IX. Midsummer Combinatorial Workshop

(M. Mareš, ed.)

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Preface

The Ninth Prague Midsummer Combinatorial Workshop was held from July 29 to August 2, 2002 at Malostranské náměstí building of Charles University which is depicted on the cover of this publication. The workshop was organized by the Department of Applied Mathematics (KAM) of Charles University jointly with the DIMATIA centre. Only a small but distinguished group of mathematicians was invited and we were particularly happy to have Hubert de Fraysseix and Stefan Felsner among us.

As it already became a tradition, the workshop benefited from participation of young researchers and PhD students, and as in the last three years, 5 selected undergraduate students from the USA and 5 students from Charles University took part in the workshop, together with student guides Martin Mareš and José Torres, in the framework of the joint DIMATIA-DIMACS REU program.

The workshop followed an informal daily routine with morning and early afternoon discussions and presentations. This report reflects some of the presentations during the workshop. Perhaps you can digest from these proceedings some of the atmosphere at the workshop and you can also see that the fruitful exchange of ideas led directly to some new results and papers.

This volume was edited by Martin Mareš. Most of the problems described here were supplied by the authors in electronic form; in a few cases, slight typographical changes were necessary. We apologize for any possible inaccuracies which might have occurred in the editing process.

The conference photos were taken in the beautiful baroque library of the Premonstratensian monsatery at Strahov where we were on a very exciting excursion. Also, the conference dinner was held in a nearby restaurant at Nebozízek.

This summer workshop was partly supported by a Charles University grant GAUK 158, and Kontakt 337. The support of our institute ITI (financed by the Ministry of Education of the Czech Republic as project LN00A056) was instrumental for the success of the workshop.

Based on our past experience and being encouraged by several participants, we hope to organize the Tenth Prague Combinatorial Workshop in the summer of 2003. We hope to meet you all there!

Jaroslav Nešetřil

Improving the crossing constant

Stefan Felsner

One of the most fundamental result about geometric graphs is the crossing lemma:

Lemma 1 (Crossing Lemma) *If G is a graph with n vertices and $m > 100n$ edges, then*

$$\mathrm{Cr}(G) \geq \gamma_x \frac{m^3}{n^2}$$

for some constant γ_x.

The crossing lemma was conjectured by Erdős and Guy in the 70's. First proofs where given by Ajtai, Chvátal, Newborn and Szemerédi and independently by Leighton in the 80's. In the 90's a very easy probabilistic proof became popular, this proof shows $\gamma_x \geq 1/64$. Pach and Tóth improved the constant from $1/64 \approx 0.015$ to 0.029. The result is based on the following notion and result:

> A geometric graph is *k-restricted* if every edge is crossed by at most k other edges.
>
> For $k = 0, 1, 2$, if G is a k-restricted geometric graph on n vertices, then
> $$m(G) \leq (k + 3)(n - 2).$$

We conjecture that 3-restricted graphs have at most $5.5(n - 2)$ edges. This implies $\gamma_x \geq 0.31$. Some ideas that may lead to a proof are discussed in the talk.

Post-Workshop Note: Radoš Radoičić has a proof of the above conjecture.

Distance labelings of trees

Jiří Fiala

Let p, q be positive intergers and T be a tree. We pose a question on the computational complexity of determining the minimal width of an interval, such that vertices of the tree can be assigned elements of the interval s.t. labels of adjacent vertices differ by at least p and labels of vertices at distance two differ by at least q.

We know that a polynomial-time algorithm for this problem exists whenever q divides p. On the other hand the precoloring version of this problem is NP-complete. (Here some vertices are already pre-assigned their labels.)

Tutte Trails

Tomáš Kaiser

A famous theorem of Tutte [6] states that all 4-connected planar graphs are hamiltonian. There is a conjecture of Thomassen [5] which resembles this theorem: All 4-connected line graphs are hamiltonian. Indeed, this is one of the many equivalent forms of the *dominating cycle conjecture*. Is the similarity superficial? Could line graphs and planar graphs share some crucial property ensuring that the 4-connectivity implies the hamiltonicity? If so, the property does not seem easy to find.

It is interesting that all known proofs ([4],[3],[2]) of extensions of the Tutte theorem follow the pattern of Tutte's original proof, being based on the notion of a Tutte cycle (as it is called today). The definition is as follows. For any subgraph H of a graph G, an *H-bridge* is any component of $G - V(H)$ together with the edges attaching it to H and their ends in H (the *attachment* vertices). A single edge with both ends in H is also regarded as an H-bridge. A cycle C in a graph is a *Tutte cycle* if every C-bridge has at most 3 attachment vertices. Note that in a 4-connected graph, a Tutte cycle is necessarily a Hamilton cycle (all bridges are just edges). Thus, since Tutte proved that every 2-connected planar graph contains a Tutte cycle, he also showed that all 4-connected planar graphs are hamiltonian.

At a workshop on the dominating cycle conjecture in Hannover this year, M. Kriesell and H. Broersma discussed some attempts to prove the conjecture along the lines of Tutte's proof. I suggested what turned out to be (more or less) an existing conjecture due to Jackson [1] in 1992:

Conjecture (Jackson) *Every bridgeless graph has a closed trail T of length at least 3 such that every T-bridge is attached by at most 3 edges.*

I propose to call trails of this sort *Tutte trails*. The conjecture would directly imply the following form of the dominating cycle conjecture: Every cyclically 4-connected cubic graph has a dominating cycle. (Here, a cubic graph is *cyclically 4-connected* if it is connected and has no nontrivial edge cuts of size at most 3; a cycle C in is *dominating* if the vertices not on C form an independent set.) To see the implication, simply note that a closed trail T in a cubic graph is a cycle, and that any bridge of this cycle contains at most one vertex outside T, for otherwise we would obtain a non-trivial edge cut of size at most 3. But then the vertices outside T form an independent set.

At present, it is not at all clear if the substantial planarity arguments in the existing proofs of the Tutte theorem can be replaced by ones with weaker assumptions (if we look for a trail, rather than a cycle). Although Jackson's conjecture is 10 years old by now, the possibility is still there.

I am indebted to M. Kriesell, H. Broersma and J. Torres for interesting conversations on this topic.

References

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Point Configurations in Almost Convex Position

Gyula Károlyi

We say that a set of points in the plane is in *general position* if no three of them are collinear. $\mathcal{X}$ will denote a set of points in the plane in general position. Let $\mathrm{vert}(\mathcal{X})$ denote the vertex set of the convex hull of $\mathcal{X}$. A polygon is said to be *empty*, if it contains no elements of $\mathcal{X}$ in its interior. If every triple in $\mathrm{vert}(\mathcal{X})$ determines an empty triangle, then $\mathcal{X} = \mathrm{vert}(\mathcal{X})$ is in *convex position* or, in short, *convex*.

According to a well known theorem of Erdős and Szekeres, for any integer $n \geq 3$, there exists $E(n) = O(4^n)$ with the property that every set $\mathcal{X}$ of at least $E(n)$ points in general position in the plane has n elements in convex position. It has been also suspected by Erdős that every sufficiently large point set contains the vertex set of an *empty* convex n-gon. This conjecture however turned out to be false even in the case $n = 7$. Thus, in order to state any positive result one has to either weaken on the emptiness condition or make some restriction on the point set in question. This is indeed possible, and best results so far, in each direction, depend on the novel notion of *almost convex* sets.

Towards to first direction, Bialostocki, Dierker, and Voxman proposed the following modular version of the original problem.

Conjecture *For any $n \geq 3$ and $p \geq 2$, there exists an integer $B(n,p)$ such that every set of $B(n,p)$ points in general position in the plane determines a convex n-gon such that the number of points in its interior is* $0 \bmod p$.

Bialostocki et al. verified this conjecture for every $n \geq p + 2$. The original upper bound on $B(n,p)$ was later improved by Caro, but his proof also relied heavily on the assumption $n \geq p+2$. In a joint work with János Pach and Géza Tóth we somewhat relaxed this condition.

Theorem 1 *For any $n \geq 5p/6 + O(1)$, there exists an integer $B(n,p)$ such that every set of $B(n,p)$ points in general position in the plane determines a convex n-gon such that the number of points in its interior is* $0 \bmod p$.

If every triple in $\mathrm{vert}(\mathcal{X})$ determines a triangle with *at most one* point in its interior, then $\mathcal{X}$ is said to be *almost convex*.

The proof of Theorem 1 was based on the following result that is an instance how one can proceed in the other direction.

Theorem 2 *For any $n \geq 3$, there exists an integer $K(n)$ such that every almost convex set of at least $K(n)$ points in general position in the plane determines an empty convex n-gon. Moreover, we have $K(n) = \Omega(2^{n/2})$.*

This result has been extended by Valtr to k-convex sets, that is, sets in which every triple determines a triangle with *at most k* points in its interior. Both proofs relied heavily on Ramsey's theorem that resulted in huge admissible bounds that have been reduced recently by Kun and Lippner and subsequently by Valtr using more refined combinatorial arguments.

An additional property of almost convex sets is that every sufficiently large k-convex set $\mathcal{X}$ contains a subset with a prescribed number of points of $\mathcal{X}$ in the interior of its convex hull, a result due to Bisztriczky, Hosono, Károlyi, and Urabe.

In this talk we overviewed these results and the methods of their proofs. The message of the talk was that in the problems related to the Erdős-Szekeres theorem, though Ramsey's theorem is a natural and powerful tool, to obtain good quantitative results one has to look for more refined methods, and also that the notion of almost convexity plays an important role.

Improved PTAS for the unit-height rectangle packing problem: a new dynamic programming procedure

Sofia Kovaleva

Abstract

We consider the following problem: given a set of weighted axis-parallel unit-height closed rectangles on the plane find a maximum weight subset in which no two rectangles intersect. This problem is equivalent to the maximum weight independent set problem in the intersection graph of rectangles of unit height. The previously known polynomial time approximation scheme described in Agarwal *et al* [1] uses a shifting technique of Hochbaum and Maass [6] in combination with a dynamic programming procedure. Its running time and memory consumption are both $O(n^{2k-1})$ with respect to approximation factor of $(1-1/k)$, where k is a fixed constant ($k \geq 2$) and n is number of rectangles in the input. We describe a new dynamic programming procedure which enables us to reduce the time and memory requirements of that PTAS to $O(k^2 n^k \log n)$ and $O(n^{k-1})$ respectively.

KEYWORDS: rectangle packing, independent set, dynamic programming

1 Introduction

The problem of choosing a maximum cardinality non-overlapping subset of a given set of axis-parallel closed rectangles on the plane is a special case of the general set packing problem. We refer to it as the *rectangle packing problem*. From a graph-theoretical point of view, the problem is the maximum independent set problem in the intersection graph of axis-parallel rectangles. It has received a great attention in the literature due to its applications, such as VLSI design [6], data mining [2] and map labeling [1].

The latter, or, more specific, automated label placement is an important tool for visualization of information on the web. For instance, imagine a web server that displays a geographic map of some area. On a user's request, the map can be dynamically enriched with information such as location of hotels, gas stations, etc. In this case it can be required to label automatically a large number of points with non-overlapping labels of prescribed shapes, sizes and positions. The labels can often be represented as axis-parallel closed rectangles and each label has typically to be attached with a corner to the point that it labels. This leaves for every label four positions, from which at most one is to choose. The objective is to choose the maximum number of labels of different points that do not overlap. Notice, that since the closed rectangles, that are attached with a corner to a common point, overlap with each other, a set of non-overlapping labels always contain at most one label for each point. Thus we obtain the rectangular packing problem.

We refer to the specified above references for a more elaborate description of the other applications.

Unfortunately, the rectangle packing problem is a notoriously difficult problem. Even the simple case restricted to squares of unit size is known to be NP-hard [5]. To our knowledge, it is still an open problem whether the general rectangle packing problem can be approximated with a constant factor in polynomial time. So far the best known approximation algorithm by Berman et al. [2] only achieves a factor of $O(\log_b n)$ for any constant b if running in time $n^{O(b)}$. However, in many cases a particular application suggests some restrictions simplifying the model. For instance, Hochbaum and Maass [6], in connection to applications in VLSI design, considered the packing problem restricted to unit squares or disks (hypercubes or balls in higher dimensions). They developed a well known shifting technique that enabled a polynomial time approximation scheme (PTAS) with running time $n^{O(k^d)}$ with respect to approximation factor $(1+1/k)$. (Here and further on in this paper we use d for the space dimension, $k \geq 2$ for a fixed integer, defining approximation factor, n for the number of objects in the input). The strategy generalizes to any collection of fat objects of roughly the same size (an object is fat if the maximum-to-minimum size ratio is bounded by a constant). By applying dynamic programming along one of the dimensions we can reduce the time complexity to $n^{O(k^{d-1})}$ and further generalize to objects, whose projection to the first $d-1$ coordinates is fat and of roughly the same size. The latter was essentially shown by Agarwal et. al. [1], who were interested in rectangles of unit height. This type of problem naturally occurs in map labeling, when one aims to use labels containing

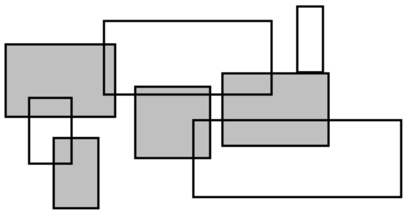

Figure 1: A problem instance and a feasible solution (shaded).

strings in a uniform typeface. The case when the objects are fat but can have possibly varying sizes was considered in [4] and [3]. These authors independently developed two PTAS using similar in spirit strategies. The running time $n^{O(k^4)}$ achieved for R^2 in [4] was improved to $n^{O(k^{d-1})}$ for R^d by [3].

In this paper we focus on packing unit-height rectangles in R^2. Moreover we consider the weighted version of the problem, which has not received much attention in the literature so far, but is of course a natural and important generalization. As was mentioned above, Agarwal et. al. [1] described a PTAS for the unweighted case of this problem using the shifting technique by Hochbaum and Maass [6] in combination with a dynamic programming subroutine. Their algorithm easily generalizes to the weighted case and requires $O(n^{2k-1})$ time and memory space. To a large extent, this value is determined by the dynamic programming procedure they use. Applying a different approach to dynamic programming we describe a routine that is more economical in time and memory consumption. This enables us to reduces the time and memory requirements of the PTAS of [1] to $O(k^2 n^k \log n)$ and $O(n^{k-1})$ respectively.

In the next section we give the formal problem statement and describe the framework of the PTAS of [1]. In section 3 we describe our dynamic programming procedure.

2 The Polynomial-Time Approx. Scheme

We consider the following problem $\mathcal{P}$:

Given is a set $\mathcal{R}$ of n axis-parallel closed rectangles of unit height: $\mathcal{R}= \{1, 2, \ldots, n\}$. Each rectangle i is specified in terms of its Cartesian coordinates $[l_i, r_i] \times [b_i, b_i + 1]$. The rectangles are ordered according to non-decreasing r_i. For each rectangle i a weight $w(i)$ is specified, $w(i) \in R^+, \forall i \in 1, \ldots, n$. The goal is to find a maximum weight subset of pairwise non-overlapping rectangles.

An example of a problem instance and a feasible solution to it is shown on figure 1. Notice that rectangles are closed and therefore two rectangles sharing an edge are considered to be overlapping.

Following [1], for any $k \in N$ we describe a polynomial time algorithm $\mathcal{A}^k$ such that $\mathcal{A}^k(I) \geq (1 - 1/k) \cdot OPT(I)$ for any instance I of $\mathcal{P}$, where $\mathcal{A}^k(I)$ and $OPT(I)$ are respectively the value delivered by $\mathcal{A}^k$ and the value of the optimum solution on I.

Algorithm $\mathcal{A}^k$:

- Consider an instance I. Draw a grid consisting of horizontal lines placed at a unit distance from each other. Without loss of generality assume that no rectangle intersects two lines with its top and bottom edges. (We can always choose a real number y so that the horizontal lines drawn through $y, y + 1, y + 2, \ldots, y + p$ intersect each rectangle only once.) Skip those lines which don't intersect any of the rectangles. Since all the rectangles have unit height, our grid will have at most n lines. This procedure can be easily implemented in $O(n)$ time.

- Number the lines in the grid consecutively starting from the top as $1, 2, \ldots, p$.

- For each $j \in \{0, \ldots, k - 1\}$ compose a new instance I_j containing all the rectangles of I except those intersecting the lines from the set $\{i : i \bmod k = j\}$.

- Solve our problem on each I_j to optimality using dynamic programming procedure $\mathcal{B}$ described in section 3.

- Return a solution with the best value.

The following result was first established in [6]:

Theorem 1 *Algorithm $\mathcal{A}^k$ delivers a solution with value $\mathcal{A}^k(I) \geq (1-1/k) \cdot OPT(I)$ for all instances I.*

Proof. Consider some optimum solution $S^{OPT}(I)$ and its intersections with $I_j, \forall j$: $S^{OPT}(I) \cap I_j$.

(We denote the total weight of rectangles in a subset X as $w(X)$.)

The key observation is the following:

$$\sum_{j=0}^{k-1} w(S^{OPT}(I) \cap I_j) = (k-1) \cdot w(S^{OPT}(I)).$$

To explain this notice that each rectangle i has to be intersected by exactly one line of the grid. Therefore i is included in all I_j-s except one, thus in exactly $k-1$ I_j-s. Therefore the weight of each rectangle $i \in S^{OPT}$ enters exactly $k-1$ terms of the sum and thus is counted $k-1$ times in the left-hand as well as in the right-hand part of the equality.

Since $S^{OPT}(I) \cap I_j$ is a feasible solution to problem $\mathcal{P}$ on instance I_j it holds: $w(S^{OPT}(I) \cap I_j) \leq OPT(I_j)$.

Recall that $w(S^{OPT}(I)) = OPT(I)$. Now we have:

$$\sum_{j=0}^{k-1} OPT(I_j) \geq \sum_{j=0}^{k-1} w(S^{OPT}(I) \cap I_j) = (k-1) \cdot OPT(I).$$

Thus one of $OPT(I_j)$-s has to be greater or equal to $\frac{k-1}{k} \cdot OPT(I)$. $\square$

3 Dynamic Programming Procedure $\mathcal{B}$

Let us now describe how to solve a subinstance I_j to optimality in time polynomial in n.

Observe that I_j was constructed so that it consists of at most $\lceil n/k \rceil$ subsets of rectangles $S_1, S_2, \ldots, S_{\lceil n/k \rceil}$ so that any two rectangles from different subsets don't intersect each other. Each subset S_i is a set of rectangles intersecting all the lines of the grid with numbers strictly between $k \cdot i + j$ and $k \cdot (i+1) + j$, thus at most $k-1$ lines. The optimal solution on I_j can be found as the union of the optimal solutions on the subsets S_i, $i = 1, .., \lceil n/k \rceil$.

After giving some definitions we describe a dynamic programming procedure $\mathcal{B}$ that finds an optimal solution to $\mathcal{P}$ on any $S \in \{S_1, S_2, \ldots, S_{\lceil n/k \rceil}\}$ in $O(k|S|^k \log |S|)$ time and $O(|S|^{k-1})$ space.

In what follows we restrict our attention to rectangles from the set S. For simplicity we change their numbering to make it consecutive while keeping the same order.

Definition 1 *Let us call an ordered collection* $\{j_1, j_2, \ldots, j_l\}$ *of pairwise non-overlapping rectangles that intersect a common vertical line a* ***pile****. The rectangles in the pile are ordered according to increasing* j_i*.*

Observe that there is no pile of size more than $k-1$ in S (since all the rectangles intersect at most $k-1$ lines of the grid).

Definition 2 *Define an order* $\prec$ *on the set of piles as following:*

$$\{j_1, j_2, \ldots, j_l\} \prec \{i_1, i_2, \ldots, i_m\}$$

if $j_l < i_m$*, or if* $j_l = i_m$ *and* $j_{l-1} < i_{m-1}$ *and so on, or if* $m < l$ *and* $j_l = i_m,\ j_{l-1} = i_{m-1}, \ldots, j_{l-m+1} = i_1$*.*

This can be understood as a 'reverse lexicographic' order; notice that $\{j_1, j_2, \ldots, j_l\} \prec \{j_2, \ldots, j_l\}$.

Definition 3 *For a set of rectangles* X *we denote the optimum solution to* $\mathcal{P}$ *on* X *as* $OPT(X)$*. For a sequence of non-overlapping rectangles* $j_1, \ldots, j_l$ *sorted in increasing order, we define* $OPT(X|j_1, \ldots, j_l)$ *to be the optimum solution to* $\mathcal{P}$ *on* X *given that rectangles* $j_1, \ldots, j_l$ *are already embedded.*

Definition 4 *For a sequence of non-overlapping rectangles* $j_1, j_2, \ldots, j_l$*, sorted in increasing order and a rectangle* t*,* $t < j_1$*, we define a function* $v_{j_1, j_2, \ldots, j_l}(t)$ *as:*

$$v_{j_1,\ldots,j_l}(t) = OPT(\{i : i \le t\}|j_1, \ldots, j_l) - OPT(\{i : i < t\}|j_1, \ldots, j_l).$$

Thus this function has the meaning of marginal contribution of rectangle t in the optimum $OPT(\{i : i \le t\})$ given that $j_1, \ldots, j_l$ are embedded.

Remark 1 *Observe that* $v_{j_1,\ldots,j_l}(t) = 0$ *if* t *overlaps with some of* $j_1, \ldots, j_l$*. Otherwise,* $v_{j_1,\ldots,j_l}(t) = v_{i_1,\ldots,i_q}(t)$*, where* $i_1, \ldots, i_q$ *are those of* $j_1, \ldots, j_l$ *that intersect the vertical line drawn through the right edge of* t *(see Figure 2). For if a rectangle lies entirely at the right side of the line, its embedding cannot influence the choice of an optimal solution for rectangles lying at the other side. Notice that* $\{t, i_1, \ldots, i_q\}$ *constitutes a pile and, if* $\{j_1, \ldots, j_l\}$ *is a pile, then* $\{t, i_1, \ldots, i_q\} \prec \{j_1, \ldots, j_l\}$*.*

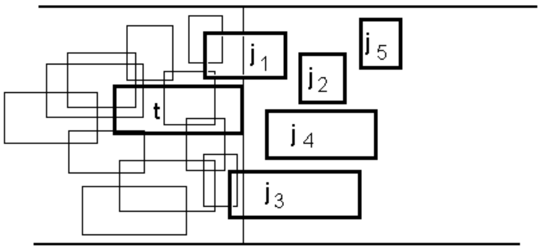

Figure 2: In this example $v_{j_1,j_2,\ldots,j_5}(t) = v_{j_1,j_3}(t)$.

Lemma 1

$$OPT(\{i : i \leq t\}|j_1, j_2, \ldots, j_l) = \sum_{i \leq t} v_{j_1,j_2,\ldots,j_l}(i)$$

Proof. Consider the sum at the right-hand side of the equality:

$$\begin{aligned}
v_{j_1,j_2,\ldots,j_l}(t) + v_{j_1,j_2,\ldots,j_l}(t-1) + \ldots + v_{j_1,j_2,\ldots,j_l}(1) = \\
& OPT(\{i : i \leq t\}|j_1, j_2, \ldots, j_l) \\
& - OPT(\{i : i < t\}|j_1, j_2, \ldots, j_l) \\
& + OPT(\{i : i \leq t-1\}|j_1, j_2, \ldots, j_l) \\
& - OPT(\{i : i < t-1\}|j_1, j_2, \ldots, j_l) \\
& \quad \ldots \\
& + OPT(\{i : i \leq 1\}|j_1, j_2, \ldots, j_l) \\
& - OPT(\{i : i < 1\}|j_1, j_2, \ldots, j_l).
\end{aligned}$$

Since $(i < t) \equiv (i \leq t-1)$, most of the terms cancel each other and we get the necessary expression. □

Lemma 2 *If rectangle t does not intersect any of $j_1, j_2, \ldots, j_l$, then:*

$$\begin{aligned}
v_{j_1,j_2,\ldots,j_l}(t) = &(w(t) + OPT(\{i : i < t\}|t, j_1, j_2, \ldots, j_l) \\
& - OPT(\{i : i < t\}|j_1, j_2, \ldots, j_l))^+.
\end{aligned}$$

Here $(x)^+ = x$ if $x \geq 0$ and 0 otherwise.

Proof. The above expression is easy to deduce from the following statement:

$$\begin{aligned}&OPT(\{i : i \le t\}|j_1, j_2, \ldots, j_l) = \\ &\quad = \max(w(t) + OPT(\{i : i < t\}|t, j_1, j_2, \ldots, j_l), \\ &\qquad\quad OPT(\{i : i < t\}|j_1, j_2, \ldots, j_l)).\end{aligned}$$

The terms under the $\max(\cdot)$ correspond to the two possible cases, namely, when rectangle t is selected and is not selected in the optimum. □

The next statement follows immediately from Lemmas 1 and 2:

Corollary 2 *If rectangle t does not intersect any of $j_1, j_2, \ldots, j_l$, then:*

$$v_{j_1,\ldots,j_l}(t) = \left(w(t) + \sum_{i:i<t} v_{t,j_1,\ldots,j_l}(i) - \sum_{i:i<t} v_{j_1,\ldots,j_l}(i) \right)^+ .$$

Let us now describe procedure $\mathcal{B}$:

- Step 1: Find all the piles in S and sort them in the order of $\prec$.
- Step 2: Take the piles one by one in the order $\prec$ starting from the least one. For each pile $\{j_1, j_2, \ldots, j_l\}$ calculate and store in the memory the value $v_{j_2,\ldots,j_l}(j_1)$ according to the formula:

  $$v_{j_2,\ldots,j_l}(j_1) = \left(w(j_1) + \sum_{i:i<j_1} v_{j_1,j_2,\ldots,j_l}(i) - \sum_{i:i<j_1} v_{j_2,\ldots,j_l}(i) \right)^+$$

  (see Remark 3).
- Step 3: When the values of $v_{\ldots}(\cdot)$ are known for all the piles, schedule the rectangles in the following way:

  Consider the rectangles from S in decreasing order.

  Take a rectangle j next in that order, suppose $j_1, \ldots, j_l$ are already scheduled, then schedule j iff $v_{j_1,\ldots,j_l}(j)$ is positive.

Remark 2 *Consider step 1. Since a pile in S can not consist of more than $k-1$ rectangles there are at most $\sum_{i=1}^{k-1} \binom{|S|}{i} = O(|S|^{k-1})$ piles. Sorting them can be implemented in $O(k|S|^{k-1} \log |S|)$ time.*

Remark 3 *In order to calculate the expression at step 2, observe that, for any* $i < j_1$, $v_{j_1,j_2,\ldots,j_l}(i)$ *equals either 0 or some previously calculated (and thus already available in the memory) value (see Remark 1). The same is true for* $v_{j_2,\ldots,j_l}(i)$. *Thus to evaluate one item under the sums one might need to search for a particular pile in the list of piles that are already processed. Since there are at most* $|S|^{k-1}$ *piles the binary search takes at most* $O(k \log |S|)$ *time. There are at most* $O(|S|)$ *items to sum up, therefore the whole expression can be calculated in* $O(k|S| \log |S|)$ *time.*

Remark 4 *Consider step 3. Using Remark 1 and binary search over the list of piles we can check whether* $v_{j_1,\ldots,j_l}(j)$ *is positive in at most* $O(k|S|)$ *time. Thus scheduling all the rectangles at step 3 requires at most* $O(k|S|^2)$ *time.*

Theorem 3 *Algorithm* $\mathcal{B}$ *finds an optimal solution to problem* $\mathcal{P}$ *on instance* S.

Proof. We have that, for some rectangle t and non-overlapping rectangles $j_1, \ldots, j_q$:

$$OPT(\{i : i \leq t\}|j_1, \ldots, j_q) = \sum_{i \leq t} v_{j_1,\ldots,j_q}(i) = v_{j_1,\ldots,j_q}(p) + \sum_{i<p} v_{j_1,\ldots,j_q}(i),$$

where p is the highest index of a positive item under the sum: $p = \max\{i : i \leq t \ \text{ and } \ v_{j_1,\ldots,j_q}(i) > 0\}$.

Keeping in mind that $v_{j_1,\ldots,j_q}(p) > 0$, we have from Lemma 2:

$$\begin{aligned} v_{j_1,\ldots,j_q}(p) = w(p) + OPT(i : i < p|_{p,j_1,\ldots,j_q}) - OPT(i : i < p|_{j_1,\ldots,j_q}) = \\ = w(p) + OPT(\{i : i < p\}|_{p,j_1,\ldots,j_q}) - \sum_{i<p} v_{j_1,\ldots,j_q}(i). \end{aligned}$$

Combining the last two formulas, obtain:

$$\begin{aligned} OPT(\{i : i \leq t\}|j_1, \ldots, j_q) = w(p) + OPT(\{i : i \leq p-1\}|p, j_1, \ldots, j_q), \\ p = \max\{i : i \leq t \ \text{ and } \ v_{j_1,\ldots,j_q}(i) > 0\}. \end{aligned}$$

We can use this recursively to calculate $OPT(S)$, as follows:

$$OPT(S) = w(p_1) + w(p_2) + \ldots + w(p_l),$$

where $p_1, p_2, \ldots, p_l$ can be found from the following recursive expression:

$$p_1 = \max\{i : v(i) > 0\}, \quad p_j = \max\{i : i < p_{j-1} \text{ and } v_{p_1,\ldots,p_{j-1}}(i) > 0\}.$$

It is easy to check that the last step of our algorithm correctly implements this recursive procedure. □

Theorem 4 *The running time of procedure $\mathcal{B}$ is at most $O(k|S|^k \log |S|)$ and it requires at most $O(|S|^{k-1})$ memory space.*

Proof. The most time consuming operations are apparently sorting the piles and calculating the value of $v_{\ldots}(\cdot)$ for each of them. The first can be implemented in at most $O(k|S|^{k-1} \log |S|)$ time (see Remark 2), the second takes at most $O(|S|^{k-1}) \cdot O(k|S| \log |S|) = O(k|S|^k \log |S|)$ time (see Remark 3). We take the largest. □

Theorem 5 *Algorithm $\mathcal{A}^k$ runs in at most $O(kn^k \log n)$ time and requires at most $O(n^{k-1})$ memory space for any $k \geq 2$.*

Proof. We have to solve subinstances I_j for $j = 1, \ldots, k$. For each of them we solve all the subinstances S_i, $i = 1, .., \lceil n/k \rceil$, each of which is solved using procedure $\mathcal{B}$ in at most $O(k|S_i|^k \log |S_i|)$ time. Since $O(k|S_1|^k \log |S_1|) + O(k|S_2|^k \log |S_2|) + \ldots + O(k|S_{\lceil n/k \rceil}|^k \log |S_{\lceil n/k \rceil}|) = O(kn^k \log n)$, each of I_j can be solved in $O(kn^k \log n)$ time. Since there is k of I_j-s, the whole algorithms takes at most $O(k^2 n^k \log n)$ time. □

We conclude with a small remark:

Remark 5 *Observe that in case when $k \geq n$, algorithm $\mathcal{A}$ becomes an exact algorithm with an exponential running time. Therefore, the only practical values of k are: $2 \leq k < n$.*

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Cyclic, Diagonal and Facial Colorings

Daniel Kráľ

(Joint work with Tomáš Madaras and Riste Škrekovski)

We introduce an extension of cyclic coloring of plane graphs which we call facial coloring and apply the extended notion to 1-diagonal coloring of plane quadrangulations.

A cyclic coloring of a plane graph is a coloring of its vertices such that any two vertices incident with the same face receive distinct colors. The number of used colors have clearly to be at least the size $\Delta^\star$ of the largest face (measured as the number of its vertices) of a plane graph. Hence any upper bounds have always to be restricted to plane graphs with bounded maximum face sizes; this is not the case in facial coloring.

Two vertices u and v are l-facially adjacent if there exists a facial walk of length (measured as its number of edges) at most l between them. In an l-facial coloring, we demand that any two different l-facially adjacent vertices receive distinct colors. If $\Delta^\star \leq 2l+1$, any cyclic coloring is an l-facial coloring and moreover if a plane graph is 2-connected, any l-facial coloring is a cyclic coloring (note that each plane graph can be changed to a 2-connected graph only by adding some edges without increasing its cyclic chromatic number).

Let $f_f(l)$ be the maximum number of colors needed for an l-facial coloring of a plane graph and $f_c(\Delta^\star)$ the maximum number of colors needed for a cyclic coloring of a plane graph with maximum face size at most $\Delta^\star$. It clearly holds that $f_c(2l+1) \leq f_f(l)$. Our upper bounds on $f_f(l)$, the best known upper bounds on $f_c(2l+1)$ and the best lower bounds for $1 \leq l \leq 10$ can be found in Table 1. We do not know a single example of a graph for which the concept of facial coloring would be more general (in sense it requires more colors) than the concept of cyclic coloring:

Problem 1 *Is it true that $f_f(l) = f_c(2l+1)$ for all $l \geq 1$?*

Ore and Plummer conjectured that $f_c(\Delta^\star) = \left\lfloor \frac{3}{2}\Delta^\star \right\rfloor$; the counterpart of this famous conjecture is the following problem:

Problem 2 *Does the equality $f_f(l) = 3l+1$ hold for all $l \geq 1$?*

l	1	2	3	4	5	6	7	8	9	10
$\Delta^\star = 2l+1$	3	5	7	9	11	13	15	17	19	21
Upper bound on $f_f(l)$	4	8	12	15	20	23	27	30	34	38
Upper bound on $f_c(\Delta^\star)$	4	8	12	15	19	22	25	29	32	35
Lower bound	4	7	10	13	16	19	22	25	28	31

Table 1: Summary of known results for cyclic and facial coloring.

The results on 2-facial coloring may be applied to 1-diagonal coloring of plane quadrangulations. Two vertices u and v are d-diagonally adjacent if there exists a set S of at most d edges such that the removal of the edges of S makes u and v to be incident with the same face. In the d-diagonal coloring, any two vertices which are d-diagonally adjacent have to receive distinct colors. A 0-diagonal coloring is clearly a cyclic coloring. A d-diagonal coloring of plane quadrangulations was first studied by Horňák and Jendrol'; they proved that 21 colors are sufficient to 1-diagonal color a plane quadrangulation. Sanders and Zhao improved this upper bound to 19. The upper bound on the number of colors needed for a 1-diagonal coloring of plane quadrangulations can be decreased to 16 by a certain reduction to 2-facial coloring. The best known lower bound for this type of coloring is 11.

Stable Matchings in Three-Sided Systems with Cyclic Preferences

Daniel Krasner

Abstract

We consider generalizations of the Gale-Shapley (1962) Stable Marriage Problem to three-sided families. Alkan (1988) gave an example which shows that in the case of general preferences stable matchings do not always exist. Here we suggest a more compact example. Danilov (2001) proved that stable matchings exist for some acyclic preferences and he raised the problem for the cyclic preferences. Here we show that the answer is negative. We construct a three-sided system with lexicographical cyclic preferences for which no stable matching exists.

KEYWORDS: stable matching, stable marriage, cyclic preferences.

1 Introduction

In 1962 Gale and Shapley proposed and solved what is called the Stable Marriage Problem. A significant amount of work, relating to algorithms, lattice structure, strategic behavior, etc. was induced by this paper. We are interested in whether the Stable Marriage Problem can be generalized with families consisting of more than two individuals, for example, men, women, and cats.

We consider three sets M, W, C such that $|M| = |W| = |C| = n$. We call these sets men, women, and cats, respectively. Let us call a triple $(m, w, c) \in M \times W \times C$ a *family*. A set of families $\mathcal{M} \subset M \times W \times C$ is called a *matching*, if every member in $M \cup W \cup C$ belong to exactly one of the families in $\mathcal{M}$. We give every man $m \in M$ a preference order, denoted by $\mathcal{P}_{(m)}$, over the woman-cat pairs, and we write $(w_i, c_j) \succ_m (w_{i'}, c_{j'})$ if m prefers (w_i, c_j) to $(w_{i'}, c_{j'})$. Also, we write $\mathcal{P}_{(m)}(w, c) = k$ if (w, c) is $m's$ $k'th$ choice. We do the same for the women and the cats. Suppose that $(m_i, w_{i'}, c_{i''})$, $(m_j, w_{j'}, c_{j''})$, $(m_k, w_{k'}, c_{k''}) \in \mathcal{M}$. A triple $(m_i, w_{j'}, c_{k''})$ is called *breaking* if $(m_i, w_{j'}, c_{k''}) \notin \mathcal{M}$ and $(w_{j'}, c_{k''}) \succ_{m_i} (w_{i'}, c_{i''})$, $(m_i, c_{k''}) \succ_{w_{j'}} (m_j, c_{j''})$,

$(m_i, w_{j'}) \succ_{c_{k''}} (m_k, w_{k'})$. A matching is called *stable* if no breaking triples exist. In other words, a family $F \notin \mathcal{M}$ is breaking for a matching $\mathcal{M}$, if every member of F is "better off" with F (with respect to his/her/its own preference) than with the family it belongs to in $\mathcal{M}$.

Alkan (1988) showed that the Gale-Shapley (1962) Stable Marriage Theorem does not generalize to the case of three-sided families. He constructed a three-sided system with $n = 3$ individuals in each class and preferences for each individual over all pairs of possible partners such that no stable matching exists. Here we construct an analogous example with only $n = 2$ individuals in each class. The natural next step was to restrict the preference orders. Danilov (2001) proved that the Gale-Shapley Theorem generalizes the case of three-sided systems with acyclic preferences: the man first cares about the woman, and a woman first cares about the man (i.e. if $w_1 \succ_m w_2$, then $(w_1, c_i) \succ_m (w_2, c_j)$ for every i, j), and each cat has a general preference order over all (m, w) pairs. Using an extension of the Gale-Shapley algorithm Danilov showed that, in such a case, a stable matching exists. Danilov (2001) also introduced lexicographical cyclic preferences: the men first care about the women and then (if the woman is the same) about the cats, the women first care about the cats and then about the men, the cats first care about the men and then about the women. For this type of preferences he left the question open. Here we prove that the answer is negative, that is stable matchings may fail to exist. We, also, show that this implies that in the case of cyclic preference orders (i.e. when the men only care about the women, the women only care about the cats, and the cats only care about the men) there do not always exist stable matchings.

2 General Preferences

We show that for $n = 2$ there exists a set of preferences that does not admit a stable matching.

Preferences:
$m_1 : \{(w_2, c_1) > (w_1, c_2) > (w_2, c_2) > (w_1, c_1)\}$
$m_2 : \{(w_2, c_1) > (w_1, c_1) > (w_2, c_2) > (w_1, c_2)\}$
$w_1 : \{(m_2, c_2) > (m_2, c_1) > (m_1, c_1) > (m_1, c_2)\}$
$w_2 : \{(m_2, c_2) > (m_1, c_1) > (m_2, c_1) > (m_1, c_2)\}$
$c_1 : \{(m_1, w_2) > (m_2, w_1) > (m_1, w_1) > (m_2, w_2)\}$
$c_2 : \{(m_1, w_1) > (m_1, w_2) > (m_2, w_2) > (m_2, w_1)\}$

Possible Matchings:
$\mathcal{M}_1 = [(m_1, w_1, c_1), (m_2, w_2, c_2)]$: breaking triple is (m_2, w_1, c_1)
$\mathcal{M}_2 = [(m_1, w_2, c_1), (m_2, w_1, c_2)]$: breaking triple is (m_2, w_2, c_2)
$\mathcal{M}_3 = [(m_1, w_1, c_2), (m_2, w_2, c_1)]$: breaking triple is (m_1, w_2, c_1)
$\mathcal{M}_4 = [(m_1, w_2, c_2), (m_2, w_1, c_1)]$: breaking triple is (m_1, w_2, c_1)

3 Lexicographical Cyclic Preferences

A preference is called *lexicographical cyclic*, or lex. cyclic for short, if $(w_i, c_{i'}) \succ_m (w_j, c_{j'})$ and $i \neq j \Rightarrow (w_i, c_{i'}) \succ_m (w_j, c_{j'})\ \forall i', j' \in [n]$, and similar for the women and cats. In other words the men first care about the women and then the cats, the women first care about the cats and then the men, and the cats first care about the men and then the women. We show that for $n = 1, 2$ there always exist a stable matching and $\forall n \geq 3$ there exists a lex. cyclic preference order that does not admit a stable matching.

The fact that for $n = 1, 2$ there is always a stable matching comes from Lemma 2, which shows that for the cyclic preference case there is always a stable matching for $n = 1, 2$, and from Lemma 3, which shows that if for some n there exists a lex. cyclic preference that does not admit a stable matching then for the same n there exists a cyclic preference that does not admit a stable matching. Consider the following example with three men, women, and cats.

Theorem 1 *No stable matching exists in the three-sided system* $\{m_1, m_2, m_3; w_1, w_2, w_3; c_1, c_2, c_3\}$ *with the following lexicographical cyclic preference order:*

men m_1 and m_3 prefer women in order $w_1 > w_2 > w_3$ and m_2 prefers in order $w_2 > w_3 > w_1$; in case of the same woman all men prefers cats in order $c_2 > c_1 > c_3$;

women w_1 and w_3 prefer cats in order $c_1 > c_2 > c_3$ and w_2 prefers in order $c_2 > c_3 > c_1$; in case of the same cat all women prefers men in order $m_2 > m_1 > m_3$;

cats c_2 and c_3 prefer men in order $m_1 > m_2 > m_3$ and c_1 prefers in order $m_2 > m_3 > m_1$; in case of the same man all cats prefers women in order $w_2 > w_1 > w_3$.

Proof. There are $(3!)^2 = 36$ possible matchings. We show that for each one a breaking triple exists. For 9 of these 36 matchings there exists a *unique* breaking triple (BT). We list these first:

$\mathcal{M}_1 = [(m_1, w_1, c_1), (m_2, w_2, c_2), (m_3, w_3, c_3)]$: BT is (m_3, w_3, c_1)
$\mathcal{M}_2 = [(m_1, w_1, c_2), (m_2, w_2, c_1), (m_3, w_3, c_3)]$: BT is (m_3, w_2, c_3)
$\mathcal{M}_3 = [(m_1, w_1, c_2), (m_2, w_2, c_3), (m_3, w_3, c_1)]$: BT is (m_3, w_1, c_1)
$\mathcal{M}_4 = [(m_1, w_1, c_2), (m_2, w_3, c_1), (m_3, w_2, c_3)]$: BT is (m_2, w_2, c_3)
$\mathcal{M}_5 = [(m_1, w_2, c_2), (m_2, w_1, c_1), (m_3, w_3, c_3)]$: BT is (m_2, w_3, c_3)
$\mathcal{M}_6 = [(m_1, w_2, c_2), (m_2, w_3, c_3), (m_3, w_1, c_1)]$: BT is (m_2, w_3, c_1)
$\mathcal{M}_7 = [(m_1, w_2, c_2), (m_2, w_3, c_1), (m_3, w_1, c_3)]$: BT is (m_1, w_1, c_3)
$\mathcal{M}_8 = [(m_1, w_3, c_3), (m_2, w_2, c_2), (m_3, w_1, c_1)]$: BT is (m_1, w_3, c_2)
$\mathcal{M}_9 = [(m_1, w_3, c_2), (m_2, w_2, c_3), (m_3, w_1, c_1)]$: BT is (m_1, w_2, c_2)

Now let $\mathcal{M}$ be a matching. Suppose $(m_2, w_2, c_2) \notin \mathcal{M}$ and $(m_1, w_i, c_2) \notin \mathcal{M}$, then it is clear that (m_2, w_2, c_2) will be breaking triple. This breaks twenty other matchings, leaving us with only 7 to break. We list these now:

$\mathcal{M}_{10} = [(m_1, w_1, c_3), (m_2, w_2, c_2), (m_3, w_3, c_1)]$: BT is (m_3, w_1, c_1)
$\mathcal{M}_{11} = [(m_1, w_1, c_2), (m_2, w_3, c_3), (m_3, w_2, c_1)]$: BT is (m_2, w_3, c_1)
$\mathcal{M}_{12} = [(m_1, w_2, c_2), (m_2, w_1, c_3), (m_3, w_3, c_1)]$: BT is (m_2, w_3, c_1)
$\mathcal{M}_{13} = [(m_1, w_3, c_2), (m_2, w_1, c_1), (m_3, w_2, c_3)]$: BT is (m_1, w_2, c_2)
$\mathcal{M}_{14} = [(m_1, w_3, c_2), (m_2, w_1, c_3), (m_3, w_2, c_1)]$: BT is (m_1, w_2, c_2)
$\mathcal{M}_{15} = [(m_1, w_3, c_2), (m_2, w_2, c_1), (m_3, w_1, c_3)]$: BT is (m_1, w_2, c_2)
$\mathcal{M}_{16} = [(m_1, w_3, c_1), (m_2, w_2, c_2), (m_3, w_1, c_3)]$: BT is (m_1, w_1, c_3) $\square$

Lemma 1 *If for some n there exists a lexicographical-cyclic preference list that does not admit a stable matching, then there exists a lexicographical-cyclic preference list for n+1 that does not admit a stable matching.*

Proof. Suppose $\mathcal{P}$ is a lexicographical-cyclic preference list that does not admit a stable matching for n men, n women, and n cats. First, we introduce m_{n+1}, w_{n+1}, c_{n+1} into the original n men's, n women's, and n cats' preference orderings such that $\mathcal{P}_{(m_i)}(w_{n+1}, c_{n+1}) = \mathcal{P}_{(w_i)}(m_{n+1}, c_{n+1}) = \mathcal{P}_{(c_i)}(m_{n+1}, w_{n+1}) = n + 1 \ \forall i \leq n$. Second, we let $\mathcal{P}_{(m_{n+1})}(w_{n+1}, c_{n+1}) = \mathcal{P}_{(w_{n+1})}(m_{n+1}, c_{n+1}) = \mathcal{P}_{(c_{n+1})}(m_{n+1}, w_{n+1}) = 1$. Clearly we can do this and ensure that our new preference order is still lexicographically cyclic. Suppose $\mathcal{M}$ is a matching of these $n + 1$ men, women, and cats, and that $(m_{n+1}, w_{n+1}, c_{n+1}) \in \mathcal{M}$. Clearly we can remove all pairs involving m_{n+1}, w_{n+1}, c_{n+1} from the new preference lists and reduce this to the case of n men, n women, and n cats. Therefore, $\mathcal{M}$ is unstable by our inductive hypothesis. Suppose $(m_{n+1}, w_{n+1}, c_{n+1}) \notin \mathcal{M}$. Then, $(m_{n+1}, w_{n+1}, c_{n+1})$ is a breaking triple, and thus $\mathcal{M}$ is unstable. Therefore, the new preference order for $n + 1$ men, $n + 1$ women, and $n + 1$ cats is unstable. $\square$

Hence, for every $n \geq 3$ there exists a lexicographical cyclic preference that does not admit a stable matching.

4 Cyclic Preferences

A preference is called *cyclic* if every man is given a preference order over the women, every woman is given a preference order over the cats, and every cat is given a preference order over the men. Given a matching $\mathcal{M}$, suppose that (m_i, w_i, c_i), (m_j, w_j, c_j), $(m_k, w_k, c_k) \in \mathcal{M}$. A triple (m_i, w_j, c_k) is called *breaking* if $(m_i, w_j, c_k) \notin \mathcal{M}$ and the following conditions hold:

(i) If $i = j$ then $(c_k) \succ_{w_i} (c_i,)$, and $(m_i) \succ_{c_k} (m_k)$.
(ii) If $j = k$ then $(w_j) \succ_{m_i} (w_i,)$, and $(m_i) \succ_{c_j} (m_j)$.
(iii) If $i = k$ then $(w_j) \succ_{m_i} (w_i,)$, and $(c_i) \succ_{w_j} (c_k)$.
(iv) If $i \neq j \neq k$ then $(w_j) \succ_{m_i} (w_i,)$, $(c_k) \succ_{w_j} (c_j)$, and $(m_i) \succ_{c_k} (m_k)$.

In other words, a family $F \notin \mathcal{M}$ is breaking for a matching $\mathcal{M}$, if at least two of the members of F are "better off" with F (with respect to their own preferences) than with the families they belong to in $\mathcal{M}$. A matching is called *stable* if no breaking triples exist. We show that for $n = 1, 2$ there is always a stable matching and that for $n \geq 3$ there exists a cyclic preference that does not admit a stable matching.

Lemma 2 *For $n = 1, 2$ every cyclic preference has a stable matching.*

Proof. For $n = 1$ there is nothing to prove. Suppose $n = 2$. Either there exists a triple, say, (m_1, w_1, c_1), such that m_1 has w_1 as his first choice, w_1 has c_1 as her first choice and c_1 has m_1 as its first choice. Then $\mathcal{M} = \{(m_1, w_1, c_1), (m_2, w_2, c_2)\}$ is stable matching, since it is clear that neither m_1,w_1, or c_1 can be in a breaking triple. Otherwise we can give all men their first choices and all women their first choices, resulting in a stable matching. $\square$

Lemma 3 *If there exists a lexicographical-cyclic preference list that does not admit a stable matching, then there exists a cyclic preference list that does not admit a stable matching.*

Proof. Suppose that $\mathcal{P}$ is a lex. cyclic preference list that does not admit a stable matching. It is clear that if we simply ignore the second member of each pair $\mathcal{P}$ can be considered as a cyclic preference list. Suppose we have a matching $\mathcal{M}$ and a breaking triple (m_i, w_j, c_k) for the lex. cyclic case, $(m_i, w_j, c_k) \notin \mathcal{M}$. Now suppose that (m_i, w_j, c_k) is not a breaking triple for the cyclic case, and w.l.o.g. assume that m_i prefers w_i to w_j. Then clearly (m_i, w_j, c_k) can not be a breaking triple in the lex. cyclic case, a contradiction. Hence, since every matching has such a breaking triple for the lexicographical-cyclic case, we are done. $\square$

Hence, since we know that for $n \geq 3$ there exist a lex. cyclic preference that does not admit a stable matching, the same holds for cyclic preferences.

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How many colors may we require?

Patrice Ossona de Mendez

(Joint work with Jaroslav Nešetřil)

1 Introduction

All graphs considered in this paper are finite and simple.

A *coloring* of a graph G is an assignment of colors to the vertices of the graph, one color to each vertex. A coloring is *proper* if each the vertices of any K_2 in G gets at least 2 colors. The minimum number of colors required for a proper coloring of G is the *chromatic number* of G, denoted $\chi(G)$.

One way of generalizing proper colorings is to consider the minimum number of colors required, so that any subgraph H of G gets at least $f(H)$ colors. For instance, star coloring corresponds to the request of any P_4 (that is: any path on 4 vertices) to get at least 3 colors. As long as $f(H)$ is less or equal to $|V(H)|$, a coloration will exist (the coloration of the vertices in $|V(G)|$ colors will do). However, usual families of graph won't get a bound on the number of colors required to color any of its members.

Important families of graphs are proper minor closed classes of graphs. These families have bounded density, bounded chromatic number, bounded star coloring number, aso. If we require this bounding property to be preserved, we may usually not ask for any subgraph H to get $|V(H)|$ colors. Therefore, we introduce the following definition:

Definition 1 *The upper chromatic number of a graph H is the greatest integer $\overline{\chi}(H)$, such that, for any proper minor closed class of graph $\mathcal{K}$, there exists a constant $k(\mathcal{K}, H)$, such that any graph $G \in \mathcal{K}$ has a coloring using at most $k(\mathcal{K}, H)$ colors so that any subgraph of G isomorphic to H gets at least $\overline{\chi}(H)$ colors.*

The following bounds are thus known:

$$\chi(G) \leq \overline{\chi}(G) \leq |V(G)|,$$
$$\overline{\chi}(P_4) \geq 3 \quad \text{(bounding of the star coloring number)}$$

Moreover, considering the class of series-parallel graphs, and particularly the graphs $K_{2,n}$, a simple argument shows that, whatever bounded number of colors we use, we fill find some n sufficiently large, such that some P_4 of $K_{2,n}$ will get at most 3 colors. Thus, $\overline{\chi}(P_4) = 3$.

Similar examples may be used to get an upper bound on $\overline{\chi}(P_{2^l})$, leading to the following question:

Problem 1 *Is it true that* $\overline{\chi}(P_{2^l}) = l + 1$*?*

Obviously, for any finite family $\mathcal{F}$ of graphs, and any proper closed class of graph $\mathcal{K}$, there will exist some bound $k(\mathcal{K}, \mathcal{F})$, such that any graph $G \in \mathcal{K}$ has a coloring using at most $k(\mathcal{K}, \mathcal{F})$ colors so that any subgraph of G isomorphic to a member H of $\mathcal{F}$ gets at least $\overline{\chi}(H)$ colors. It is also obvious that the same statement would probably fail for the family $\mathcal{F}$ of all graphs, as $\overline{\chi}(G)$ is very probably not bounded on all proper minor closed classes of graphs. However, if we bound the maximum number of colors we may require on the subgraph, the statement might be true:

Problem 2 *For any positive integer N and any proper minor closed class of graph $\mathcal{K}$, does there exists a constant $k(\mathcal{K}, N)$, such that any graph $G \in \mathcal{K}$ has a coloring using at most $k(\mathcal{K}, N)$ colors so that any subgraph H of G gets at least* $\min(N, \overline{\chi}(H))$ *colors?*

In this paper, we add some little stone to the edifice, by proving that one may require $\chi(H) + 1$ colors whenever H does not include a clique of size $\chi(H)$:

Theorem 1 *Let t be a positive integer. For any proper minor closed class $\mathcal{K}$ there exists $k = k(\mathcal{K}, t)$ such that any graph $G \in \mathcal{K}$ has a proper k-coloring with the property that any subgraph with chromatic number t either contains a K_t in it, or gets at least $t + 1$ colors.*

Corollary 2 *For any graph G, if $\chi(G) > \omega(G)$, then $\overline{\chi}(G) > \chi(G)$.*

In order to prove this theorem, we will introduce a special kind of homomorphisms of directed graphs, the foldings.

2 Foldings

Let $\vec{G}, \vec{H}$ be directed graphs , let $x \in V(\vec{G})$ be a vertex and let $d > 0$ be a positive integer.

- A *homomorphism* $f : \vec{G} \to \vec{H}$ is a mapping $f : V(\vec{G}) \to V(\vec{H})$ satisfying $(f(x), f(y)) \in E(\vec{H})$ whenever $(x, y) \in E(\vec{G})$.

- The *d-down set* $D^d_{\vec{G}}(x)$ of x in $\vec{G}$ is the set of all those vertices y of $\vec{G}$ for which there exists a directed path of length $\leq d$ from y to x. Thus $D^1_{\vec{G}}(x)$ is just closed in-neighborhood of x in $\vec{G}$.

- A *d-folding* from $\vec{G}$ to $\vec{H}$ is a homomorphism from $\vec{G}$ to $\vec{H}$, which restriction of f to any d-down set in G is injective. If $\vec{K}_k$ is an orientation of K_k, a d-folding $\vec{G} \to \vec{K}_k$ is then a d-folding which uses k colors.

It appears that, for any proper minor closed class of graph, the number of colors required for the existence of a d-folding may be bounded:

Theorem 3 ([1]) *Let $\mathcal{K}$ be a proper minor closed class of graphs, d a positive integer. Then there exists a positive integer $k = k(d, \mathcal{K})$ and an orientation $\vec{K}_k$ of K_k, such that for any graph $G \in \mathcal{K}$ there exists an acyclic orientation $\vec{G}$ and a d-folding $f : \vec{G} \to \vec{K}_k$ (i.e. a d-folding using at most k colors).*

3 Proof of theorem 1

Proof. Let $k(\mathcal{K}, t)$ be the bound for the existence of a 3-folding. Let G be any graph in $\mathcal{K}$ colored using a 3-folding f to some K_N.

The requested property may be restated as follows: "for any subgraph H in G which is colored by p colors, either $\chi(H) < p$ or H includes a clique of size p."

We proceed by induction on p. Assume the statement is proved for $p-1$. Let $\vec{G}$ be a given graph with stated 3-folding into $\vec{K}_N$. Consider the homomorphic image of $\vec{G}$ as a subgraph $\vec{K}'$ of $\vec{K}_N$ and consider the subgraph $\vec{K}''$ of $\vec{K}'$ induced by any p colors. Then, we can assume without loss of generality that $\vec{K}''$ contains a monotone path with p vertices (for otherwise $\chi(K'') \leq p-1$ and we are done). We number the colors according to this monotone path from 1 to p for convenience. Consider now any subgraph $\vec{H}$ of $\vec{G}$ with all its vertices colored by $1, \ldots, p$. If the subgraph of $\vec{H}$ induced by vertices with colors $1, \ldots, p-1$ has no clique of size $p-1$, then it has chromatic number at most $p-2$. Hence, in this case we have $\chi(\vec{H}) \leq p-1$. Otherwise, we may assume $\vec{H}$ is p-color critical. Let $\vec{K}$ be a clique of size

$p-1$ of this subgraph. Then, as $\vec{G}$ is acyclic oriented, so is $\vec{K}$. According to the existence of a monotone path $1, 2, \ldots, p-1$ the orientation of $\vec{K}$ follows. Let u be the vertex with color $p-1$. Thus u has an incoming edge from the vertices of $\vec{K}$ colored $1, \ldots, p-2$. As the degree of u in $\vec{H}$ is at least $p-1$ (because $\vec{H}$ is p-color-critical), and as u may not have other incoming edges from vertices colored $1, \ldots, p-2$ other than those in K (here we use 3-folding) and as the edge of $\vec{K}'$ linking $p-1$ to p is oriented from $p-1$ to p (this is the last edge of the monotone path), $\vec{H}$ contains an edge (u, v) with the vertex v is colored by color p. The vertex v may not have an outgoing edge (for otherwise we would get an oriented cycle of form $i, p-1, p, i$) and the vertex v may not have incoming edges from vertices outside of $\vec{K}$. Thus, as the degree of v is at least $p-1$ (as $\vec{H}$ is p-color-critical), the vertex v is adjacent to all the vertices of the clique $\vec{K}$ and thus $\vec{K}$ together with the vertex u form a clique of size p. $\square$

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Anti-Ramsey type problems

Juan José Montellano-Ballesteros

Given an edge colouring of a graph G, a subgraph M of G will be called *totally multicoloured* (in literature also known as *heterochromatic*, *rainbow* or *polychromatic*) if no two edges of M receive the same colour. Given a graph G and a family of subgraphs $\mathcal{F}$ of G, determining the minimum integer $h(G, \mathcal{F})$ such that for every edge colouring of G using $h(G, \mathcal{F})$ colours there is at least one totally multicoloured element of $\mathcal{F}$, constitutes what is generally described as an anti-Ramsey problem. Such problems were introduced by Erdös, Simonovits and Sós [1].

There are some interesting open questions about this sort of problems; for instance:

- Given n and p integers, determine the minimum integer m such that for every edge colouring of the complete graph of order n using m colours there is at least one totally multicoloured p-cube (the hypercube of dimension p). In this case G is the complete graph of order n and $\mathcal{F}$ is the set of all the subgraphs isomorphic to the p-cube in the complete graph of order n.
- Given n and p integers, determine the minimum integer m such that for every edge colouring of the n-cube using m colours there is at least one totally multicoloured p-cube. In this case G is the n-cube and $\mathcal{F}$ is the set of all the subgraphs isomorphic to the p-cube in the n-cube.

Another type of anti-Ramsey problems, which seems to have been introduced by Galvin in [2], can be stated as follows:

An edge colouring $\Gamma : E(G) \to \{c_i : 1 \leq i < \omega\}$ of a graph G will be called *k-bounded* if for every i, $|\Gamma^{-1}(c_i)| \leq k$; in other words, if no colour appears more that k times in G. Then, given a graph H and an integer k, the problem is to determine the minimum integer $sr(H, k) = m$ (which has been called *sub-Ramsey number*) such that for every k-bounded edge-colouring of K_m there is a totally multicoloured copy of H.

An interesting example of this type of problems is when H is the complete graph of order n; in other words:

Given $n \geq 3$ and $k \geq 2$, determine the minimum integer $sr(K_n, k) = m$ such that for every k-bounded edge colouring of K_m there is a totally multicoloured copy of K_n.

It is already known [3] that the function

$$sr(K_n,k) \text{ is } \begin{cases} \mathcal{O}(kn^2) \text{ and } \Omega(n^{3/2}) & \text{if } k \geq 15, \\ \mathcal{O}(kn^2) \text{ and } \Omega(n^{4/3}) & \text{if } 15 > k \geq 3, \\ \mathcal{O}(kn^2) \text{ and } \Omega(n) & \text{if } k = 2. \end{cases}$$

Regarding the above, an interesting open problem consists in closing the gap between the upper and lower bounds, specially for the case $k = 2$.

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Canonical coloring of forest

Diana Piguetová

Let $\mathbb{N}$ denotes the set of natural numbers and $\mathcal{P}(\mathbb{N})$ the set of all finite subsets of $\mathbb{N}$. Then an *ω-forest of height n* is a pairwise disjoint union of ω many rooted trees, where every maximal chain has $n+1$ vertices and every vertex that is not a leaf has degree ω. An *arithmetical copy of an ω-forest* $\mathcal{F}$ is a subset $\mathcal{S} \subseteq \mathcal{P}(\mathbb{N})$ such that $\exists a, d \in \mathbb{N}$ and a bijection $\varphi : V(\mathcal{F}) \to \mathcal{S}$ such that $\forall x, y \in \mathcal{F}$:

1. $x \leq y \Leftrightarrow \varphi(x) \subseteq \varphi(y)$
2. $\exists x \wedge y \Rightarrow \varphi(x \wedge y) = \varphi(x) \cap \varphi(y)$
3. x and y are from different trees $\Rightarrow \varphi(x) \cap \varphi(y) = \emptyset$
4. $x < y$ and $\nexists z \; x < z < y \Rightarrow |\varphi(y) \backslash \varphi(x)| = d$
5. if x is a root then $|\varphi(x)| = a$

Brown proved that if we finitely colored $\mathcal{P}(\mathbb{N})$ we find a monochromatic arithmetical copy in $\mathcal{P}(\mathbb{N})$ of an ω-forest of any finite height. In this talk we showed a canonical version of this theorem, it is : Let χ be a coloring of $\mathcal{P}(\mathbb{N})$ with arbitrarily many colors. Then we find a canonicaly colored arithmetical copy $\mathcal{S}$ of an ω-forest of any finite height. The 'pattern' of the canonical coloring is the following:

1. $\chi \restriction \mathcal{S}$ is injective
2. $\chi \restriction \mathcal{S}$ is constant
3. $\forall A, B \in \mathcal{S} \quad \chi(A) = \chi(B) \Leftrightarrow |A| = |B|$
4. $\forall A, B \in \mathcal{S} \quad \chi(A) = \chi(B) \Leftrightarrow A \cap B = \emptyset$
5. $\forall A, B \in \mathcal{S} \quad \chi(A) = \chi(B) \Leftrightarrow (|A| = |B| \;\&\; A \cap B = \emptyset)$

Then we presented some surprising particularity of the proof and finished by exposing an open problem: a polynomial version of above theorem or the Brown's theorem.

Generalization of the polygon-crossing problem

Pavel Podbrdský

(Joint work with Jakub Černý, Zdeněk Dvořák and Vít Jelínek)

Let $k, l \geq 3$ be given integer numbers. We are interested in the problem of determining the maximum possible number $f(k,l)$ of intersections of a simple k-gon and a simple l-gon. This problem was studied in [2] — the cases when k or l is even are solved there, but an unrecoverable error appears in the case of k and l being both odd. For k and l both even is $f(k,l) = kl$ and for k even and l odd is $f(k,l) = kl - k$. For k and l both odd easy bounds $kl - k - l + 3 \leq f(k,l) \leq kl - k$ are proved in [2]. The same problem was studied in [1] for odd k and l. The problem was solved for $l = 5$ when $f(k,5) = 4k - 2$ and for $k, l \geq 7$ the upper bound from [2] was improved to $f(k,l) \leq kl - k - \lceil l/6 \rceil$. The conjecture that the lower bound is tight was formulated in [1].

We extend the definition of polygon to a larger class of objects and generalize the polygon-crossing problem. This defines the function $g(k,l) \geq f(k,l)$. The more general definition of the problem makes it easier to handle by induction. As a main result we prove that

$$g(k,l) \leq kl - k - (l-3)/2.$$

for $k \geq 1$ and $l \geq 3$ both odd. As an easy consequence we get the same upper bound for the function $f(k,l)$, $k, l \geq 3$ both odd.

Note that the obtained upper bound is exactly in the middle between the easy upper and lower bounds proved in [2].

This result improves the upper bound proved in [1] and together with the fact that $f(k,l)$ is even gives also an alternative proof of the equality $f(k,5) = 4k - 2$ also proved in [1].

The general problem of determining the exact value of $f(k,l)$ (and $g(k,l)$) for $k, l \geq 7$ odd remains open. We also do not know if $f(k,l) = g(k,l)$ in general and if the function $g(k,l)$ is symmetric.

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Variations on Hamiltonian Themes

Moshe Rosenfeld

(Joint work with Tomáš Kaiser, Daniel Král',

Zdeněk Ryjáček and Heinz-Jürgen Voss)

1 Introduction

A good workshop brings together researchers from different institutions, forms groups to pursue a well-defined project. Hopefully this will be such a project. Following the first presentation in the workshop by M. Rosenfeld, and accelerated by the floods in Dresden (where Moshe was visiting H.-J. Voss) the project described below is surging ahead as fast as the Vltava.

Finding hamiltonian paths or cycles in graphs is a "proven" very difficult problem. There are probably hundreds of papers devoted to results of the form "If G satisfies property P, then G is hamiltonian". There are also many surveys, book chapters, web sites and many other sources devoted to the search of hamiltonian cycles and paths in graphs. Recent trends suggest that since it is difficult to find hamiltonian cycles (paths) in a graph, one could modify the Hamilton cycle (path) and search for related structures. Though these structures might still be difficult to find, they are more ubiquitous. This is motivated by looking at the cycle as a walk or the path as a tree, both with bounded degree.

One approach to modify the hamiltonian cycles or paths is highlighted in Mark Ellingham's [4] extensive survey on the topic. By viewing the hamiltonian cycle as a closed spanning 1-walk, or the hamiltonian path as a spanning 2-tree, one is naturally led to the following definitions:

Define a *k-walk* in a graph to be a spanning closed walk in which every vertex is visited at most k times and an *m-tree* to be a spanning tree T in which $\Delta(T) \leq m$.

G is *hamiltonian* if it has a hamiltonian cycle and *traceable* if it has a hamiltonian path. Clearly, if G is hamiltonian it admits a 1-walk and if it is traceable it admits a 2-tree. Thus, we can search for 2-walks or 3-trees, 3-walks, 4-trees... It is very easy to see that the class of graphs having a

2-walk properly contains all hamiltonian graphs and that deciding whether a graph admits a 2-walk is in NPC. The same holds for deciding whether G has a spanning 3-tree. It is also easy to see that if G has a 2-walk then it has a spanning 3-tree but the opposite is not necessarily true. The existence of a 2-walk in a graph G has a very simple intepretation: replace every edge in G by a pair of parallel edges. In this multi-graph, search for a spanning Eulerian subgraph E with $\Delta(E) \leq 4$. As pointed out in [4], graphs with k-walks or $(k+1)$-trees produce a hierarchical chain.

An alternative approach to modifying (mutilating) the hamiltonian cycle is to modify the underlying graph. We adopt this approach in the present note. We study the hamiltonicity of the prism over G.

The prism over a graph G, denoted by $G \square K_2$, is the Cartesian product of G and K_2. It consists of two disjoint copies of G and a perfect matching connecting a vertex in one copy of G to its clone in the other copy.

Prisms are "natural" extensions of the underlying graph. First, they contain the graph as an induced subgraph and second, they are "natural" geometric objects. If G is d-polyhedral then the prism over G is $d+1$-polyhedral.

Studying the hamiltonicity of prisms was initially motivated by an attempt to tackle Barnette's conjecture that all simple 4-polytopes are hamiltonian (this conjecture is still open). For more details see the introduction in [3] which contains a brief historical description of this problem and related results. We first observe that if $G \square K_2$ is hamiltonian then G admits a 2-walk. On the other hand, Fig. 1 shows a line graph (thus a claw-free graph) which has a 2-walk but does not have a hamiltonian prism. This fact was also noted by G. Pruesse and F. Ruskey [9]. Highly connected examples are also easy to come by. The graphs consisting of 3 disjoint copies of $K_{n,2n-1}$ and the 3 mutually connected $\left[\frac{n}{2}\right]$-sets (see Fig. 2) admit a 2-walk but the prisms over them are not hamiltonian.

Thus the question whether $G \square K_2$ is hamiltonian is "*sandwiched*" between hamiltonicity and having a 2-walk. This leads us naturally to investigate the interaction between these notions. In this note we discuss problems related to hamiltonicity, hamiltonian cycles in $G \square K_2$ and 2-walks.

We approach these problems from different directions. First, we try to tackle conjectures that certain graphs are hamiltonian by trying the weaker conclusion that the underlying graphs have a hamiltonian prism. Second we look at theorems that imply hamiltonicity and try to see whether a reduction of the assumptions can imply hamiltonian prisms, and third we try to resuscitate false conjectures about hamiltonian cycles by trying to

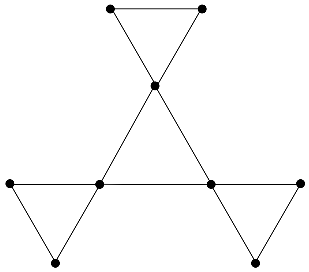

Figure 1: A (line) graph which admits a 2-walk but has a non-hamiltonian prism.

prove that the assumptions imply that the graphs have a hamiltonian prism. In the next section we give a sample of 3 examples, one from each approach.

2 Hamiltonian cycles in $G \square K_2$

Hamiltonian cycles in prisms were sporadically investigated. The following is a list (very partial) of notable results on the hamiltonicity of $G \square K_2$. Barnette and Rosenfeld [8] observed that the prisms over 2-connected, cubic planar graphs are hamiltonian if the four color conjecture was true. Later this assertion was proved by Fleischner [5] independent of the four color theorem. Paulraja proved that $G \square K_2$ is hamiltonian if G is a 3-connected cubic graph. An alternative proof is given in [3]. In this paper a much stronger result was proved (for a smaller class of graphs): the prisms over cubic, 3-connected, bipartite planar graphs admit a hamiltonian decomposition. G. Pruesse and F. Ruskey [9] proved that the prism of the acyclic orientation graph is hamiltonian.

Planar Graphs

One of the most remarkable results in hamiltonicity is Tutte's thorem: 4-connected planar graphs are hamiltonian [10]. Reducing 4-connectivity to 3-connectivity yields of course non hamiltonian graphs. D. Barnette [1] proved that all planar 3-connected graphs have a 3-tree (actually the proof holds for a bigger family of graphs, circuit graphs). B. Richter and Z. Gao [7] strengthened Barnette's result by proving that all planar 3-connected graphs contain a 2-walk. This leads to the natural question: can this be further strengthened?

Do all 3-connected planar graphs have hamiltonian prisms?

We proved that all Kleetopes have a hamiltonian prism and so do Halin graphs. (Actually even generalized Halin graphs, that is graphs obtained by taking a tree with n leaves and adding an n-cycle through the leaves, have a hamiltonian prism.) Even general plane triangulations are still waiting for a proof that they admit a hamiltonian prism.

4-connected, 4-regular graphs

Nash-Williams conjectured that all 4-connected, 4-regular graphs are hamiltonian. Meredith produced the first counterexample to this conjecture [2]. It is not difficult to check that Meredith's graph, as well as many other non-hamiltonian 4-connected, 4-regular graphs built similarly, admit a hamiltonian prism. This led us to conjecture:

If G is a 4-connected, 4-regular graph then it has a hamiltonian prism.

Another interesting related question is the complexity of finding a hamiltonian cycle in $G \square K_2$. Given G, 4-connected and 4-regular, the question "does G has a hamiltonian cycle?" is NP-complete. On the other hand, "does G have a 2-walk?" is polynomial. Indeed an Eulerian cycle in G is a 2-walk. Where does the "sandwiched" decision problem "does G have a hamiltonian cycle?" lie?

Line graphs, claw-free graphs

C. Thomassen conjectured that all 4-connected line graphs are hamiltonian. The classical examples of 3-connected non-hamiltonian line graphs include the line graph of a graph obtained from the Petersen graph by adding 10 vertices of degree one connected by a pendant edge to the vertices of the Petersen graph, or the line graph of the Petersen graph modified by subdividing five edges that form a 1-factor. It is not difficult to check that both graphs have hamiltonian prisms. So we propose:

All 3-connected claw-free graphs have a hamiltonian prism.

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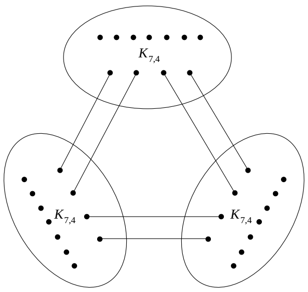

Figure 2: Highly connected graphs with 2-walks having non-hamiltonian prisms ($n = 4$).

Representation of separoids and a Tverberg-type problem

Ricardo Strausz

Abstract

A *separoid* is a symmetric relation $\dagger \subset \binom{2^S}{2}$ defined on pairs of disjoint subsets which is closed as a filter in the natural partial order (i.e., $A \dagger B \preceq C \dagger D \iff A \subseteq C$ and $B \subseteq D$). We discuss the Geometric Representation Theorem for separoids: *every separoid $(S, \dagger)$ can be represented by a family of (convex) polytopes, and their Radon partitions, in the Euclidean space of dimension $|S| - 1$.* Furthermore, we introduce a new kind of separoids' morphisms — called *chromomorphisms*— which allow us to study Tverberg's generalisation (1966) of Radon's theorem (1921) in the context of convex sets. In particular the following Tverberg-type theorem is proved: *Let S be a separoid of order $|S| = (k-1)(\mathrm{d}(S)+1)+1$, where $\mathrm{d}(S)$ denotes the (combinatorial) dimension of S. If there exists a monomorphism $S \to P$ into a separoid of points in general position in $\mathbb{E}^d$, then there exists a chromomorphism $S \longrightarrow K_k$ onto the complete separoid of order k.* This theorem is, in a sense, dual to the Hadwiger-type theorem proved by Arocha, Bracho, Montejano, Oliveros & Strausz [Disc. & Comp. Geom., **27**, 2002, 377–385].

KEYWORDS: separoids; abstract convexity; Tverberg's theorem; Hadwiger's theorem; chromomorphisms.

1 Introduction

Combinatorial convexity has its origins in the theorems of Helly (1923) [10] and Radon (1921) [14]; because of their simplicity, they have been applied and generalised in several directions (see [5]). The present exposition centres around one generalisation of each: Hadwiger's Transversal Theorem (1957) [9] and Tverberg's Theorem (1966) [19], which generalise Helly's and Radon's theorems, respectively (see [6, 8]).

In the one hand, Hadwiger generalises Helly's theorem in $\mathbb{E}^2$ changing the notion of "non-empty intersection" by the notion of "0-transversal"; then, he finds sufficient conditions, in the spirit of Helly's theorem, for a

"1-transversal" to exist. However, he had to add additional conditions; in particular, he uses the notion of a "linear order" to which every "partial" transversal has to be consistent. This was further generalised by Katchalski (1980) [11], Goodman & Pollack (1988) [7] and Arocha, Bracho, Montejano, Oliveros & Strausz (2002) [1].

On the other hand, Tverberg generalises Radon's theorem giving the sufficient condition

$$|\mathcal{P}| \geq (k-1)(d+1)+1,$$

on a family of points $\mathcal{P} \subset \mathbb{E}^d$, for a k-partition to exist in such a way that the convex hull of the parts have a point in common. Tverberg's theorem is Radon's for $k=2$. Observe that Tverberg's theorem is easily extendable to families of convex sets (instead of points, cf. Corollary 4). Furthermore, if we can guarantee the existence of a d-transversal, under suitable conditions, then we can guarantee the existence of a "Tverberg partition".

Using categoric notions of *separoids* [1, 3, 12, 13, 16, 17, 18] we find sufficient conditions —in the spirit of Hadwiger's and Tverberg's theorems— for a family of convex sets to have a k-partition such that the family of convex hulls have the same separation structure as if they have a common point (in [1] this is called a *virtual 0-transversal*).

Observe that the separation structure defined by hyperplanes, of a family of convex sets, is the same if every two have a common point than if all have a common point —no subset of the family can be separated from another. So, in terms of the combinatorial structure of separation —in terms of separoids— it is the same if we cannot separate any pair of singletons than if we have a "Tverberg partition".

The main theorem can be formulated as follows (cf. Theorem 5):

Theorem 1 *Let $\mathcal{S}$ be a family of $(k-1)(d+1)+1$ convex sets in $\mathbb{E}^n$ such that every $d+2$ convex sets admits a d-transversal. Suppose that in addition, there is an injective map $\varphi\colon \mathcal{S} \longrightarrow \mathcal{P}$ into a family of points $\mathcal{P} \subset \mathbb{E}^d$ in general position such that, if A is separated from B, subsets of $\mathcal{P}$, then $\varphi^{-1}(A)$ is separated from $\varphi^{-1}(B)$. Then, there exists a k-coloration of $\mathcal{S}$ in such a way that the convex hulls of each pair of colours, have a common point.*

2 Separoids

A *separoid* is a relation $\dagger \subset 2^S \times 2^S$ defined on the family of subsets of a set S with three simple properties: for every $A, B \subseteq S$

$$\begin{array}{rl} \circ & A \dagger B \Longrightarrow B \dagger A, \\ \circ\circ & A \dagger B \Longrightarrow A \cap B = \phi, \\ \circ\circ\circ & A \dagger B \text{ and } B \subset B' \; (\subseteq S \setminus A) \Longrightarrow A \dagger B'. \end{array}$$

The separoid is identified with the set S and is sometimes denoted as a pair $(S, \dagger)$. An element $A \dagger B$ is called a *Radon partition*. The *order* of the separoid is the cardinal $|S|$ and the *size* is half of the Radon partitions $\frac{1}{2}|\dagger|$. If the separoid is finite, the third axiom implies that it is enough to know the *minimal* Radon partitions to reconstruct the separoid; they encode all the structure.

A *separation* $A \mid B$ is a pair of disjoint sets that are not a Radon partition. When talking of a separation $A \mid B$ we often say *"A is separated from B"*. The separoid is *acyclic* if $\phi \mid S$ and the separations with the empty set are considered *trivial*.

The separation relation $| \subset 2^S \times 2^S$ satisfies the following properties

$$\begin{array}{rl} \circ & A \mid B \Longrightarrow B \mid A, \\ \circ\circ & A \mid A \Longrightarrow A = \phi, \\ \circ\circ\circ & A \mid B \text{ and } A' \subset A \Longrightarrow A' \mid B. \end{array}$$

In other words the relation $|$ is a *symmetric, quasi-anti-reflexive* and *ideal* relation on the subsets of the given set S. It is clear that, in the finite case, it is enough to know *maximal* separations to reconstruct the separoid. The separoid is denoted sometimes as a pair $(S, |)$.

Clearly $\dagger$ and $|$ determine each other. They are related by the following equivalence

$$A \dagger B \iff A \not| B \quad \text{and} \quad A \cap B = \phi.$$

Finite separoids have also an intrinsic notion of dimension. The *d-dimensional simploid* is the separoid of order $d+1$ and size 0 and it will be denoted by σ^d. Since there are no Radon partitions, every pair of disjoint subsets are separated and then, it can be represented with the vertex set of a simplex —hence the name. The *dimension* of a separoid S is the maximum dimension of its induced simploids

$$\mathrm{d}(S) = \max_{\sigma^d \hookrightarrow S} d.$$

S is called *complete* if for every $i, j \in S$ follows that $i \dagger j$. The complete separoid of order n is denoted by K_n.

Examples:

1. Consider a subset $X \subseteq \mathbb{E}^d$ of the d-dimensional Euclidean space and define the following relation

$$A \dagger B \iff \langle A \rangle \cap \langle B \rangle \neq \phi \quad \text{and} \quad A \cap B = \phi,$$

where $\langle A \rangle$ denotes de convex hull of A. The pair $\mathcal{P} = (X, \dagger)$ is a separoid and will be called a *point separoid.* In fact, the name of separoids arises as an abstraction of the fact that $A \mid B$ is a separation if and only if there exists a hyperplane strictly separating $\langle A \rangle$ from $\langle B \rangle$.

2. Consider a family $\mathcal{F}$ of convex sets in $\mathbb{E}^d$ and define the separoid $S(\mathcal{F})$ as above, that is, two subsets of the family $A, B \subset \mathcal{F}$ are separated $A \mid B$ if there exists a hyperplane that leaves all members of A on one side of it and those of B on the other. If $\mathcal{F}$ is finite and its members are compact, $S(\mathcal{F}) = (\mathcal{F}, |)$ is an acyclic separoid and will be called a *separoid of convex bodies* (see Theorem 2).

3. All acyclic separoids on three elements arise from one of the eight families of convex bodies in Figure 1. Those labelled **a**, **b**, **e** and **h** are the point separoids of order 3; in fact, they come from the four essentially different order types with three elements.

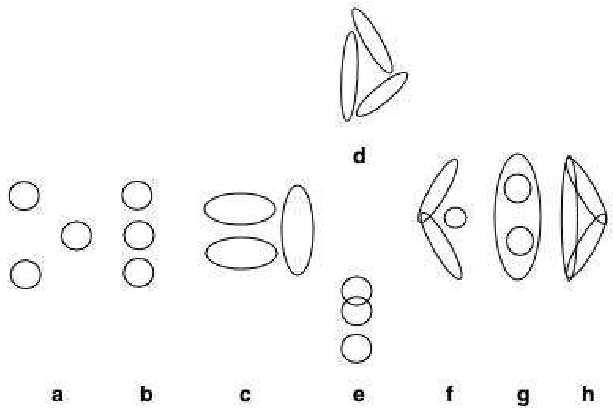

Figure 1. *The acyclic separoids of order 3*

Let S and T be two separoids. A *separoid morphism* is a function $\varphi\colon S \longrightarrow T$ with the property that for all $A, B \subseteq T$,

$$A \mid B \Longrightarrow \varphi^{-1}(A) \mid \varphi^{-1}(B).$$

The *category of separoids* is defined with such morphisms between separoids. Other kind of "maps" —and therefore another category— will be defined in Section 4 (see also [13]). Two separoids are *isomorphic* $S \approx T$ if there is a bijective morphism between them whose inverse function is also a morphism.

More examples:

4. Consider a family of convex sets $\mathcal{F}$, choose a point in each of its elements to construct a point separoid $\mathcal{P}$ and define the obvious bijection $\varphi\colon \mathcal{P} \longrightarrow \mathcal{F}$. This is a morphism since every hyperplane that separates A from B, subsets of $\mathcal{F}$, also separates their respective points $\varphi^{-1}(A)$ and $\varphi^{-1}(B)$. This construction is quite useful and we will say that *$\varphi\colon \mathcal{P} \longrightarrow \mathcal{F}$ is a choice on $\mathcal{F}$.*

5. Consider a family of convex sets $\mathcal{F}$ in $\mathbb{R}^d$ and let $\pi\colon \mathbb{R}^d \to \mathbb{R}^e$ be an affine projection ($e \leq d$). The obvious bijection $\hat{\pi}\colon \mathcal{F} \to \pi(\mathcal{F})$ is a morphism between their separoids $S(\mathcal{F}) \longrightarrow S(\pi(\mathcal{F}))$.

6. Consider a family of convex sets $\mathcal{F}$ and give them a coloration $\varsigma\colon \mathcal{F} \to \to \{c_1, \ldots, c_k\}$. If we denote by $\mathcal{F}' = \{\langle \varsigma^{-1}(c_i) \rangle\}$ the convex hulls of the colour classes' family, the obvious map $\mathcal{F} \longrightarrow \mathcal{F}'$ is a morphism. This is a key ingredient in our study of Tverberg's theorem (Section 4).

7. In Figure 1, monomorphisms go from left to right between every pair of separoids. Observe that there is no monomorphism in either directions between the separoids labelled **d** and **e**.

3 The Geometric Representation Theorem

We show now how Example 2 is indeed the more general example; this is, every finite and acyclic separoid is a separoid of convex bodies (for the non-acyclic case, see [3]).

Theorem 2 *Every acyclic separoid of order n can be represented by a family of convex polytopes in the $(n-1)$-dimensional Euclidean space.*

Proof. Let $\dagger$ be an acyclic separoid on $S = \{1, \ldots, n\}$ (i.e., $A \dagger B \Longrightarrow |A||B| > 0$). To each element $i \in S$ and each (minimal Radon) partition

$A \dagger B$ such that $i \in A$, we assign a point of $\mathbb{R}^n$

$$\rho^i_{A\dagger B} = \mathbf{e}_i + \frac{1}{2}\left[\frac{1}{|B|}\sum_{b\in B}\mathbf{e}_b - \frac{1}{|A|}\sum_{a\in A}\mathbf{e}_a\right],$$

(where $\{\mathbf{e}_j\}$ denotes the canonical basis) and realise each element $i \in S$ as the convex hull of all such points

$$i \mapsto \mathcal{K}_i = \langle \rho^i_{A\dagger B} : i \in A \text{ and } A \dagger B\rangle.$$

Observe that these convex sets are in the $(n-1)$-dimensional affine subspace spanned by the basis.

To see that this family of convex polytopes realises the separoid observe that, for each partition $A \dagger B$, the vertex set of the simplices $\langle \mathbf{e}_a : a \in A\rangle$ and $\langle \mathbf{e}_b : b \in B\rangle$ "moves" —half of the way each— to realise such a partition intersecting one to the other precisely in their barycenter. That is, let $A \dagger B$ be fixed; in order to prove that

$$\langle \mathcal{K}_a : a \in A\rangle \cap \langle \mathcal{K}_b : b \in B\rangle \neq \phi,$$

it is enough to prove that $\langle \rho^a_{A\dagger B} : a \in A\rangle \cap \langle \rho^b_{B\dagger A} : b \in B\rangle \neq \phi$ because $\rho^a_{A\dagger B} \in \mathcal{K}_a$ and therefore $\langle \rho^a_{A\dagger B} : a \in A\rangle \subset \langle \mathcal{K}_a : a \in A\rangle$ (analogously with B).

Now, if we let $\rho : \mathbb{R}^n \to \mathbb{R}^n$ be the translation

$$\rho(\mathbf{x}) = \mathbf{x} + \frac{1}{2}\left[\frac{1}{|B|}\sum_{b\in B}\mathbf{e}_b - \frac{1}{|A|}\sum_{a\in A}\mathbf{e}_a\right],$$

we have that $\rho^a_{A\dagger B} = \rho(\mathbf{e}_a)$ and the barycenter of $\langle \rho^a_{A\dagger B} : a \in A\rangle$ is

$$\begin{aligned}\frac{1}{|A|}\sum_{a\in A}\rho^a_{A\dagger B} &= \frac{1}{|A|}\sum_{a\in A}\rho(\mathbf{e}_a) = \rho\left(\frac{1}{|A|}\sum_{a\in A}\mathbf{e}_a\right)\\ &= \frac{1}{2}\left[\frac{1}{|B|}\sum_{b\in B}\mathbf{e}_b + \frac{1}{|A|}\sum_{a\in A}\mathbf{e}_a\right].\end{aligned}$$

Analogously, using that $\rho^b_{B\dagger A} = -\rho(-\mathbf{e}_b)$, we have that

$$\frac{1}{|B|}\sum_{b\in B}\rho^b_{B\dagger A} = \frac{1}{2}\left[\frac{1}{|B|}\sum_{b\in B}\mathbf{e}_b + \frac{1}{|A|}\sum_{a\in A}\mathbf{e}_a\right]$$

and therefore

$$\langle \rho^a_{A\dagger B} : a \in A\rangle \cap \langle \rho^b_{B\dagger A} : b \in B\rangle \neq \phi.$$

On the other hand, given a separation $\alpha \mid \beta$, define the affine extension of the equations

$$\psi_{\alpha|\beta}(\mathbf{e}_j) = \begin{cases} -1 & j \in \alpha \\ 1 & j \in \beta \\ 0 & \text{otherwise} \end{cases} \qquad \text{for } j = 1, \ldots, n.$$

Now, it is enough to prove that for every $i \in \alpha$ (resp. β), we have that $\psi_{\alpha|\beta}(\rho^i_{A\dagger B}) < 0$ (resp. > 0). For this, observe that, if $i \in \alpha$ then

$$\psi_{\alpha|\beta}\left(\rho^i_{A\dagger B}\right) = \psi_{\alpha|\beta}\left(\mathbf{e}_i + \frac{1}{2}\left[\frac{1}{|B|}\Big(\sum_{B\cap\alpha}\mathbf{e}_b + \sum_{B\cap\beta}\mathbf{e}_b\Big) - \frac{1}{|A|}\Big(\sum_{A\cap\alpha}\mathbf{e}_a + \sum_{A\cap\beta}\mathbf{e}_a\Big)\right]\right)$$

$$= -1 + \frac{(|B\cap\beta| - |B\cap\alpha|)}{2|B|} + \frac{(|A\cap\alpha| - |A\cap\beta|)}{2|A|} \leq 0.$$

Equality holds if and only if $B \cap \beta = B$ and $A \cap \alpha = A$ which leads to a contradiction. □

This result allow us to introduce an important invariant of separoids (see [1, 3]): the *geometric dimension* of a separoid S, denoted as $\mathrm{gd}(S)$, is the minimum dimension where it can be represented as a family of convex sets.

4 Tverberg's theorem

In the year of 1966, Tverberg published a generalisation of the classic Radon's theorem which can be written, in separoids terms, as follows

Theorem 3 (Tverberg [19]) *Let $\mathcal{P}$ be a point separoid in $\mathbb{E}^d$. If S is of order*

$$|\mathcal{P}| = (k-1)(d+1) + 1$$

then $\mathcal{P}$ can be divided in k disjoint subsets A_i, $i = 1, \ldots, k$ such that there is a point in common into their convex hulls: $\bigcap_{i=1}^k \langle A_i\rangle \neq \phi$.

The proof of this result is far from trivial. The simplest one known by the author is based on Sarkaria's argument [15] and uses Bárány's generalisation [2] of Caratheodory's theorem [4].

Clearly, Theorem 3 can be extended to convex sets instead of points (just take a point in each of the convex sets, look for the partition using Theorem 3 and ... voilà!). This motivates the following discussion (and definition); Consider a family of convex bodies S and an effective 3-coloration of them $\varsigma: S \twoheadrightarrow \{R, G, B\}$ of them (recall Example 6). If we consider now the convex hull of each color class, $T = \{\langle\varsigma^{-1}(R)\rangle, \langle\varsigma^{-1}(G)\rangle, \langle\varsigma^{-1}(B)\rangle\}$, the resulting separoid most be one of the eight separoids depicted in Figure 1. More over, there is a canonic (natural) epimorphisms $\varphi: S \longrightarrow T$ between these separoids. Such an epimorphism satisfies an extra condition: for $A, B \subset T$,

$$(*) \qquad A \dagger B \Longrightarrow \varphi^{-1}(A) \dagger \varphi^{-1}(B).$$

An epimorphism which satisfies the extra condition $(*)$ will be called a *chromomorphism*. So, Theorem 2 implies the following

Corollary 4 *If S is a separoid of order $|S| = (k-1)(\mathrm{gd}(S)+1)+1$ then there exists a chromomorphism $S \longrightarrow K_k$ onto the complete separoid of order k.*

Proof. Consider S as a family of convex sets in $\mathbb{E}^d$, where $d = \mathrm{gd}(S)$. Now, let $\varphi: P \longrightarrow S$ be a choice on S. Applying Theorem 3 to P we conclude that there exists a partition $P = A_1 \cup \ldots \cup A_k$ such that the convex hulls of the parts have a common point. If we denote by $K_k = \{1, \ldots, k\}$ (where $i \dagger j$ for all i and j) the elements of the complete separoid, clearly the function $\varsigma: S \to \{1, \ldots, k\}$ defined as $\varsigma(s) = i \iff \varphi^{-1}(s) \in A_i$ is a chromomorphism onto K_k. $\square$

However, observe that the conclusion in the proof is a bit stronger than that of the statement; we had conclude that there is a point in common to *all* color classes' convex hulls. But in order to be isomorphic to a complete separoid it is enough that every pair of convex sets intersect (see the realisation of K_3 in Figure 1.**h**). So, it may be that the hypothesis of Corollary 4 can be weakened. This motivated our main result:

Theorem 5 *Let S be a separoid of order $|S| = (k-1)(\mathrm{d}(S)+1)+1$. Suppose that in addition, there exists a monomorphism $\mu: S \to P$ into a separoid of points in general position, and $\mathrm{d}(P) = \mathrm{d}(S)$. Then there exists a chromomorphism $S \longrightarrow K_k$ onto the complete separoid of order k.*

Proof. Let S be a d-dimensional separoid of order $(k-1)(d+1)+1$. Suppose there is a monomorphism $\mu: S \to P$ into a d-dimensional point separoid in general position. By Tverberg's theorem, there exists a chromomorphism $\tau: P \longrightarrow K_k$. We will show that $\varsigma = \tau \circ \mu$ is a chromomorphism.

For, let $i \dagger j$ be an edge of K_k (or a minimal Radon partition if you will). Since τ is chromomorphism, we have that $\tau^{-1}(i) \dagger \tau^{-1}(j)$. Then, there exist $A \subseteq \tau^{-1}(i)$ and $B \subseteq \tau^{-1}(j)$ such that $A \dagger B$ is minimal. Since P is in general position, $|A \cup B| = d + 2$. Since μ is injective, $|\mu^{-1}(A \cup B)| = d + 2$ and there exist $C \dagger D$ such that $C \cup D = \mu^{-1}(A \cup B)$. Therefore, since μ is a monomorphism, $\mu(C) \dagger \mu(D)$. Since P is a point separoid, it is a Radon separoid and we may suppose that $\mu(C) = A$ and $\mu(D) = B$. Finally, since $C \subseteq \varsigma^{-1}(i)$ and $D \subseteq \varsigma^{-1}(j)$, we have that $\varsigma^{-1}(i) \dagger \varsigma^{-1}(j)$, which concludes the proof. □

The Hadwiger-type hypothesis added to Theorem 5 still is "geometric" in nature... Is there a purely combinatorial Tverberg-type theorem?

That motivates the following

Problem 1 *Determine the minimum number $n(d,k)$ such that, if $|S| \geq n(\mathrm{d}(S),k)$ then there exists a chromomorphism $S \longrightarrow K_k$ onto the complete separoid of order k.*

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On 2-connected spanning subgraphs of given maximum degree

Heinz-Jürgen Voss

(Joint work with Michal Tkáč)

Finding hamiltonian paths and cycles in graphs is a well investigated subject in graph theory. there are probably hundreds of papers devoted to results of the form "If G satisfies property P, then G is hamiltonian". If P is replaced by a weaker condition P' then G is perhaps no longer hamiltonian, but has a generalized structure near to hamiltonicity. Such generalizations have been stated in [6]. Here we generalize a hamiltonian cycle in an other way. A hamiltonian cycle in a graph G is a 2-connected spanning subgraph of G of maximum degree 2.

Define a *k-trestle* to be a 2-connected spanning subgraph of maximum degree at most k. Thus a 2-trestle is a hamiltonian cycle.

The property P investigated here is the toughness of a graph G. The concept *toughness* has been introduced by Chvátal [4]. Let G be a connected graph. A subset S of the vertex set of G *separates* G if the graph $G - S$ obtained from G by deleting the vertices of S is disconnected. G is said to be *t-tough* if for every separating set $S \subseteq V(G)$ the number $\omega(G - S)$ of components of $G - S$ is at most $\frac{|S|}{t}$. The toughness $\tau(G)$ of a non-complete graph G is defined to be the largest real number $t > 0$ such that G is t-tough. For a complete graph G let $\tau(G) = \infty$. It is easy to see that every graph with toughness less than one has no 2-trestles. In general for k-trestles it can easily be proven (see [7]).

Every graph G with toughness $\tau(G) < \frac{2}{k}$ has no k-trestle.

In [4] Chvátal presented his famous conjecture.

There is a real number $t_0 > 0$ such that every t_0-tough graph has a 2-trestle.

This conjecture is open till today. For the class of all chordal graphs Chen, Jacobson, Kézdy, and Lehel [3] proved the validity of Chvátal's Conjecture with $t_0 = 18$. A graph G is said to be a *chordal* if each cycle C

of length at least 4 has a chord e, i. e., e is an edge not in C but its end vertices are on C.

Theorem 1 *Each chordal polyhedral graph G of toughness > 18 has a 2-trestle, i. e., a Hamiltonian cycle.*

By Steinitz's theorem a polyhedral graph is a planar and 3-connected graph. Let G be a connected graph. Here we will discuss a special class of polyhedral graphs, namely, the chordal polyhedral graphs.

In the class of all chordal polyhedral graphs Böhme, Harant, and Tkáč [2] found the sharp bound $t_0 = 1$ for the toughness.

Theorem 2 *Each chordal polyhedral graph G of toughness > 1 has a 2-trestle, i. e., a Hamiltonian cycle. Moreover, there are 1-tough chordal polyhedral graphs without a Hamiltonian cycle.*

There are several papers which deal with k-trestled polyhedral graphs, $k \geq 3$. In [1] Barnette showed that there is a polyhedral graph with no 5-trestles. In [5] Gao proved that every 3-connected graph on the plane, projective plane, torus and Klein bottle has a 6-trestle.

Recently we have shown in [8]:

Theorem 3 *Each chordal polyhedral graph G of toughness greater than $5/6$, or $7/10$, or $11/18$ has a 3-trestle, or a 4-trestle, or a 5-trestle, respectively. There are chordal polyhedral graphs of toughness $21/29 > 0.72$, or $51/86 > 0.593$, or $139/259 > 0.536$, without a 3-trestle, or a 4-trestle, or a 5-trestle, respectively.*

The bound of Theorem 2 is sharp.

Open problems are the determination of the precise bounds in Theorems 1 and 3.

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