Free Planar Graphs on Torus: examining triconnected graphs for unbounded augmentability

Dainis ZEPS *†

Abstract

Free toroidal graphs are examined whether they may be augmented unboundedly retaining freeness. Two types of augment are distinguished where both so called punkah and ladder augments applied unboundedly can retain freeness for graphs that are not projective planar.

Most of definitions of the topological graph theory are the same as in [1].

Let us denote by A_{Pl}, A_P, A_T classes of graphs on plane, projective plane and torus respectively. Of course these are minor closed classes. Let us say that a graph is *free-planar* with respect to surface S if it itself and augmented with arbitrary edge can be embedded in S. There are only two non-trivial triconnected free-planar graphs[one graph - envelope graph and one graph class: wheel graphs] on sphere that are shown in fig.1, where only the second one[class], $W_k, k > 2$ can be unboundedly augmented with a *punkah* like augment retaining its triconnectivity and freeness on the sphere. First one, the envelope graph may be augmented with ladder like augment only when it is on Mobius strip.

Further we are going to examine triconnected graphs from class $Free(A_T)$ for unbounded augmentability.

Let two auxiliary graphs (see fig.2) be called *punkah-graph* and *ladder-graph*, in punkah-graph a vertex being connected with vertices on subdivided edge adding new edges, but in ladder-graph vertices on two subdivided edges coupled connected adding new edges.

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†Author's address: Institute of Mathematics and Computer Science, University of Latvia, 29 Rainis blvd., Riga, Latvia. dainize@mii.lu.lv

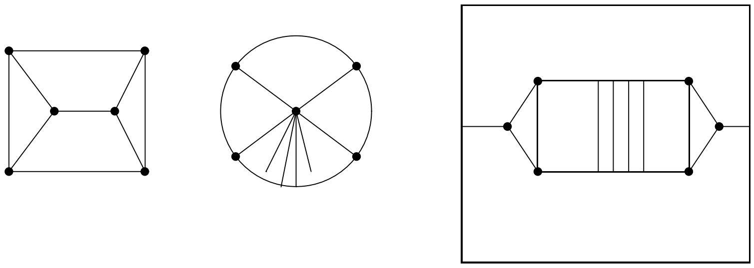


Figure 1: The only possible triconnected free planar graphs on sphere: a) envelope graph can not be augmented retaining triconnectivity and free planarity; b) wheel graphs may be augmented only with punkah like augment.; c) envelope graph on Mobius strip may be augmented with ladder like augment.

Let a graph belonging to free-minor closed class $Free(A)$ be called *punkah-augmentable* [or *ladder-augmentable*] if a vertex and an edge[or two edges] in it may be replaced with arbitrary large punkah-graph [or ladder-graph] and graph will remain to belong to $Free(A)$.

For $A = A_{Pl}$ the only triconnected punkah-augmentable graph is wheel-graph $W_k, k > 2$. No triconnected graph there is ladder-augmentable. Envelope-graph is ladder augmentable on Mobius strip.

Let us study the class A_T.

Picture fig. 3 shows two simple examples of ladder-augment, i.e. in $K_{3,3}$ and in K_5.

It is easy to see that free toroidal graph that is ladder augmentable to the extent of some anticipated supposition should be non-projective-planar. Nevertheless there exists a graph [graph D_{17} in [2]] that violates this supposition, i. e. it is ladder augmentable unboundedly on torus and it is not projective planar in the same time.

Let us first go through considerations that suppose nonexistence of the last situation as more anticipated.

If graph G is not projective planar then there always exist two paths p_1 and p_2 that are non-contractible. If graph is ladder augmentably free-

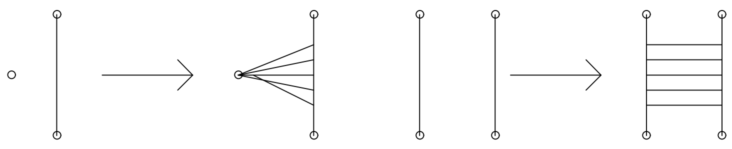
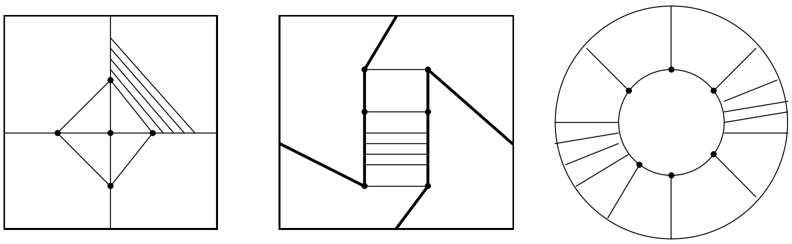


Figure 2: a) Punkah augment subgraph replacing operation. b)Ladder augment subgraph replacing operation. According both augments' definitions, after these replacements graph is supposed to retain its freeness.

toroidal then $G+e$ is toroidal but e we choose to join two opposite ends of two rungs[r_1, r_2] of ladder with at least one rung[r_3] between them. Then in $G+e$ there should be two non-contractible cycles [because of $K_{3,3}$ that necessarily arise]: one [q_e] going through e and r_1, r_2 and second [q] through r_3 and all ends of rungs on one side before r_3 and other side after r_3. Then q must necessarily cross both paths p_1 and p_2 otherwise path q_e would not be possible in toroidal implementation: either q is disjoined from p_1 and p_2 or cross only one of them q_e would be forced to cross it at least in two places but it is impossible[because it goes through changing part of ladder and e]. Thus, to construct counterexample to supposition, q_e should cross both p_1 and p_2. Just such pathes configuration is procured in the graph D_{17} in [2]. This consideration shows that these counterexample graphs are rather rear. Nevertheless the graph D_{17} is not unique. This same graph may be punkah augmented as in 5. Either graph C_7 from [2] is ladder-augmentably free-toroidal.

An interesting example to the more anticipated case when G is free ladder augmentable on torus and projective planar is demonstrated in fig. 6. In fig. 6. a) there is shown one forbidden minor on A_T. This graph is deferred 3*3+1-lattice on torus with one torus planar edge replaced with torus non-planar edge. It is easy to see that it contains two disjoined $K_{2,3}$-graphs [1] and therefore it is not projective planar. G without two edges (s,u) and (u,v) is free-toroidal ($=G'$) [in b)]. As a consequence G' can not be ladder-augmented but it can be punkah-augmented, see c).

Non-trivial example is shown in fig. 7. Here in a) projective planar



K_5 ladder
augmented

$K_{3,3}$
ladder augmented

$K_{3,3}$
ladder augmented
on projective plane

Figure 3: In a) inner vertex may be placed outside if necessary.

4*4-lattice is shown that is minimal forbidden minor for torus [1]. In b) this same graph with two eliminated horizontal edges, becoming toroidal free planar. But it is not projective free planar. Toroidal freeness is checked using computer.

Some generalization of the above result is possible. Both types of augmentability may be assumed to be applied not only to single punkah and ladder subgraph but for set of such subgraphs.

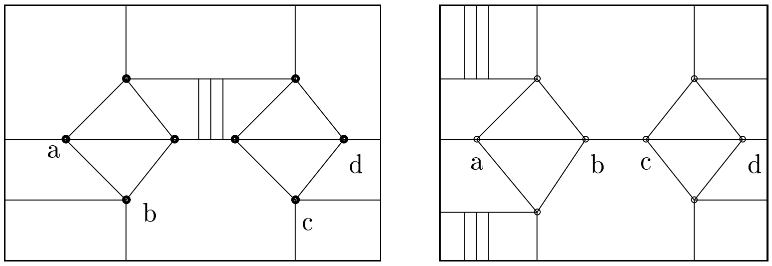


Figure 4: Counterexample to the supposition: forbidden graph for projective plane that is free ladder-augmentable on torus, in too embeddings. Free planarity is easy discernable.

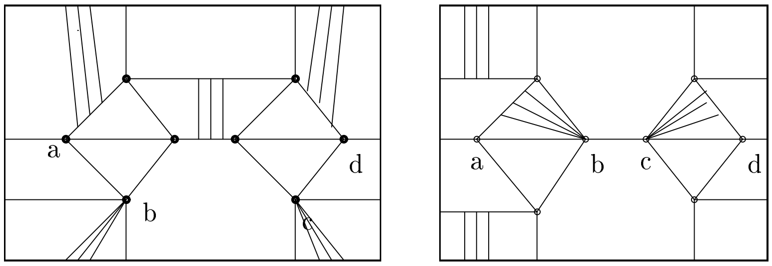


Figure 5: Counterexample graph punkah augmented.

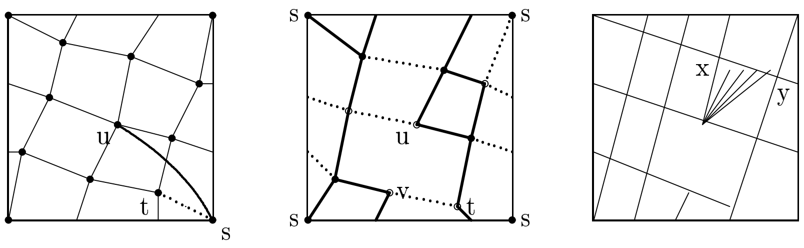
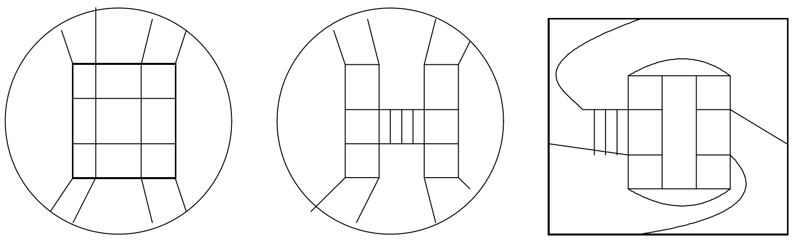


Figure 6: a) Toroidal planar lattice with 10 vertices with one edge replacement becomes forbidden minor for torus. b) Its toroidal free subgraph contains two disjoined $K_{2,3}$-graphs[1][drawn bold], thus it is not projective planar. c) It can not be ladder augmented, but can be punkah augmented.



a) b) c)

Figure 7: a) 4*4-lattice on projective plane that is forbidden minor for torus. c) Its free subgraph [on torus] can be ladder augmented[checked on computer]. b) It is not projective free planar.

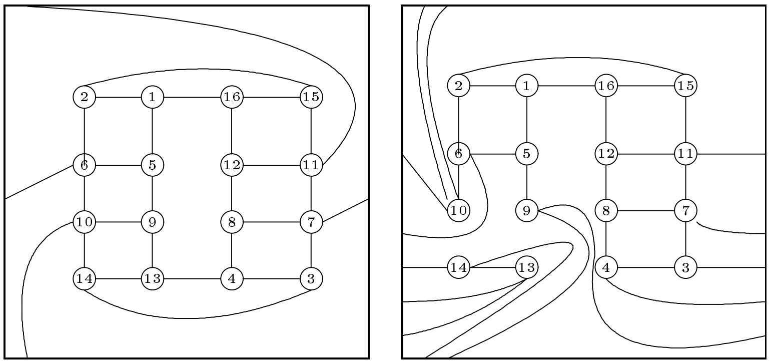


Figure 8: Illustration to fig. 7 c) that, say, edge (2,13) may be added.

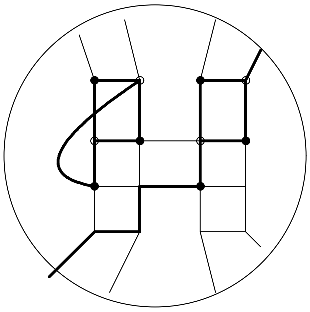


Figure 9: Illustration to fig. 7 b) that graph is not projective free planar. Two $K_{3,2}$-graphs are drawn bold.

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