Hamiltonian threshold for strong products of graphs*

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Abstract

We prove that the strong product of any at least $(\ln 2)\Delta + O(\sqrt{\Delta})$ non-trivial connected graphs of maximum degree at most Δ is pancyclic. The obtained result is asymptotically best possible since the strong product of $\lfloor(\ln 2)D\rfloor$ stars $K_{1,D}$ is not even hamiltonian.

1 Introduction

Hamilton cycles in graphs form one of the most intensively studied topics in graph theory. There are numerous surveys on the subject, e.g., [4, 5, 8, 9]. Gray codes [12] which are just special types of Hamilton cycles are important in the computer science because of their relation to the efficient enumeration of combinatorial objects. In the present paper, we study what is the least number $h_{\max}(\Delta)$ such that the $h_{\max}(\Delta)$-th strong power of each non-trivial connected graph G of maximum degree Δ is hamiltonian. We treat this problem in a more general setting by considering strong products of connected graphs of maximum degree Δ.

The *strong product* of graphs $G_1, \ldots, G_k$ is the graph $G_1 \times \cdots \times G_k$ with the vertex set $V(G_1) \times \cdots \times V(G_k)$ where two distinct vertices $[v_1, \ldots, v_k]$ and $[v'_1, \ldots, v'_k]$ are joined by an edge in $G_1 \times \cdots \times G_k$ if for every coordinate

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i, $1 \leq i \leq k$, $v_i = v_i'$ or $v_i v_i' \in E(G_i)$. Since we consider throughout the paper only strong products of graphs, we often leave the adjective "strong". The *k-th power* G^k of a graph G is the strong product of k copies of G. A graph G is *hamiltonian* if it contains a *Hamilton cycle*, i.e., a cycle through all the vertices of G. The *order* of a graph G is the number of its vertices; the *size* of G is the number of its edges. A graph is said to be *non-trivial* if its order is at least two. A graph G of order n is *pancyclic*, if it contains cycles of all lengths between 3 and n.

We show that the (strong) product of any $(\ln 2)\Delta + (10 + \ln 4)\sqrt{\Delta} + 0.5 \log_2 \Delta + O(1)$ non-trivial connected graphs with maximum degree at most Δ is pancyclic. This proves Conjecture 2 from [10, 11]. Our proof is based on the concept of fractional factors in graphs which we introduce in Section 3. Let us remark that the obtained constant at the linear term is the best possible since the graph $K_{1,\Delta}^{\lfloor (\ln 2)\Delta \rfloor}$ is not hamiltonian [15], In fact, the graph $K_{1,\Delta}^{\lfloor (\ln 2)\Delta \rfloor}$ contains an independent set of size greater than half of its order. Since the product of slightly more than $(\ln 2)\Delta$ stars $K_{1,\Delta}$ still contains a large independent set and its toughness is close to one, our results might seem quite surprising at the first sight.

Before we proceed further, let us survey the history of the considered problem and previous results.

1.1 Previous results

The problem which we address can be traced back to the 1970's to the work of Barnette and Rosenfeld [2]. Zaks [15] in 1975 asked whether for each non-trivial connected graph G, there exists a number $h(G)$ such that the graph $G^{h(G)}$ is hamiltonian. Note that the following theorem from [2, 3] implies that if the number $h(G)$ exists, then for every $k \geq h(G)$, the graph G^k is also hamiltonian:

Theorem 1 *If G is a connected hamiltonian graph of order at least Δ and H is a connected graph of maximum degree at most Δ, then the product $G \times H$ is hamiltonian.*

Therefore, we define $h(G)$ to be the least integer such that $G^{h(G)}$ contains a Hamilton cycle.

Bermond, Germa and Heydemann [3] answered the question of Zaks in affirmative by showing the existence of the number $h(G)$ for each non-trivial connected graph G. In their paper, they also conjectured that the following should hold:

Conjecture 1 *If G is a non-trivial connected graph of maximum degree Δ, then the Δ-th power of G is hamiltonian.*

If we define $h_{\max}(\Delta)$ to be the maximum of $h(G)$ taken over all non-trivial connected graphs G of maximum degree Δ, then Conjecture 1 asserts that $h_{\max}(\Delta) \leq \Delta$. Conjecture 1 have been proved only recently by Král', Maxová, Podbrdský and Šámal [10] in the following stronger form:

Theorem 2 *Let $G_1, \ldots, G_\Delta$ be any non-trivial connected graphs of maximum degree at most Δ. The product $G_1 \times \cdots \times G_\Delta$ is hamiltonian.*

However, the function $h_{\max}(\Delta)$ is not equal to Δ. In the subsequent paper, Král' et al. [11] showed that $\limsup_{\Delta\to\infty} \frac{h_{\max}(\Delta)}{\Delta} \leq \ln \frac{25}{12} + \frac{1}{64} < 1$ and conjectured [10, 11] the following:

Conjecture 2 *The following equality holds:*

$$\lim_{\Delta\to\infty} \frac{h_{\max}(\Delta)}{\Delta} = \ln 2.$$

Note that the result of Zaks [15] on the powers of stars implies that the limit cannot be smaller than $\ln 2$, since $h_{\max}(\Delta) \geq (\ln 2)\Delta - O(1)$.

In the present paper, we prove Conjecture 2 in the following stronger form (Theorem 12): the product of any $(\ln 2)\Delta + (10+\ln 4)\sqrt{\Delta} + 0.5\log_2 \Delta + O(1)$ non-trivial connected graphs of maximum degree at most Δ is pancyclic.

2 Preliminaries

Throughout the paper, we use the standard notation from graph theory which can be found, e.g., in [7]. In this section, we just introduce less standard concepts which we use and we also recall some results on factors in strong products of graphs mostly from [10, 11].

The *star* S_k of size $k \geq 1$ is the complete bipartite graph $K_{1,k}$. The vertices of the star S_k are denoted by $\{*, 1, \ldots, k\}$ where $*$ stands for the central vertex and $1, \ldots, k$ for the peripheral vertices. The set of all stars of size at most D is denoted by $\mathcal{S}(D)$.

A k-factor of a graph G is a k-regular spanning subgraph of G. If Γ is a set of graphs, a Γ-*factor* of G is a spanning subgraph of G whose each component is isomorphic to a graph from the set Γ. In particular, a $\{K_2\}$-factor is just a perfect matching (which is just a 1-factor).

The proof of our main result is divided into several steps. In the first step, we show the existence of "good" $\mathcal{S}(D)$-factors in products of graphs of bounded maximum degree for suitable small D. The tools to reach this goal are developed in Section 4. Let us now recall some previous results on the existence of $\mathcal{S}(D)$-factors in graphs and their strong products. It is not hard to see that each non-trivial connected graph of maximum degree at most Δ has an $\mathcal{S}(\Delta)$-factor [1, 10, 11]:

Lemma 3 *Let G be a non-trivial connected graph of maximum degree at most Δ. The graph G has an $\mathcal{S}(\Delta)$-factor.*

A sufficient and necessary condition on the existence of $\mathcal{S}(D)$-factors for $D \geq 2$ was proved by Amahashi and Kano [1]. The condition found by Amahashi and Kano when applied to the strong products of stars readily yields the following [11]:

Lemma 4 *Let $k \geq 1$, $\ell \geq 2$ and $n_1, \ldots, n_k$ be any integers such that $1 \leq n_i \leq k\ell$ for each i, $1 \leq i \leq k$. The graph $S_{n_1} \times \cdots \times S_{n_k}$ has an $\mathcal{S}(\ell)$-factor.*

In the second step (Section 5), we show how to find 2-factors comprised by long cycles in products of good stars. One of the cases, which we will need to handle is the case that all the stars in the considered product have small sizes. At this point, the following corollary of Theorem 2 becomes helpful:

Corollary 5 *The product of any Δ stars of size at most Δ is hamiltonian.*

In the final step of the proof, presented in Section 6, we find a 2-factor comprised by cycles of lengths at least Δ and apply the following lemma which was implicitly proven in [11] (see the proof of Theorems 9 and 10 in [11]):

Lemma 6 *Let G be a connected graph which has a 2-factor comprised by cycles of lengths at least Δ and let G_1 and G_2 be two connected graphs of maximum degree at most Δ. The product $G \times G_1 \times G_2$ is pancyclic.*

3 Fractional matchings and factors

In the proofs in Sections 4 and 5, we use the concept of fractional matchings and factors in graphs. In this section we provide a gentle introduction to this concept. The interested reader is referred to any monograph on polyhedral combinatorics and combinatorial optimization, e.g., [6, 13, 14], for more details.

Let G be a fixed graph for the rest of the paragraph. If $f : V(G) \to \mathbb{N}$ is a function which assigns vertices of G non-negative integers, a spanning subgraph H of G is said to be an *f-factor* if each vertex v of G has degree $f(v)$ in H. Note that if f is a function constantly equal to an integer k, then an f-factor is just a k-factor. A *fractional factor* of G is a function $x : E(G) \to \mathbb{R}$ which assigns each edge of G a real number between 0 and 1 (inclusively). If the sum of the values $x(e)$ taken over all edges e incident with a vertex v is equal to $f(v)$, the fractional factor is a *fractional f-factor.*

The concept of fractional factors is a relaxation of ordinary factors: if the values $x(e)$ are restricted to be only integers, then the fractional factor x corresponds to the factor of G (include to the factor the edges e with $x(e) = 1$). If a graph G has an f-factor, then it has also a fractional f-factor, but the converse is not true in general: any odd cycle has a fractional 1-factor (assign each edge 1/2) and clearly it does not have a 1-factor. However, if a graph G is bipartite, the converse is also true [6, 13, 14]. We decided to include a proof of this simple fact for the sake of completeness:

Lemma 7 *Let G be a bipartite graph and let $f : V(G) \to \mathbb{N}$ be any function assigning the vertices of G non-negative integers. If the graph G has a fractional f-factor, then G has also an f-factor.*

Proof: Let x be a fractional f-factor such that the least number of edges of G are assigned non-integral values. If G contains no edge whose value $x(e)$ is an integer, then the edges e with $x(e) = 1$ form an f-factor of G. Assume in the rest that G contains an edge $e = v_1v_2$ with $0 < x(v_1v_2) < 1$. Since the sum of the edges incident with the vertex v_2 is an integer, the vertex v_2 must be incident with another edge v_2v_3 whose value is not an integer. Similarly, the vertex v_3 is incident, besides the edge v_2v_3, with an edge v_3v_4 whose value is not an integer, etc. Since the graph G is finite, we eventually reach a vertex v_ℓ which is already among the vertices $v_1, \ldots, v_{\ell-1}$.

Assume now that $v_k = v_\ell$ and all the vertices $v_k, \ldots, v_{\ell-1}$ are distinct. Since the graph G is bipartite, $\ell - k$ is an even integer. Let us define δ to be the following number:

$$\begin{aligned}\delta \;=\min\; & \{x(v_kv_{k+1}), x(v_{k+2}v_{k+3}), \ldots, x(v_{\ell-2}v_{\ell-1})\} \cup \\ & \{1 - x(v_{k+1}v_{k+2}), 1 - x(v_{k+3}v_{k+4}), \ldots, 1 - x(v_{\ell-1}v_\ell)\}\end{aligned}$$

Modify the fractional factor x to another fractional factor x': decrease the values $x(v_kv_{k+1})$, $x(v_{k+2}v_{k+3})$, ..., $x(v_{\ell-2}v_{\ell-1})$ by δ and increase the values

$x(v_{k+1}v_{k+2})$, $x(v_{k+3}v_{k+4})$, ..., $x(v_{\ell-1}v_\ell)$ by δ. Clearly, the obtained assignment x' is a fractional f-factor. In addition, one of the values $x'(v_kv_{k+1})$, $x'(v_{k+1}v_{k+2})$, ..., $x'(v_{\ell-1}v_\ell)$ is an integer. However, this contradicts the fact that fractional f-factor x has the least number of non-integral values among all fractional f-factors.

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4 Star factors in the products of stars

In the next section, we show that the product of stars whose sizes are multiples of s has a 2-factor comprised by large cycles. In order to be able to apply this result, we need a tool to construct a factor comprised by stars whose sizes are either multiples of a given number s or are small. We call such stars s-good: a star is *s-good* if its size is a multiple of s or it is smaller than $4s$. The set of all s-good stars of size at most D is denoted by $\mathcal{S}_s(D)$.

The next lemma provides us the wanted tool to construct a factor comprised by s-good stars:

Lemma 8 *Let Δ and s be two positive integers. The product of any two stars of size at most Δ has an $\mathcal{S}_s(\Delta/2+s)$-factor.*

Proof: Fix integers A and B between 1 and Δ. We aim to show that the product $S_A \times S_B$ has an $\mathcal{S}_s(\Delta/2+s)$-factor. If either $A < 4s$ or $B < 4s$, the claim trivially holds (e.g., if $A < 4s$, consider the factor formed by the $B+1$ copies of S_A in $S_A \times S_B$). In the rest, we assume that both $A \geq 4s$ and $B \geq 4s$.

Let $\alpha = \left\lfloor \frac{A}{2s} \right\rfloor$ and $\beta = \left\lfloor \frac{B}{2s} \right\rfloor$. In addition, let $\delta = AB - \alpha Bs - \beta As$. The integer δ is non-negative:

$$\begin{aligned}\delta = AB - \alpha Bs - \beta As = AB - \left\lfloor \tfrac{A}{2s} \right\rfloor Bs - \left\lfloor \tfrac{B}{2s} \right\rfloor As \geq \\ AB - \tfrac{ABs}{2s} - \tfrac{ABs}{2s} = 0 \end{aligned} \qquad (1)$$

Note that $\delta = 0$ if and only if both A and B are multiples of $2s$. Next, we bound the integer δ from above:

$$\begin{aligned}\delta = AB - \alpha Bs - \beta As \leq AB - \left(\tfrac{A}{2s} - 1\right) Bs + \left(\tfrac{B}{2s} - 1\right) As = \\ AB - \tfrac{ABs}{2s} + Bs - \tfrac{ABs}{2s} + As = (A+B)s \end{aligned} \qquad (2)$$

Fix integers A', $0 \le A' \le A$, and B', $0 \le B' \le B$, such that $\frac{\delta}{s} - 1 \le A' + B' < \frac{\delta}{s}$. Such integers A' and B' exist by the inequalities (1) and (2) unless $\delta = 0$. If $\delta = 0$, set both A' and B' to zero. Note that if $\delta \neq 0$, then $\delta - (A' + B')s$ is a positive integer smaller or equal to s.

Let us now consider the following partition of the vertex set of $S_A \times S_B$:

$$\begin{aligned} V &= \{[a, b], 1 \le a \le A \;\&\; 1 \le b \le B\} \\ W &= \{[a, *], 1 \le a \le A\} \cup \{[*, b], 1 \le b \le B\} \cup \{[*, *]\} \end{aligned}$$

Next, we construct a fractional factor in the bipartite subgraph of $S_A \times S_B$ with the parts V and W. The fractional factor x is defined as follows:

$$\begin{aligned} x([i,j][i,*]) &= \begin{cases} \frac{(\beta+1)s}{B} + \frac{(B'-B)s}{AB} & \text{if } i \le A' \text{ and } j \le B', \\ \frac{\beta s}{B} + \frac{(B'-B)s}{AB} & \text{if } i > A' \text{ and } j \le B', \\ \frac{(\beta+1)s}{B} + \frac{B's}{AB} & \text{if } i \le A' \text{ and } j > B', \\ \frac{\beta s}{B} + \frac{B's}{AB} & \text{otherwise,} \end{cases} \\ x([i,j][*,j]) &= \begin{cases} \frac{(\alpha+1)s}{A} + \frac{(A'-A)s}{AB} & \text{if } i \le A' \text{ and } j \le B', \\ \frac{(\alpha+1)s}{A} + \frac{A's}{AB} & \text{if } i > A' \text{ and } j \le B', \\ \frac{\alpha s}{A} + \frac{(A'-A)s}{AB} & \text{if } i \le A' \text{ and } j > B', \\ \frac{\alpha s}{A} + \frac{A's}{AB} & \text{otherwise,} \end{cases} \\ x([i,j][*,*]) &= \frac{\delta - (A'+B')s}{AB}, \end{aligned}$$

where i ranges from 1 to A and j from 1 to B. Since both the integers A and B are at least $4s$, we infer that:

$$\frac{\alpha s}{A} \ge \frac{\left(\frac{A}{2s} - 1\right) s}{A} = \frac{1}{2} - \frac{s}{A} \ge \frac{1}{4},$$

and similarly:

$$\frac{\beta s}{B} \ge \frac{1}{4}.$$

On the other hand, we have also the following bounds:

$$\frac{(B'-B)s}{AB} \ge -\frac{s}{A} \ge -\frac{1}{4} \quad \text{and} \quad \frac{(A'-A)s}{AB} \ge -\frac{s}{B} \ge -\frac{1}{4}.$$

These bounds together with (2) imply that each $x(e)$ is a non-negative real number.

Fix integers i, $1 \le i \le A$, and j, $1 \le j \le B$. The sum $x([i,j][i,*]) + x([i,j][*,j])$ is equal to the following:

$$\frac{\beta s}{B} + \frac{B's}{AB} + \frac{\alpha s}{A} + \frac{A's}{AB} = \frac{\alpha B s + \beta A s}{AB} + \frac{(A'+B')s}{AB} =$$

$$\frac{AB - \delta + (A'+B')s}{AB} = 1 - \frac{\delta - (A'+B')s}{AB} \ .$$

Therefore, the sum of the values assigned to all the three edges $[i,j][i,*]$, $[i,j][*,j]$ and $[i,j][*,*]$ incident with the vertex $[i,j]$ is equal to one. Since all the values $x(e)$ are non-negative, we infer that $0 \le x(e) \le 1$ for each edge e.

In this paragraph, we compute the sums of the values $x(e)$ taken over the edges e incident with the vertices of W. First, let us consider a vertex $[i,*]$ with $i \le A'$. Out of all the B edges between the vertex $[i,*]$ and the vertices of V, B' of them are assigned $\frac{(\beta+1)s}{B} + \frac{(B'-B)s}{AB}$ and $B - B'$ of them are assigned $\frac{(\beta+1)s}{B} + \frac{B's}{AB}$. Hence, the total sum of the values assigned to the edges incident with the vertex $[i,*]$ is the following:

$$B' \cdot \left(\frac{(\beta+1)s}{B} + \frac{(B'-B)s}{AB}\right) + (B - B') \cdot \left(\frac{(\beta+1)s}{B} + \frac{B's}{AB}\right) = (\beta+1)s \ .$$

Similarly, the sum of the values assigned to the edges incident with a vertex $[i,*]$, $i > A'$, is βs, the sum of the values of the edges incident with a vertex $[*,j]$ is $(\alpha+1)s$ if $j \le B'$, and αs, otherwise. Finally, each of the AB edges incident with the vertex $[*,*]$ is assigned $\frac{\delta-(A'+B')s}{AB}$, and thus the sum of the values of the edges incident with $[*,*]$ is $\delta - (A'+B')s \le s$.

If $\delta \ne 0$, then x_e is a fractional $\mathcal{S}_s(\Delta/2 + s)$-factor of the bipartite subgraph of $S_A \times S_B$ with the parts V and W. By Lemma 7, this bipartite subgraph of $S_A \times S_B$ has also a $\mathcal{S}_s(\Delta/2+s)$-factor which immediately yields the statement of the theorem.

If $\delta = 0$, each edge incident with the vertex $\{[*,*]\}$ is assigned 0 by x. Therefore, x is a fractional $\mathcal{S}_s(\Delta/2 + s)$-factor of the bipartite subgraph of $(S_A \times S_B) \setminus \{[*,*]\}$ with the parts V and $W \setminus \{[*,*]\}$. By Lemma 7, the graph $(S_A \times S_B) \setminus \{[*,*]\}$ has an $\mathcal{S}_s(\Delta/2 + s)$-factor. Moreover, since $A' = 0$ and $B' = 0$, the degree of each vertex $[i,*]$, $1 \le i \le A$, is $B/2$ and the degree of each vertex $[*,j]$, $1 \le j \le B$, is $A/2$. Since $B \ge 4s$, the degree of the vertex $[1,*]$ in the constructed factor is at least $2s$. We now turn the obtained factor into an $\mathcal{S}_s(\Delta/2 + s)$-factor of $S_A \times S_B$. Let X be a set of any s neighbors of the vertex $[1,*]$ in the factor. Reconnect each vertex of X from the vertex $[1,*]$ to the vertex $[*,*]$. The degrees of all the vertices of X remain equal to one, the degree of the vertex $[1,*]$ drops to $B/2-s$, which is still a positive multiple of s, and the degree of the vertex $[*,*]$ is now equal to s. Therefore, we have obtained an $\mathcal{S}_s(\Delta/2+s)$-factor of the entire product $S_A \times S_B$.

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5 Cycle factors in the products of stars

In this section, we show that the product of s-good stars contains a 2-factor comprised by large cycles. Before we do so, we introduce several definitions which will simplify our presentation. The vertices of the product of stars $S_{n_1} \times \cdots \times S_{n_k}$ are denoted by k-tuples whose entries are from the set $\{*, 1, 2, \ldots\}$. A *pattern* of a vertex v is a k-tuple in which each entry of v which is not a star is replaced with the circle ($\circ$). In particular, a pattern is always k-tuple whose entries are stars ($*$) and circles ($\circ$). A *k-pattern* is a pattern which contains exactly k stars, and a *k-vertex* is a vertex whose pattern is a k-pattern. Note that the 0-vertices form an independent set in the product.

As the first step towards the main goal of this section, we show that the product of stars whose sizes are positive multiples of s contains a 2-factor comprised by large cycles:

Lemma 9 *Let $s \geq 2$ and $D > s$ be positive integers, and let $k = \lceil (D+1)\ln(2+\frac{s}{D-1})\rceil$. The product of any k stars of sizes $n_1, \ldots, n_k$, where each size n_i is a positive multiple of s smaller or equal to D, has a 2-factor comprised by cycles of lengths at least $2s$.*

Proof: Let G be the product $S_{n_1} \times \cdots \times S_{n_k}$. The proof of the lemma is divided into two major steps. In the first step, we find a subgraph $\overline{F}$ of G formed by vertex-disjoint cycles whose lengths are at least $2s$ such that each 0-vertex of G is contained in one of the cycles of $\overline{F}$. In the second step, we modify the subgraph $\overline{F}$ to get the desired 2-factor of G.

Let P_i be the set of all i-patterns of length k. Note that the set P_i contains $\binom{k}{i}$ patterns. Split now each P_i into disjoint sets $P_i^1, \ldots, P_i^s$ of size $\left\lfloor \frac{1}{s}\binom{k}{i} \right\rfloor$ each and discard (at most s) remaining patterns contained in P_i.

Let W^j, $1 \le j \le s$, be the set of all the vertices of G whose pattern is in the union $\bigcup_{i=1}^{k} P_i^j$. Note that all the sets W^j are disjoint. Let $\overline{W}^j$ be the set of cardinality $|W^j|$ whose vertices one-to-one correspond to the vertices of W^j. Finally, let V^j, $1 \le j \le s$, be the set of all the 0-vertices of the graph G whose sum of entries modulo s is equal to j. Note that $|V^j| = (\prod_{i=1}^{k} n_i)/s$ for every j.

Fix j, $1 \le j \le s$, and let us consider an auxiliary bipartite graph H^j with the vertex set $V^j \cup W^j \cup \overline{W}^j \cup V^{j+1}$ (indices are taken modulo s if necessary). A vertex $v \in V^j$ is joined to a vertex $w \in W^j$ if the edge vw is contained in G. Similarly, a vertex $v \in V^{j+1}$ is joined to a vertex $\overline{w} \in \overline{W}^j$ if the graph G contains the edge vw where w is the counterpart of $\overline{w}$. Finally, H^j contains a perfect matching between W^j and $\overline{W}^j$ consisting of edges which join the pairs of the corresponding vertices. Clearly, H^j is a bipartite graph with the parts $V^j \cup \overline{W}^j$ and $W^j \cup V^{j+1}$.

We now show that the graph H^j contains a fractional 1-factor. For each pattern $\pi \in \bigcup_{i=1}^{k} P_i^j$, define a variable y_π which is initially equal to s/D^i if π is an i-pattern. We establish the following inequality:

$$\sum_{\pi \in \bigcup_{i=1}^{k} P_i^j} y_\pi \ge 1 \tag{3}$$

Since each P_i^j contains exactly $\left\lfloor \frac{1}{s}\binom{k}{i} \right\rfloor$ patterns, we can infer the following:

$$\begin{aligned}
\sum_{\pi \in \bigcup_{i=1}^{k} P_i^j} y_\pi &= \sum_{i=1}^{k} |P_i^j| \frac{s}{D^i} = \sum_{i=1}^{k} \left\lfloor \frac{1}{s}\binom{k}{i} \right\rfloor \frac{s}{D^i} \ge \\
\sum_{i=1}^{k} \left(\frac{1}{s}\binom{k}{i} - 1\right)\frac{s}{D^i} &= \sum_{i=1}^{k} \binom{k}{i}\frac{1}{D^i} - \sum_{i=1}^{k} \frac{s}{D^i} = \\
\sum_{i=0}^{k} \binom{k}{i}\frac{1}{D^i} - 1 - \frac{s(D^k-1)}{D^k(D-1)} &\ge \left(1+\frac{1}{D}\right)^k - 1 - \frac{s}{D-1} = \\
\left(1+\frac{1}{D}\right)^{\left\lceil (D+1)\ln\left(2+\frac{s}{D-1}\right)\right\rceil} &- 1 - \frac{s}{D-1} \ge \\
\left(1+\frac{1}{D}\right)^{(D+1)\ln\left(2+\frac{s}{D-1}\right)} &- 1 - \frac{s}{D-1} \ge \\
e^{\ln(2+\frac{s}{D-1})} - 1 - \frac{s}{D-1} &= 2 + \frac{s}{D-1} - 1 - \frac{s}{D-1} = 1
\end{aligned}$$

The inequality (3) now readily follows.

In this paragraph, we construct the fractional 1-factor of H^j. First, decrease the value of some of the variables y_π so that the sum $\sum_{\pi\in\bigcup_{i=1}^k P_i^j} y_\pi$ becomes equal to one but all the variables y_π remain non-negative. The value $x(vw)$ for an edge vw between $v \in V^j$ and $w \in W^j$ is set to y_π where π is the pattern of w. Similarly, the value $x(v\overline{w})$ for an edge $v\overline{w}$ between $v \in V^{j+1}$ and $w \in \overline{W}^j$ is set to y_π where π is the pattern of the counterpart of $\overline{w}$. Finally, choose the value of $x(w\overline{w})$ so that the total sum of the values assigned to all the edges incident with the vertex w is equal to one.

We verify that we have constructed a fractional 1-factor. The number of neighbours of a vertex $w \in W^j$ that are in V^j is equal to $(\prod_{i\in I} n_i)/s$ where I are the indices of the star entries of w. Indeed, there are $\prod_{i\in I} n_i$ choices how to replace star entries of w in order to obtain a 0-vertex, but only the fraction of $1/s$ of them have the sum of their entries equal to j modulo s. Since $(\prod_{i\in I} n_i)/s \le D^{|I|}/s$, the value of $x(w\overline{w})$ is non-negative. Similarly, the number of neighbors of a vertex $\overline{w} \in \overline{W}^j$ in V^{j+1} is equal to $(\prod_{i\in I} n_i)/s$ where I are the indices of the star entries of the counterpart vertex w. Note that the vertex w and its counterpart $\overline{w}$ have the same degree in H^j. Therefore, the sum of the values assigned to the edges between V^j and w is equal to the sum of the values assigned to the edges between V^{j+1} and $\overline{w}$. By the choice of the value of $x(w\overline{w})$, the sum of the values of the edges incident with each of w and $\overline{w}$ is equal to one.

It remains to verify that the sum of the values assigned to the edges incident with each vertex $v \in V^j \cup V^{j+1}$ is equal to one. For each pattern $\pi \in \bigcup_{i=1}^k P_i^j$, the vertex $v \in V^j \cup V^{j+1}$ is adjacent to precisely one vertex of $W^j \cup \overline{W}^j$ with the pattern π. Hence, the sum of the values assigned to the edges incident with v is equal to $\sum_{\pi\in\bigcup_{i=1}^k P_i^j} y_\pi = 1$.

We have shown that each H^j, $1 \le j \le s$, has a fractional 1-factor. Since the graph H^j is bipartite, it has a 1-factor by Lemma 7. Fix for the rest of the proof a 1-factor F^j of H^j. In addition, contract in each H^j the edge between a vertex w and its counterpart $\overline{w}$. Let $\overline{H}^j$ be the resulting graph with the vertex set $V^j \cup W^j \cup V^{j+1}$ and let $\overline{F}^j$ be the spanning subgraph of $\overline{H}^j$ comprised by the edges contained in the 1-factor F^j of H^j. Observe that, in $\overline{F}^j$, each vertex $v \in V^j \cup V^{j+1}$ has degree exactly one and each vertex $w \in W^j$ has degree either zero or two.

Since the sets W^j, $1 \le j \le s$, are mutually disjoint, the union $\overline{F}^1 \cup \cdots \cup \overline{F}^s$ induces a 2-regular subgraph $\overline{F}$ of G which contains all the 0-vertices of G (the vertices of degree zero are not included to $\overline{F}$). In each cycle of $\overline{F}$, the 0-vertices alternate with the other vertices of G and the sum of the entries of the 0-vertices increases by one (modulo s) when traversing the cycle. We conclude that the length of each cycle of $\overline{F}$ is a multiple of $2s$, in particular, it is at least $2s$.

As announced in the beginning of the proof, we now turn the subgraph $\overline{F}$ into a 2-factor of G. Assume that a vertex v of G is not contained in $\overline{F}$. Note that v is not a 0-vertex. Let v' be the vertex of G obtained from v by substituting all the star entries of v with 0. Since v' is a 0-vertex, it is contained in a cycle of $\overline{F}$. Let v'' be a neighbor of v' in the cycle. Replace the edge $v'v''$ by the edges $v'v$ and vv'' (observe that G contains the edge vv''). In this way, insert all the vertices to the cycles of $\overline{F}$. Since the length of the cycles cannot decrease during this process, we eventually obtain a 2-factor of G whose each cycle has length at least $2s$.

■

We now extend Lemma 9 to the case of products of s-good stars:

Lemma 10 *Let D and s be positive integers such that $2 \leq s < D$ and let $k = \lceil (D+1)\ln(2+\frac{s}{D-1})\rceil + 4s - 2$. The product of any k stars from $\mathcal{S}_s(D)$ has a 2-factor comprised by cycles of lengths at least $2s$.*

Proof: Consider k stars from $\mathcal{S}_s(D)$ and let $n_1, \ldots, n_k$ be their sizes. Note that at least $4s-1$ of the numbers $n_1, \ldots, n_k$ are smaller than $4s$ or at least $\lceil (D+1)\ln(2+\frac{s}{D-1})\rceil$ of them are divisible by s. We deal with the two cases separately.

Assume first that at least $4s-1$ of the numbers $n_1, \ldots, n_k$ are smaller than $4s$. Without loss of generality, let us say that these numbers are $n_1, \ldots, n_{4s-1}$. By Corollary 5, the product of $S_{n_1} \times \cdots \times S_{n_{4s-1}}$ is hamiltonian. Clearly, the order of $S_{n_1} \times \cdots \times S_{n_{4s-1}}$ is $\prod_{i=1}^{s}(n_i+1) \geq 2s$. The copies of the obtained Hamilton cycle in the product of $S_{n_1} \times \cdots \times S_{n_{4s-1}}$ with $S_{n_{4s}} \times \cdots \times S_{n_k}$ comprise a 2-factor whose cycles have length at least $2s$.

We now consider the case that at least $k-4s+2 = \lceil (D+1)\ln(2+\frac{s}{D-1})\rceil$ of the numbers $n_1, \ldots, n_k$ are divisible by s. Assume without loss of generality that the numbers $n_1, \ldots, n_{k-4s+2}$ are multiples of s. By Lemma 9, the graph $S_{n_1} \times \cdots \times S_{n_{k-4s+2}}$ has a 2-factor comprised by cycles of lengths at least $2s$. The copies of the cycles of the 2-factor in the product of $S_{n_1} \times \cdots \times S_{n_{k-4s+2}}$

with $S_{n_{k-4s+3}} \times \cdots \times S_{n_k}$ form a 2-factor comprised by cycles of lengths at least $2s$.

■

6 The main result

As the final step towards our main result, we show that the product of approximately $(\ln 2)\Delta$ stars of sizes at most Δ has a 2-factor comprised by cycles of lengths at least Δ:

Lemma 11 *Let $\Delta \geq 1$ be an integer. The product of any $(\ln 2)\Delta + (10 + \ln 4)\sqrt{\Delta} + 0.5 \log_2 \Delta + O(1)$ stars of sizes at most Δ has a 2-factor comprised by cycles of lengths at least Δ.*

Proof: Let $s = \lceil \sqrt{\Delta} \rceil$ and $D = \frac{\Delta}{2} + s$. In addition, let k be the following number:

$$k = 2\left(\left\lceil (D+1)\ln\left(2 + \frac{s}{D-1}\right)\right\rceil + 4s - 2\right) + \sum_{i=1}^{\lceil 0.5 \log_2 \Delta \rceil} \left\lceil \frac{\Delta}{2^i s} \right\rceil .$$

We show that the product of any k stars of sizes at most Δ has a 2-factor comprised by cycles of lengths at least Δ. Note that the number k is of the magnitude claimed in the statement of the lemma:

$$k = 2\left(\left\lceil (D+1)\ln\left(2 + \frac{s}{D-1}\right)\right\rceil + 4s - 2\right) + \sum_{i=1}^{\lceil 0.5 \log_2 \Delta \rceil} \left\lceil \frac{\Delta}{2^i s} \right\rceil =$$

$$2(D+1)\ln\left(2 + \frac{s}{D-1}\right) + 8s + \sum_{i=1}^{\lceil 0.5 \log_2 \Delta \rceil} \left\lceil \frac{\Delta}{2^i s} \right\rceil + O(1) =$$

$$2\left(\frac{\Delta}{2} + \sqrt{\Delta} + O(1)\right)\left(\ln 2 + \frac{s}{2D} + O\left(\frac{s^2}{D^2}\right)\right) + 8s +$$

$$\sum_{i=1}^{\lceil 0.5 \log_2 \Delta \rceil} \left\lceil \frac{\Delta}{2^i \left\lceil \sqrt{\Delta} \right\rceil} \right\rceil + O(1) =$$

$$2\left(\frac{\Delta}{2} + \sqrt{\Delta} + O(1)\right)\left(\ln 2 + \frac{1}{\sqrt{\Delta}} + O\left(\frac{1}{\Delta}\right)\right) + 8\sqrt{\Delta} +$$

$$\sum_{i=1}^{\infty} \frac{\Delta}{2^i\sqrt{\Delta}} + 0.5\log_2\Delta + O(1) =$$

$$(\ln 2)\Delta + \sqrt{\Delta} + 2(\ln 2)\sqrt{\Delta} + O(1) + 8\sqrt{\Delta} + \sqrt{\Delta} + 0.5\log_2\Delta + O(1) =$$

$$(\ln 2)\Delta + (10+\ln 4)\sqrt{\Delta} + 0.5\log_2\Delta + O(1)\ .$$

Let $n_1, \ldots, n_k$ be any positive integers between 1 and Δ. We aim to find a 2-factor of $S_{n_1} \times \cdots \times S_{n_k}$. Let $k' = \left\lceil (D+1)\ln\left(2 + \frac{s}{D-1}\right)\right\rceil + 4s - 2$. For each i, $1 \le i \le k'$, apply Lemma 8 to the product of $S_{n_{2i-1}}$ and $S_{n_{2i}}$ to get an $\mathcal{S}_s(\Delta/2+s)$-factor of $S_{n_{2i-1}} \times S_{n_{2i}}$. Let $S^i_1, \ldots, S^i_{m_i}$ be the components of the obtained $\mathcal{S}_s(\Delta/2+s)$-factor of $S_{n_{2i-1}} \times S_{n_{2i}}$. By Lemma 10, each individual product $S^1_{j_1} \times \cdots \times S^{k'}_{j_{k'}}$, $1 \le j_i \le m_i$, has a 2-factor comprised by cycles of lengths at least $2s$. Since the stars $S^i_1, \ldots, S^i_{m_i}$ form a factor of $S_{n_{2i-1}} \times S_{n_{2i}}$ for each i, $1 \le i \le k'$, we infer that the product $S_{n_1} \times \cdots \times S_{n_{2k'}}$ has a 2-factor comprised by cycles of length at least $2s$.

For the rest of the proof, fix a 2-factor of $S_{n_1} \times \cdots \times S_{n_{2k'}}$ formed by cycles of length at least $2s$ and let $C_1, \ldots, C_m$ be the cycles which comprise it. The product $S_{n_{2k'+1}} \times \cdots \times S_{n_{2k'+\lceil \frac{\Delta}{2s}\rceil}}$ has an $\mathcal{S}(2s)$-factor by Lemma 4 used with parameters $k = \lceil \frac{\Delta}{2s} \rceil$ and $\ell = 2s$. Similarly, the product $S_{n_{2k'+\lceil \frac{\Delta}{2s}\rceil+1}} \times \cdots \times S_{n_{2k'+\lceil \frac{\Delta}{2s}\rceil+\lceil \frac{\Delta}{4s}\rceil}}$ has an $\mathcal{S}(4s)$-factor, the product of the next $\lceil \frac{\Delta}{8s} \rceil$ stars has an $\mathcal{S}(8s)$-factor, etc. Let m_i denote the number of the components (stars) of the $\mathcal{S}(2^i s)$-factor, $1 \le i \le \lceil 0.5\log_2\Delta\rceil$, and let $\overline{S}^i_1, \ldots, \overline{S}^i_{m_i}$ be the components.

Let $k'' = \lceil 0.5\log_2\Delta\rceil$. Since the length of each cycle C_{j_0}, $1 \le j_0 \le m$, is at least $2s$, the product $C_{j_0} \times \overline{S}^1_{j_1}$ is hamiltonian for every $1 \le j_1 \le m_1$ by Theorem 1. Recall that the size of the star $\overline{S}^1_{j_1}$ is at most $2s$. Since the length of the Hamilton cycle of $C_{j_0} \times \overline{S}^1_{j_1}$ is at least $4s$, the product $C_{j_0} \times \overline{S}^1_{j_1} \times \overline{S}^2_{j_2}$ is also hamiltonian for every $1 \le j_2 \le m_2$ by Theorem 1. In this way, we infer that the product $C_{j_0} \times \overline{S}^1_{j_1} \times \cdots \times \overline{S}^{k''}_{j_{k''}}$ is hamiltonian for every choice of $j_0, j_1, \ldots, j_{k''}$, and the length of its Hamilton cycle is at least $2^{k''+1}s$. Since $2^{k''+1}s \ge 2^{0.5\log_2\Delta}\sqrt{\Delta} = \Delta$, the Hamilton cycles of $C_{j_0} \times \overline{S}^1_{j_1} \times \cdots \times \overline{S}^{k''}_{j_{k''}}$ form a 2-factor of $S_1 \times \cdots \times S_k$ comprised by cycles of lengths at least Δ.

■

We are now ready to prove the main result of this paper:

Theorem 12 *Let $\Delta \geq 1$ be an integer. The product of any $(\ln 2)\Delta + (10 + \ln 4)\sqrt{\Delta} + 0.5 \log_2 \Delta + O(1)$ non-trivial connected graphs of maximum degree at most Δ is pancyclic.*

Proof: Let $k = (\ln 2)\Delta + (10 + \ln 4)\sqrt{\Delta} + 0.5 \log_2 \Delta + O(1)$ be the number of stars from the statement of Lemma 11. We show that the product of any $k + 2$ non-trivial connected graphs of maximum degree at most Δ is pancyclic.

Let $G_1, \ldots, G_{k+2}$ be any non-trivial connected graphs of maximum degree at most Δ. Each G_i, $1 \leq i \leq k$, has a $S(\Delta)$-factor by Lemma 3. Let $S^i_1, \ldots, S^i_{m_i}$ the components of the $S(\Delta)$-factor of G_i. Each of the products $S^1_{j_1} \times \cdots \times S^k_{j_k}$, $1 \leq j_i \leq m_i$, has a 2-factor comprised by cycles of lengths at least Δ by Lemma 11. The cycles of these 2-factors form a 2-factor of $G_1 \times \cdots \times G_k$ whose each cycle has length at least Δ. By Lemma 6, the product $(G_1 \times \cdots \times G_k) \times G_{k+1} \times G_{k+2}$ is pancyclic.

■

7 Conclusion

The original motivation for our research was the question related to the existence of Hamilton cycles in strong powers of graphs. We have managed to improve the bound $h_{\max}(\Delta) \leq \left(\ln \frac{25}{12} + \frac{1}{64}\right) \Delta + o(\Delta)$ from [11] to $h_{\max}(\Delta) \leq (\ln 2)\,\Delta + O(\sqrt{\Delta})$. As noted in Introduction, the coefficient at the linear term is the best possible since the graph $S_\Delta^{\lfloor (\ln 2)\Delta \rfloor}$ is not hamiltonian. Therefore, $h_{\max}(\Delta) \geq \lfloor (\ln 2)\Delta \rfloor + 1$. On the other hand, the gap between the lower bound and the upper is still non-constant, namely, it is of order $\Theta(\sqrt{\Delta})$. Using our technique based on fractional factors, we are not able to close the gap further. Therefore, the following remains as an open problem:

Problem 1 *Is it true that $h_{\max}(\Delta) \leq (\ln 2)\,\Delta + O(1)$?*

Or, in the stronger form:

Problem 2 *Is it true that the product of any $(\ln 2)\Delta + O(1)$ non-trivial connected graphs of maximum degree at most Δ is pancyclic?*

Note that in order to answer Problem 2 in affirmative, by Lemma 6, it is enough to show that the product of any $(\ln 2)\Delta + O(1)$ non-trivial connected graphs of maximum degree at most Δ has a 2-factor whose each cycle has length at least Δ.

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