Tverberg-type theorems for separoids

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Abstract

Let S be a d-dimensional separoid of $(k-1)(d+1)+1$ convex sets in some 'large-dimensional' Euclidean space $\mathbb{E}^N$. We prove a theorem that can be interpreted as follows: *if the separoid S can be mapped with a monomorphism to a d-dimensional separoid of points P in general position, then there exists a k-colouring $\varsigma\colon S \to K_k$ such that, for each pair of colours $i, j \in K_k$, the convex hulls of their preimages do intersect —they are not separated.* Here, by a monomorphism we mean an injective function such that the preimage of separated sets are separated. In a sense, this result is 'dual' to the Hadwiger-type theorems proved by Goodman & Pollack (1988) and Arocha, Bracho, Montejano, Olivereros & Strausz (2002).

We also introduce $\vartheta(k,d)$, the minimum number n such that all d-dimensional separoids of order at least n can be k-coloured as before.

By means of examples and explicit colourings, we show that for all $k > 2$ and $d > 0$,

$$(k-1)(d+1)+1 < \vartheta(k,d) < \binom{k}{2}(d+1)+1.$$

Furthermore, by means of a probabilistic argument, we show that for each d there exists a constant $C = C(d)$ such that for all k, $\vartheta(k,d) \le Ck \log k$.

Key Words: Abstract Convexity; Hadwiger's theorem; Tverberg's theorem; Graphs; Separoids; Order Types; Oriented Matroids; Pseudoachromatic number.

1 Introduction and statement of results

As suggested by Danzer, Grünbaum & Klee (1963) [4], the relationship between Helly's, Radon's and Carathéodory's theorems "could be best understood by formulating various axiomatic settings for the theory of convexity". The first attempt to give such an axiomatic setting was made by Levi (1951) [11], who uses Helly's theorem (1923) [10] as a starting point. More recently, the concept of a separoid was introduced [1, 3, 12, 13, 16, 17, 18] as a new attempt in this direction that is instead based on Radon's theorem (1921) [14].

A *separoid* is a (finite) set S endowed with a symmetric relation $\dagger \subset \binom{2^S}{2}$ defined on its family of subsets, which satisfies the following properties for all $A, B \subseteq S$:

$$\begin{array}{ll} \circ & A \dagger B \implies A \cap B = \emptyset \\ \circ\circ & A \dagger B \text{ and } B \subset B' \ (\subseteq S \setminus A) \implies A \dagger B' \end{array}$$

A pair $A \dagger B$ is called a *Radon partition*. Each part (A and B) is called a *component* and the union $A \cup B$ is the *support* of the partition. The (*combinatorial*) *dimension* of S, denoted by $\mathrm{d}(S)$, is the minimum d such that every subset of S with at least $d+2$ elements is the support of a Radon partition. By the second condition, the *minimal* Radon partitions determines the separoid. A pair of disjoint subsets $\alpha, \beta \subseteq S$ that are not a Radon partition are said to be *separated*, and denoted by $\alpha \mid \beta$ (cf. [1]).

Now, given a family of convex sets $\mathcal{F} = \{C_1, \ldots, C_n\}$ in some Euclidean space $\mathbb{E}^d$, a separoid $S(\mathcal{F})$ on $\{1, \ldots, n\}$ can be defined by the following relation: for all $\alpha, \beta \subseteq S(\mathcal{F})$,

$$\alpha \mid \beta \iff \langle\{C_i : i \in \alpha\}\rangle \cap \langle\{C_j : j \in \beta\}\rangle = \emptyset,$$

where $\langle\cdot\rangle$ denotes the convex hull. Analogously, the Radon partitions are defined by

$$A \dagger B \iff A \cap B = \emptyset \text{ and } \langle\{C_i : i \in A\}\rangle \cap \langle\{C_j : j \in B\}\rangle \neq \emptyset.$$

Conversely, as proved by Arocha et al. [1], every (abstract) separoid can be represented in such a way by a family of convex sets in some Euclidean space. Therefore each separoid S has a minimum dimension where it can be represented called the *geometric dimension* of S, denoted by $\mathrm{gd}(S)$. Furthermore, as proved by Strausz [18], if the separoid S is *acyclic* (i.e., if $\emptyset \mid S$), then $\mathrm{gd}(S) \leq |S| - 1$ (see also [17]).

The following theorem is an easy corollary of Tverberg's theorem [19] (see also [5] and the references therein).

Theorem 1 *Let S be a separoid of order $|S| = (k-1)(d+1)+1$, where $d = \mathrm{gd}(S)$. Then there exists a k-colouring $\varsigma : S \to K_k$ such that every pair of colour classes are not separated; i.e., the preimage of every pair of colours $i, j \in K_k$, are a Radon partition $\varsigma^{-1}(i) \dagger \varsigma^{-1}(j)$.*

Indeed, a stronger conclusion can be reached. Represent the separoid S with convex sets in $\mathbb{E}^d$, where $d = \mathrm{gd}(S)$. If we choose a point in each convex set and apply Tverberg's theorem to this set of points, then we can find a k-colouring of S such that there is a point that is in the convex hull of every chromatic class. •

Following [16], if the conclusion of Theorem 1 holds, we will say that there exists a *chromomorphism* onto the *complete separoid* K_k of order k (cf. Figure 1). If such a chromomorphism exists for a given S, we write $S \longrightarrow K_k$; otherwise we write $S \not\longrightarrow K_k$. In this note we are interested in purely combinatorial conditions that guarantee the existence of chromomorphisms onto complete separoids.

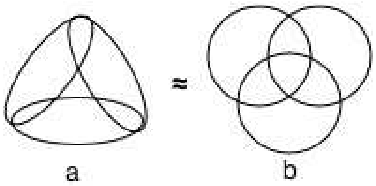

Figure 1. *Two representations of* K_3 *in* $\mathbb{E}^2$.

As shown by Figure 2, in Theorem 1 the geometric dimension cannot be replaced by the combinatorial dimension without adding a new ingredient —observe that $\mathrm{d}(S) \leq \mathrm{gd}(S)$. Thus, while replacing $\mathrm{gd}(S)$ by $\mathrm{d}(S)$, we may add a Hadwiger-type hypothesis that allows us to prove the following

Theorem 2 *Let* S *be a* d*-dimensional separoid of order* $|S| = (k-1)(d+1)+1$. *Suppose that in addition, there exists a monomorphism* $\mu\colon S \to P$ *into a* d*-dimensional separoid of points in general position. Then* $S \longrightarrow K_k$.

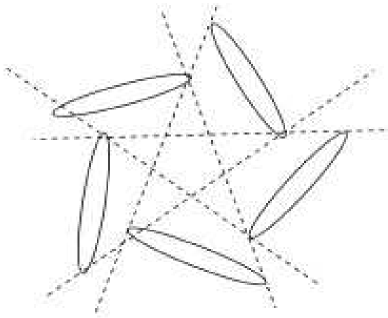

Figure 2. *A* 1*-dimensional separoid* S *of* 5 *convex sets in* $\mathbb{E}^2$ *such that* $S \not\longrightarrow K_3$.

Arocha et al. [1] proved a Hadwiger-type theorem that, supposing the existence of a monomorphism 'from the left' $\nu\colon P \to S$, concludes the existence of a *virtual* ℓ*-transversal*. That is, there are "as many" hyperplanes transversal to the family as there are hyperplanes through an ℓ-flat (e.g., while the family in Figure 1(b) has a 0-transvesal, that of Figure 1(a) has a virtual 0-transversal). This result extends ideas from Goodman & Pollack

[7] who used the notion of *order type* to characterise the existence of hyperplane transversals. On the other hand, Theorem 2 supposes the existence of a monomorphism 'to the right', and concludes that there is a *virtual Tverberg partition* (i.e., a partition with a virtual 0-transversal). Thus these theorems may be seen as 'dual' —at least in the case $\ell = \mathrm{d}(S)$.

However, the Hadwiger-type hypotheses are 'geometric' in nature; that is, they restrict the convex sets that represent the separoid to be in some 'special position'. (See [4] for the early work on such 'special position' hypotheses, and see [5, 8] for excelent updates on the subject.) The following questions arise. How far can the hypothesis of Theorem 2 be weakened without changing the conclusion? Is there a purely combinatorial Tverberg-type theorem?

We now introduce the following new concept. The *(k,d)-Tverberg number* $\vartheta(k,d)$, is the minimum number $n \in \mathbb{N}$ such that every d-dimensional separoid of order at least n maps onto K_k with a chromomorphism; that is, $\vartheta(k,d)$ is minimal with the property

$$|S| \geq \vartheta(k, \mathrm{d}(S)) \implies S \longrightarrow K_k \ .$$

Analogously, if $\mathcal{S}$ denotes a class of separoids, we denote by $\vartheta_{\mathcal{S}}(k,d)$ the (k,d)-Tverberg number restricted to the class $\mathcal{S}$. Thus, Tverberg's theorem can be rewriten as

$$\vartheta_{\mathcal{P}}(k,d) = (k-1)(d+1)+1,$$

where $\mathcal{P}$ denotes the class of separoids of points. Analogously, using the notion of *pseudoconfiguration of points*, Roudneff [15] proved that

$$\vartheta_{\mathcal{M}}(k,2) = 3k-2,$$

where $\mathcal{M}$ denotes the class of oriented matroids. In this direction, we prove the following

Proposition 1 *If $\mathcal{G}$ denotes the class of (simple) graphs —thought of as separoids whose minimal Radon partitions are pairs of singletons— then*

$$\vartheta_{\mathcal{G}}(k,d) \leq (k-1)(d+1)+1.$$

(Observe the close relation between $\vartheta_{\mathcal{G}}(k,d)$ and the so-called *pseudoachromatic number* [9]).

However, in general the (k,d)-Tverberg number is greater than that. Indeed, we will prove that

Theorem 3 *For all pairs of natural numbers $k > 2$ and $d > 0$ it follows that*

$$(k-1)(d+1)+1 < \vartheta(k,d) < \binom{k}{2}(d+1)+1.$$

Furthermore, by means of a probabilistic argument, we will prove that

Theorem 4 *For each $d > 0$, the constant $C = 2^{d+4}$ is such that for all $k \geq d+2$,*

$$\vartheta(k,d) \leq Ck\log k.$$

2 Definitions and Proof of Theorem 2

In order to be self-contained, we start with some basic notions and examples. Every (finite and acyclic) separoid S can be represented by a family of (convex) polytopes in the $(|S|-1)$-dimensional Euclidian space [17, 18]. The construction is as follows. Let S be identified with the set $\{1,\ldots,n\}$. For each element $i \in S$ and each minimal Radon partition $A \dagger B$ such that $i \in A$, consider the point

$$\rho^i_{A\dagger B} = \mathbf{e}_i + \frac{1}{2}\left[\frac{1}{|B|}\sum_{b\in B}\mathbf{e}_b - \frac{1}{|A|}\sum_{a\in A}\mathbf{e}_a\right],$$

where $\mathbf{e}_i$ denotes the i-th vector of the canonical basis of $\mathbb{R}^n$. Then, each element i is represented by the convex hull of all such elements:

$$i \mapsto \mathcal{K}_i = \left\langle \rho^i_{A\dagger B} : i \in A \text{ and } A \dagger B \right\rangle.$$

Observe that the convex sets $\mathcal{K}_i$ live in the affine hyperplane spanned by the basis. It is simple to verify that this construction is correct and that the implicit bound $n-1$ is tight.

Thus there is a minimum dimension in which S can be represented, called the *geometric dimension* of S and denoted by $\mathrm{gd}(S)$. Furthermore, if the separoid can be represented by a family of points in some Euclidian space, it is called a *point separoid* [3, 12] (also known as a *linear oriented matroid* [2] or as an *order type* [6]).

The *order* of the separoid S is the cardinal $|S|$ and its *size* is the cardinal $|\dagger|$ (i.e., the number of Radon partitions). The separoid of order $d+1$ and size 0 is called the *d-dimensional simploid*; that is, a separoid is a simploid if every subset is separated from its complement. Simploids can be represented by the vertex sets of simplices —hence the name. The (*combinatorial*) *dimension* of a separoid S is the maximum dimension of its induced simploids and is denoted by $\mathrm{d}(S)$.

We say that the separoid is in *general position* if every set of $\mathrm{d}(S)+1$ elements induce a simploid. Thus, a d-dimensional separoid of convex sets is in general position if (and only if) every $d+2$ elements admit a d-flat transversal but no $d+1$ elements do. Furthermore, d is the minimum number with that property.

A separoid is called a *Radon separoid* if for each minimal Radon partition is unique in its support; i.e., if $A \dagger B$, $C \dagger D$ are minimal then

$$A \cup B = C \cup D \implies \{A, B\} = \{C, D\}.$$

Observe that if S is a point separoid, then $\mathrm{d}(S) = \mathrm{gd}(S)$ and it is a Radon separoid. Furthermore, a separoid S in general position is a point separoid if and only if $\mathrm{d}(S) = \mathrm{gd}(S)$ (see [3]).

The (acyclic) separoid K is *complete* if for all $i, j \in K$ we have that $i \dagger j$; i.e., if its size is as big as possible. We denote by K_k the complete separoid of order k. Observe that a separoid is complete if and only if its dimension is zero.

Given two separoids S and P, a function $\varphi : S \to P$ is a *morphism* if the preimage of separations are separations (see [1, 17] for several important examples of morphisms); that is, for all $\alpha, \beta \subseteq P$,

$$\alpha \mid \beta \implies \varphi^{-1}(\alpha) \mid \varphi^{-1}(\beta).$$

If the function φ is injective (resp. surjective), the morphism is called a *monomorphism* (resp. an *epimorphism*). An epimorphism is a *chromomorphism* if the preimage of minimal Radon partitions are Radon partitions.

The main example to have in mind while thinking about chromomorphisms is the following —it motivates the name of such morphisms. Consider a family of convex sets $S = \{C_1, \ldots, C_n\}$. Given an (effective) k-colouring $\varsigma : S \to \{1, \ldots, k\}$, let $D_i = \langle \varsigma^{-1}(i) \rangle$ be the convex hull of the union of those convex sets coloured i, for $i = 1, \ldots, k$. Let $T = \{D_1, \ldots, D_k\}$. The induced function, also denoted by $\varsigma : S \longrightarrow T$, is a chromomorphism between those separoids.

Given a (simple and undirected) graph $G = (V, E)$, a separoid S on V can be defined with the relation, for $i, j \in V$

$$i \dagger j \text{ is minimal} \iff ij \in E.$$

Indeed this definition induces a functoral embedding from the category of graphs into that of separoids when both classes are endowed with *homomorphisms* (see [13]). Conversely, given a separoid S, we say that S is a *graph* if, for $A, B \subseteq S$,

$$A \dagger B \text{ is minimal} \implies |A||B| = 1.$$

Clearly K_k is the complete graph of order k —hence the notation. Observe that a graph H is a *minor* of a connected graph G if and only if there exists a chromomorphism $G \longrightarrow H$ with all its fibers connected.

Proof of Proposition 1. Let G be a d-dimensional graph. We need to prove that

$$|G| \geq (k-1)(d+1)+1 \implies G \longrightarrow K_k.$$

For, denote by $\alpha(G) = d+1$ the independence number and by $\chi(G)$ the chromatic number. Using the well-known Erdős inequality, $|G| \leq \chi(G)\alpha(G)$, we have that

$$(k-1)\alpha(G)+1 \leq |G| < \chi(G)\alpha(G)+1,$$

which implies that $k \leq \chi(G)$. Observe that any homomorphism —or proper colouring if you will—

$$\varphi: G \to K_{\chi(G)}$$

is also a chromomorphism. Furthermore, for all $n \leq m$ there is a chromomorphism $K_m \longrightarrow K_n$. Therefore, there is a chromomorphism $\psi: K_{\chi(G)} \longrightarrow K_k$ and we have that $\varsigma = \psi \circ \varphi$ is the desired chromomorphism. •

Proof of Theorem 2. Let S be a d-dimensional separoid of order $(k-1)(d+1)+1$. Suppose there is a monomorphism $\mu: S \to P$ into a d-dimensional point separoid in general position. Due to Tverberg's theorem, there exists a chromomorphism $\tau: P \longrightarrow K_k$. We now show that $\varsigma = \tau \circ \mu$ is a chromomorphism.

For, let $i \dagger j$ be an edge of K_k. Since τ is chromomorphism, we have that $\tau^{-1}(i) \dagger \tau^{-1}(j)$. Then there exist $A \subseteq \tau^{-1}(i)$ and $B \subseteq \tau^{-1}(j)$ such that $A \dagger B$ is minimal. Since P is in general position, $|A \cup B| = d+2$. Since μ is injective, $|\mu^{-1}(A \cup B)| = d+2$ and there exist $C \dagger D$ such that $C \cup D = \mu^{-1}(A \cup B)$. Therefore, since μ is a monomorphism, $\mu(C) \dagger \mu(D)$. Since P is a point separoid, it is a Radon separoid and we may suppose that $\mu(C) = A$ and $\mu(D) = B$. Finally, since $C \subseteq \varsigma^{-1}(i)$ and $D \subseteq \varsigma^{-1}(j)$, we have that $\varsigma^{-1}(i) \dagger \varsigma^{-1}(j)$, which concludes the proof. •

Figure 3 shows that the hypothesis of *general position* cannot be dropped without adding a new ingredient. On the other hand, if we suppose —as did Goodman & Pollack [7] and Arocha et al. [1]— that the monomorphism comes 'from the left' $\mu: P \to S$, then such a hypothesis is not needed and the argument is much simpler (see the proof of Lemma 1). Observe that Figure 3 also shows that the existence of a *virtual line* does not imply the existence of the corresponding chromomorphism.

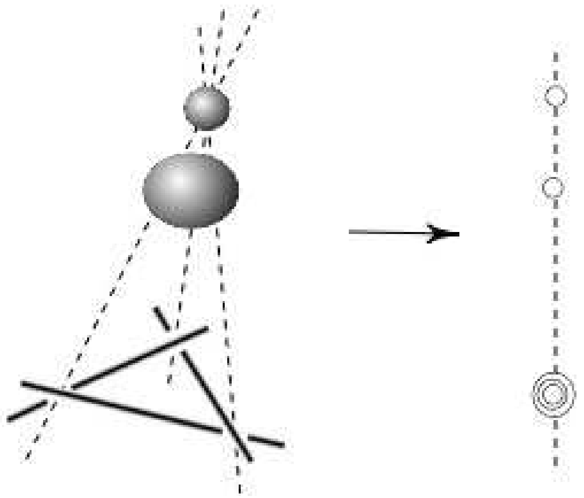

Figure 3. *A* 1*-dimensional separoid* S *of 5 convex sets in* $\mathbb{E}^3$ *such that* $S \not\to K_3$.

3 Proofs of Theorems 3 and 4

We start this section with a simple, but useful, structural result that allows us to restrict our attention to Radon separoids in general position.

Lemma 1 *Given a d-dimensional separoid S, there exists a d-dimensional Radon separoid R in general position such that*

$$R \longrightarrow K_k \implies S \longrightarrow K_k.$$

For, let R be defined on the same set as S, and with the following set of minimal Radon partitions: for each subset $X \in \binom{S}{d+2}$, choose a single

Radon partiton $A \dagger (X \setminus A)$ of S to be in R. Clearly R is a Radon separoid in general position and $\mathrm{d}(R) = \mathrm{d}(S)$. Furthermore, the identity map $\nu: R \to S$ is a monomorphism.

Now, suppose that $\varsigma: R \longrightarrow K_k$ is a chromomorphism; that is, suppose that for each edge $i \dagger j$ of K_k it follows that $\varsigma^{-1}(i) \dagger \varsigma^{-1}(j)$. Since ν is a monomorphism, we have that $\nu \circ \varsigma^{-1}(i) \dagger \nu \circ \varsigma^{-1}(j)$. Therefore, $\varsigma \circ \nu^{-1}: S \longrightarrow K_k$ is a chromomorphism which concludes the proof. •

In the sequel, given a partition $X_1, \ldots, X_k$ we say that it has *type* $(|X_1|, \ldots, |X_k|)$. In particular, given a Radon partitions $A \dagger B$ we denote its type as the pair $(|A|, |B|)$.

Theorem 3 follows immediately from the following two lemmas.

Lemma 2 *For all $k > 2$ and $d > 0$ there exists a d-dimensional separoid S of order*

$$|S| = (k-1)(d+1) + 1$$

such that $S \not\longrightarrow K_k$.

For, let $S = \{0, 1, \ldots, (k-1)(d+1)\}$ be endowed with the following minimal Radon partitions: for each $A \in \binom{S}{d+2}$ let $x \dagger (A \setminus x)$, where $x \in A$ is chosen such that

$$x = \begin{cases} 0 & \text{if } 0, 1 \in A, \\ 1 & \text{if } 0 \notin A \text{ and } 1 \in A, \\ a \neq 0 & \text{if } 0 \in A \text{ and } 1 \notin A, \\ a & \text{if } 0, 1 \notin A. \end{cases}$$

Since each minimal Radon partition has type $(1, d+1)$, if $S \longrightarrow K_k$ then the induced partition must have type $(1, d+1, \ldots, d+1)$. That isolated element in the partition will be called *the singleton*.

Now, suppose that $S \longrightarrow K_k$ and look at the partition induced by such a colouring. The singleton cannot be 0 because $0 \dagger (A \setminus 0)$ only if $1 \in A$, and 1 can only be in one part. The singleton cannot be 1 because $1 \dagger (A \setminus 1)$ only if $0 \notin A$ and 0 must be in some part. Thus suppose $a \notin \{0, 1\}$ is the singleton. But then, $a \mid [\![1]\!]$, where $[\![1]\!]$ denotes the chromatic class of 1; this is a contradiction. •

Lemma 3 *Let S be a d-dimensional separoid. Then*

$$|S| = \binom{k}{2}(d+1) \implies S \longrightarrow K_k.$$

For, let $H = H_k$ denote the graph resulting from deleting an edge to K_k; that is, H is the set (of colours) $\{1, \ldots, k\}$ and for each pair $ij \in \binom{k}{2}$ except for one, say $1k$, there is an edge $i \dagger j$. Let

$$S = \bigcup_{ij \in \binom{k}{2}} S_{ij}$$

be a partition of type $(d + 2, d + 1, \ldots, d + 1, d)$. Furthermore, suppose that $|S_{12}| = d + 2$ and $|S_{1k}| = d$. Below we exhibit a chromomorphism $\varsigma\colon S \setminus S_{1k} \longrightarrow H$. Then we extend it to a chromomorphism onto K_k using the remaining d elements of S.

Since $|S_{12}| = d + 2$, there is a Radon partition $A \dagger B$ whose support is S_{12}. Assign colours respectively (i.e., let $\varsigma(A) = 1$ and $\varsigma(B) = 2$).

Remark. We may suppose that A has the maximum size that a component may have in $S \setminus S_{1k}$.

Choose any element of colour 1, say $a \in A$. We now use a to extend the colouring to the parts S_{1j} (with $1 < j \neq k$) so that ς becomes onto the edges incident to 1. That is, for each pair $1j \in \binom{k}{2} \setminus 1k$, the set $S_{1j} \cup a$, which consists of $d + 2$ elements, defines a Radon partition $A' \dagger B'$ (we may suppose $a \in A'$). Thus, we can assign colours by $\varsigma(A') = 1$ and $\varsigma(B') = j$.

Now, choose an element coloured 2, say $b \in \varsigma^{-1}(2)$, and use it to extend all parts S_{2j}, with $2 < j$ (i.e., consider $S_{2j} \cup b$, take its Radon partition, and assign colours). Then repeat for colour 3, colour 4, and so on. At the end of such a process the colouring ς is the desired chromomorphism.

In order to extend ς to a chromomorphism onto K_k, there is one possible obstruction: if we choose two elements coloured 1, say $a, b \in \varsigma^{-1}(1)$, and consider the set $S_{1k} \cup \{a, b\}$ of $d + 2$ elements, then the defined Radon partition $A' \dagger B'$ is such that $a \in A'$ and $b \in B'$ (analogously if both elements are coloured k). Also, if we choose one element of each colour, say $a \in \varsigma^{-1}(1)$ and $b \in \varsigma^{-1}(k)$, and consider $S_{1k} \cup \{a, b\}$, then the respective Radon partition $A' \dagger B'$ contains both elements on the same component, say $\{a, b\} \subset A'$.

However, by the Remark, we may suppose that $|\varsigma^{-1}(1) \cup \varsigma^{-1}(k)| \geq d+2$. Then there is a minimal Radon partition $C \dagger D$ whose support is contained in the preimage of this "missing edge". Furthermore, since $\varsigma^{-1}(1) \mid \varsigma^{-1}(k)$, there exist a pair of elements $a \in \varsigma^{-1}(1)$ and $b \in \varsigma^{-1}(k)$ which appears on the same side of that Radon partition, say $\{a, b\} \subset C$. Therefore, we can apply the previous method, but starting with $S_{12} = C \cup D$, to find another chromomorphism $\varsigma'\colon S \setminus S_{12} \longrightarrow H$ such that $\varsigma'(C) = 1$. Then ς' can be extended to a chromomorphism $S \longrightarrow K_k$. $\bullet$

Given a separoid S of order $2^t k$, we will denote by Ω the set of all k-colourings of S such that each chromatic class consists of exactly 2^t elements. Analogously, given a Radon separoid T of order 2^{t+1} in general position we denote by $\widehat{\Omega}$ the set of all 2-partitions of T into two sets of order 2^t. Furthermore, a pair $(\alpha, \beta) \in \widehat{\Omega}$ is called a *halving* of T. We denote by $\mathbf{p}_T$ the probability that $\alpha \mid \beta$, for a randomly and uniformly choosen $(\alpha, \beta) \in \widehat{\Omega}$.

The proof of Theorem 4 is mainly based on the following

Lemma 4 *If T is a d-dimensional Radon separoid of order 2^{t+1} in general position, then*

$$\mathbf{p}_T \leq (d+2)/2^{2^{t-d-1}}.$$

Before proving this lemma, we see how it is used.

Proof of Theorem 4. Let $t = d + 4 + \lfloor \log \log k \rfloor$ and let S be a d-dimensional Radon separoid of order $2^t k$. Since $2^t \leq 2^{d+4} \log k$ it is enough to prove that $S \longrightarrow K_k$.

Let $S = S_1 \cup \cdots \cup S_k$ be a random partition of S. Let $E_{i,j} \subset \Omega$ be the event that $S_i \mid S_j$. We claim that

$$\Omega \setminus \bigcup_{ij \in \binom{k}{2}} E_{i,j} \neq \emptyset.$$

It is clear that all events $E_{i,j}$ have the same probability. Now, we can obtain S_1 and S_2 as follows. Randomly choose a set $T \subset S$ of order 2^{t+1} and let $(S_1, S_2) \in \widehat{\Omega}$ be a random halving of T. By Lemma 4, the probability that $S_1 \mid S_2$ is at most $(d+2)/2^{2^{t-d-1}}$. Therefore

$$\mathrm{Prob}\left[\bigcup_{ij \in \binom{k}{2}} E_{i,j}\right] \leq \sum_{ij \in \binom{k}{2}} \mathrm{Prob}\,[E_{i,j}] = \binom{k}{2} \mathrm{Prob}\,[E_{1,2}]$$

$$\leq \frac{k^2}{2}(d+2)2^{-2^{t-d-1}} \leq \frac{k^2}{2}(d+2)2^{-2^{2+\log\log k}} = \frac{k^2}{2}(d+2)k^{-4} < 1.$$

$\bullet$

Proof of Lemma 4. We prove the lemma by induction on d. If $d = -1$ and $t \geq 0$ then every element of T is not separated from the empty set. Thus $\mathbf{p}_T = 0$. Now, let $d \geq 0$ and suppose the lemma is true for all $d' < d$. If $t \leq d$ we have that $(d+2)/2^{2^{t-d-1}} > 1$; thus let $t > d$.

We can achieve a random halving of T as follows: in the first step, halve the set T into two equal parts T_1 and T_2; later, randomly halve each T_i $(i = 1, 2)$ into two equal sets α_i and β_i. Let $\alpha = \alpha_1 \cup \alpha_2$ and $\beta = \beta_1 \cup \beta_2$. The halving $(\alpha, \beta) \in \widehat{\Omega}$ is random (and uniform).

Let us define a $(d-1)$-dimensional separoid on T_1 as follows (we will use $\|$ and $\ddagger$ to denote the separations and Radon partitions on T_1, respectively). For each $X \in \binom{T_1}{d+1}$ and each $A \subset X$, let

$$T(X, A) = \{x \in T_2 : (A \cup x) \dagger (X \setminus A)\}.$$

Observe that for two different subsets $A, A' \subset X$ we have $T(X,A) \cap T(X,A') = \emptyset$. Furthermore, since for each element $x \in T_2$ there is a Radon partition of the set $X \cup x$, the sets $T(X, A)$, with $A \in 2^X$, are a partition of T_2. Thus, for some $A_X \subset X$ we have that $|T(X, A_X)| \geq \frac{|T_2|}{|2^X|} = 2^{t-d-1}$. We define $\ddagger$ as the relation $A_X \ddagger (X \setminus A_X)$, for all subsets $X \subset T_1$ of order $d+1$. Observe that $\ddagger$ defines a $(d-1)$-dimensional Radon separoid in general position on T_1.

Let E be the event that $\alpha \mid \beta$ and E_1 be the event that $\alpha_1 \| \beta_1$. By the induction hypothesis we have that $\text{Prob}\,[E_1] \leq (d+1)/2^{2^{(t-1)-(d-1)-1}} = (d+1)/2^{2^{t-d-1}}$.

Take the halving (α, β) in $E \setminus E_1$. Since $\alpha_1 \ddagger \beta_1$, there exists $X \subset T_1$ of order $d+1$ such that $(X \cap \alpha_1) \ddagger (X \cap \beta_1)$. We may assume that $A_X = X \cap \alpha_1$.

We claim that $T(X, A_X) \cap \alpha_2 = \emptyset$. For, suppose that there exists an $x \in T(X, A_X) \cap \alpha_2$. Since $(A_X \cup x) \dagger (X \setminus A_X)$, $A_X \cup x \subset \alpha$ and $X \setminus A_X \subset \beta$, we have that $\alpha \dagger \beta$; this is a contradiction and the claim follows.

Now, let $s = 2^{t-d-1} \leq |T(X, A_X)|$ and let $\overline{E}_1 = \Omega \setminus E_1$. In order for $\alpha \mid \beta$, the set $T(X, A_X)$ has to be contained in β_2. Thus the conditional probability is

$$\frac{\text{Prob}\left[E \cap \overline{E}_1\right]}{\text{Prob}\left[\overline{E}_1\right]} = \text{Prob}\left[E \mid \overline{E}_1\right] \leq \frac{\binom{2^t - s}{2^{t-1} - s}}{\binom{2^t}{2^{t-1}}} \leq 2^{-s} = 2^{-2^{t-d-1}}.$$

Therefore,

$$\begin{aligned} \mathbf{p}_T = \text{Prob}\,[E] &\leq \text{Prob}\,[E_1] + \text{Prob}\,[E \setminus E_1] \\ &\leq \text{Prob}\,[E_1] + \frac{\text{Prob}\,[E \setminus E_1]}{\text{Prob}\,[\overline{E}_1]} \leq (d+2)/2^{2^{t-d-1}}. \end{aligned}$$

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