the same size in G_1 or G_2.*

Note that $G_1(C)$ and $G_2(C)$ are bridgeless and v_2 and v_1 are vertices of degree 4.

We may assume that $v \in V(G_1)$. Let $E_i = E(G_i)$ $(i = 1, 2)$. By the minimality of G, there is a 2-set $X_2 \subset E_2(v_1)$ such that any even subset of $E_2(v_1)$ crossing X_2 can be extended to a good cycle of G_2.

Let $G_1^* = G_1(v_2, X_2, C \setminus X_2)$ be the result of the splitting operation defined at the beginning of this section. Thus, v_2 is split into two adjacent vertices. Let us write v_2^1 for the vertex incident with the edges in X_2 and v_2^2 for the other vertex. Furthermore, we let E_1^* denote the set $E(G_1^*)$. Note that G_1^* has fewer vertices than G. By the minimality of G, G_1^* contains a 2-set $X_1 \subset E_1^*(v)$ such that any even set $Y_1 \subset E_1^*(v)$ crossing X_1 can be extended to a good cycle of G_1. We claim that $X = X_1$ satisfies the analogous condition for G in place of G_1. Let $Y \subset E(v)$ be an even set. Extend the corresponding set of edges of G_1^* to a good cycle F_1^* of G_1^*. This determines a good cycle F_1 of G_1. Consider the set Y_2 of edges of $E_2(v_1)$ corresponding to the edges in $F_1 \cap C$.

Due to the way we split v_2, the even set Y_2 crosses X_2, for otherwise F_1^* would not pass through either v_2^1 or v_2^2. By the choice of X_2, Y_2 can be extended to a good cycle F_2 of G_2. The good cycles F_1 and F_2 agree on C. By (*), $F_1 \cup F_2$ is a good cycle of G.

Claim 5 *The graph G is cyclically 5-edge-connected.*

Assume, to the contrary, that G contains a cyclic 4-cut C. By Claim 4, C is interlaced with a 4-cut C'. The structure of G with respect to C and C' is shown in Figure 4. Let $\mathcal{E} = (E_0, E_1, \ldots, E_{k-1})$ be an inclusion-maximal collection of pairwise disjoint 2-subsets of $E(G)$ with the following properties:

(1) $C = E_r \cup E_s$ and $C' = E_{r'} \cup E_{s'}$ for some $r, s, r', s' \in \{0, 1 \ldots, k-1\}$,

(2) $E_i \cup E_j$ is a 4-cut for all distinct $i, j \in \{0, 1 \ldots, k-1\}$.

Notice that $k \geq 4$ (since r, s, r' and s' are all distinct). Let E^* be the union of all the sets E_i. We may assume that $E_0, E_1, \ldots E_{k-1}$ are enumerated in such a way that for each $j \in \{0, 1, \ldots, k-1\}$, the graph $G - (E_j \cup E_{j+1})$ has a component A_j containing no edge from E^* (the indices being taken modulo k). Clearly, the structure of G with respect to the sets E_i is "cyclic" as shown in Figure 6.

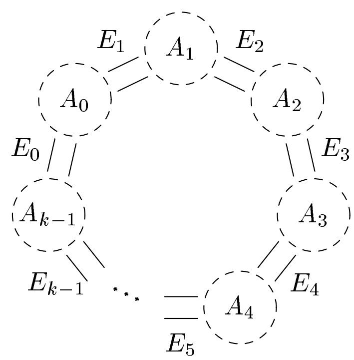

Figure 6: The structure of G with respect to the sets E_i.

We claim that for each j, $E_j \cup E_{j+1}$ is an acyclic 4-cut. Assume not. By Claim 4, $E_j \cup E_{j+1}$ is interlaced with a 4-cut C^*. From Figure 4, one sees that the subgraph A_j contains a 2-cut $E' = C^* \cap E(A_i)$ such that $E' \cup E_j$ and $E' \cup E_{j+1}$ are 4-cuts. Since the collection $(E', E_0, \ldots, E_{k-1})$ satisfies (1) and (2) above, we obtain a contradiction with the maximality of k.

Since each $E_j \cup E_{j+1}$ is an acyclic cut, each A_j is either a single vertex or a copy of K_2. If A_j is a single vertex, then this vertex is v and it is adjacent to each of the four vertices of $A_{j-1} \cup A_j$. In the other case, each of the two vertices $\{a_j, a'_j\}$ of A_j is adjacent to a vertex from A_{j-1} and a vertex from A_{j+1}. Finally, a_j and a'_j have no common neighbor except possibly for v. Thus, we have completely determined the structure of G. It can be described as follows. Let B_k be the k-prism (the Cartesian product of a cycle C_k with K_2). There are two copies of C_k in B_k; let $a_0, \ldots, a_{k-1}$ be the vertices of one copy and $b_0, \ldots, b_{k-1}$ be the vertices of the other copy, with a_i adjacent to b_i in B_k for each i. By the above, G is (up to isomorphism) the graph obtained from B_k by contracting the edge a_0b_0 (see Figure 7).

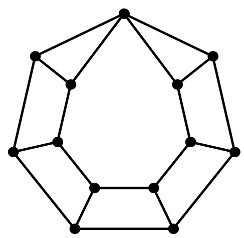

Figure 7: The prism $C_7 \square K_2$ with a contracted edge.

Let us determine which even subsets of $E(v)$ can be extended to a good cycle. Let $Y \subseteq E(v)$ be an even subset. Firstly, if $Y = E(v)$, then the cycle consisting of all the edges $a_i a_{i+1}$ and $b_i b_{i+1}$ (indices modulo k) is good and extends Y. A good cycle extending $\{va_1, vb_1\}$ is $va_1a_2 \dots a_{k-1}b_{k-1}b_k \dots b_1v$, while a good cycle extending $\{va_{k-1}, vb_{k-1}\}$ is obtained in a symmetric manner. The remaining cases depend on the parity of k.

Assume that k is odd and $Y = \{va_1, va_{k-1}\}$. A good cycle extending Y is $va_1b_1b_2a_2 \dots b_{k-1}a_{k-1}v$. The case $Y = \{vb_1, vb_{k-1}\}$ is symmetric. Thus, if we define $X = \{va_1, vb_{k-1}\}$, then any even subset of $E(v)$ crossing X can be extended to a good cycle.

If k is even, then $Y = \{va_1, vb_{k-1}\}$ extends to the good cycle $va_1b_1b_2a_2 \dots a_{k-1}b_{k-1}v$, and the case $Y = \{vb_1, va_{k-1}\}$ is symmetric again. We put $X = \{vb_1, vb_{k-1}\}$ and observe that, as above, every even subset of $E(v)$ that crosses X extends to a good cycle. This contradiction with the assumption that G is a counterexample establishes Claim 5.

By Claim 5, each cut C of size 3 or 4 in G is acyclic. Thus, one component of $G - C$ is a copy of K_2 (if $|C| = 4$) or a single vertex (if $|C| = 3$).

Let $E(v) = \{a, b, c, d\}$. If every even subset Y' of $E(v)$ can be extended to a good cycle of G, then we are done. Therefore, suppose that for some Y', this is not the case.

If $Y' = E(v)$, then consider the cubic bridgeless graph $G(v, \{a, b\}, \{c, d\})$. By Theorem 4.1, $\{a, b\}$ can be extended to a 2-factor F of $G(v, \{a, b\}, \{c, d\})$. Note that F contains c and d. Thus, F corresponds to a good cycle of G that contains all the edges of Y', a contradiction.

We may therefore assume that Y' is a 2-subset of $E(v)$, say $Y' = \{a, b\}$, and define $X = Y'$. Let Y be an even subset of $E(v)$ which crosses X. Since we have already observed that $E(v)$ extends to a good cycle of G, it may be assumed that $|Y| = 2$, say (without loss) $Y = \{a, d\}$. Consider the graph $G(v, \{a, c\}, \{b, d\})$. Let v_1, v_2 be the adjacent vertices into which v is split. Using Theorem 4.1, extend $\{a, v_1v_2\}$ to a 2-factor F of $G(v, \{a, c\}, \{b, d\})$. Notice that F contains precisely one of the edges b, d. It cannot be b, for otherwise we obtain a good cycle of G with $F(v) = X$, and this contradicts the choice of X. Thus, F must contain d. Since $F(v) = \{a, d\}$ and F is a good cycle of G, we obtain the right extension of Y. This establishes the theorem. □

Corollary 4.4 *Every bridgeless graph with maximum degree at most* 3 *has a* $\{3, 4\}$*-covering cycle.*

Theorem 4.3 implies the following strengthening of Theorem 4.1:

Corollary 4.5 *Every cubic bridgeless graph has a* 2*-factor which intersects all cuts of size* 3 *and* 4*. Moreover, any two incident edges can be extended to such a* 2*-factor.*

5 $\{3, 4\}$-covering cycles in arbitrary graphs

In this section, we generalize Corollary 4.4 to graphs with unrestricted degrees which, moreover, do not need to be bridgeless.

Let v be a vertex of a connected graph G and let E_1, E_2 be a partition of $E(v)$. Recall that the graph $G(v, E_1, E_2)$ was defined at the beginning of Section 4. The following is an easy consequence of Fleischner's Splitting Lemma [4].

Lemma 5.1 *Let* G *be a bridgeless graph and let* $v \in V(G)$ *be of degree* ≥ 4*. Then there is a partition* E_1, E_2 *of* $E(v)$ *with* $|E_1|, |E_2| \geq 2$ *such that* $G(v, E_1, E_2)$ *is bridgeless.* □

With this lemma at hand, we proceed to prove Theorem 1.2.

Proof of Theorem 1.2. For a graph H, let $\Delta(H)$ denote the maximum degree of H and let $V_\Delta(H)$ be the set of vertices of degree $\Delta(H)$. Let G be a counterexample to Theorem 1.2 chosen so that the triple $(\Delta(G), |V_\Delta(G)|, |V(G)|)$ is minimal in the lexicographic ordering.

If G contains a bridge e, then let G_1 and G_2 be the components of $G-e$. By the minimality of G, each G_i has a $\{3,4\}$-covering cycle C_i. Since $\{e\}$ is the only cut in G containing e, every cut in G, except for $\{e\}$, is a cut in some G_i. It follows that the cycle $C_1 \cup C_2$ is $\{3,4\}$-covering in G. Thus, G is bridgeless.

By Corollary 4.4, G contains a vertex v of degree at least 4. Using Lemma 5.1, split v into two vertices to obtain a bridgeless graph G^*. Every cut of G, apart from $E(v)$, corresponds to a cut (of the same size) in G^*. Thus, a $\{3,4\}$-covering cycle in G^* (which exists by the minimality of G) induces a $\{3,4\}$-covering cycle in G, a contradiction. □

6 Concluding remarks

By Theorem 1.2, both the sets $\{3\}$ and $\{4\}$ are coverable. On the other hand, $\{1\}$ and $\{2\}$ are not. How about the other single-element sets?

Question 6.1 *Is it true that for all $k \geq 3$, $\{k\}$ is coverable?*

We are unable to say anything for $k \geq 5$, except that Conjecture 1.4 clearly implies an affirmative answer to this question. On the other hand, since (as we noted in Section 1) the conjecture is true for planar graphs, any set $\{k\}$ is coverable in the class of planar graphs (which will be denoted by $\mathcal{P}$ throughout this section).

Having determined which sets of size 1 are coverable in $\mathcal{P}$, we may attempt the same for sets of size 2. Let $A = \{a,b\}$ be a pair of positive integers with $a < b$. If $a \leq 2$ then A is not coverable in $\mathcal{P}$, and if $a \geq 4$, then the planar case of Conjecture 1.4 implies that A is coverable in $\mathcal{P}$. Thus, we may assume that $a = 3$. Since the set $2\mathbb{N}+1 = \{3,5,7,\ldots\}$ is coverable in $\mathcal{P}$ (by the Four Color Theorem), we may assume that b is even and $b \geq 6$.

Question 6.2 *Let $k \geq 3$. Is $\{3,2k\}$ coverable in $\mathcal{P}$?*

In fact, this is an equivalent form of a question posed by Broersma et al. [2] in connection with Theorem 1.1: For which $k \geq 3$ can one 2-color the vertices of every planar graph in such a way that there is no monochromatic circuit of length 3 or $2k$? To our knowledge, the question is open. It may even be that $\{3,2k\}$ $(k \geq 3)$ is coverable in the class of all graphs.

One might speculate that even the set consisting of 3 and all the numbers $2k$ $(k \geq 2)$ is coverable. We show that this is not the case:

Proposition 6.1 *The set* $A = \{3, 4, 6, 8, 10, \ldots\}$ *is not coverable.*

Proof. Let G be a 3-connected, non-hamiltonian, cubic bipartite graph (which exists by a result of J. D. Horton, see [1]). Let C be an A-covering cycle in G. As $3 \in A$, C is a 2-factor. Since C has more than one component, there exists a cut K contained in the complement of C. Let G_1 denote a component of $G \setminus K$. The size of K must be odd since $|K| \neq 2$ and C is A-covering. Thus, the number of vertices of G_1 is odd. However, C covers the vertices of G_1 by disjoint circuits of even length, a contradiction. □

We remark that in the class $\mathcal{P}$, the above argument does not apply, for there is no known example of a 3-connected planar cubic bipartite graph which is not hamiltonian. Indeed, a well-known conjecture of D. Barnette [6, Section 2.12] states that there is no such graph. Thus, we conclude our paper with the following question:

Question 6.3 *Is* $\{3, 4, 6, 8, 10, \ldots\}$ *coverable in* $\mathcal{P}$*?*

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