Generalized star packing problems I.

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Abstract

Results of Las Vergnas, Hell and Kirkpatrick imply that packing an undirected graph by a set of stars is polynomial if and only if this set is $\{S_1, S_2, \ldots, S_k\}$. That is, if we forbide some stars from this 'sequential' set of stars then we get an NP-complete problem. This arises the question if it is possible to allow some other new graphs to be a component of the packing to maintain polynomiality. In this and in the sibling paper "... II." we show two types of graph sets which can be added to the packing to maintain polynomiality. These new graphs (called 'superstars') are trees such that some of the leaf vertices of a star are replaced by forbidden stars of the packing. These two papers show different methods for solving the two type of packing problems. The present paper introduces an Edmonds-type algorithm together with a Berge-type theorem.

Keywords: graph packing, star, (f, g)-factor

1 Introduction

Let $\mathcal{F}$ be a family of graphs (in this paper $K_2 \in \mathcal{F}$). An *$\mathcal{F}$-packing* of a graph G is a set of vertex disjoint subgraphs of G, each isomorphic to a

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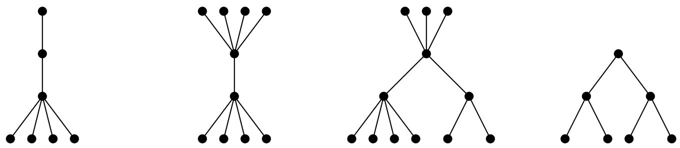
member of $\mathcal{F}$. An $\mathcal{F}$-packing is called *maximum* if it covers a maximum number of vertices of G and it is called *perfect* if it covers every vertex of G. The *$\mathcal{F}$-deficiency* of a graph G is the minimum number of uncovered vertices, for any $\mathcal{F}$-packing of G. A graph with $\mathcal{F}$-deficiency equal to zero is called *$\mathcal{F}$-saturable*. Several authors [2, 3, 4, 5, 6, 7, 8, 12, 13, 14] studied this kind of packing problem. Polynomial time algorithms for finding perfect or maximum $\mathcal{F}$-packings have been found for various special families $\mathcal{F}$ of graphs.

In [10], Las Vergnas proved that the $\{S_1, \ldots, S_k\}$-packing problem is polynomial, where S_i is a *i-star*, i.e. a simple graph with a specified vertex, called *center*, whose deletion results in a graph consisting of i isolated vertices, called *leaves*. On the contrary, Hell and Kirkpatrick [7] proved that every other set of stars gives an NP-complete packing problem. In other words, omitting some stars from the star-sequence $\{S_1, \ldots, S_k\}$ with keeping S_k yields an NP-complete problem. This arises the question, is it possible to add some other graphs to the packing to recover polynomiality? In this and in the sibling paper "... II." we consider such packing problems, called the 'superstar' packing problems.

For introducing this, an integer h is called a *gap* of a set $H \subseteq \mathbb{N}$ if $h \notin H$ but H contains an integer at least h and an integer at most h. H is said to have *no two consecutive gaps* if $\min H \leq i \leq \max H$, $i \notin H$ implies $i+1 \in H$. This paper shows that we may add some 'superstars' to the set $\{S_i : i \in H\}$ to recover polynomiality in case $1 \in H$ *and H has no two consecutive gaps*. For this, denote $u = \max H$ and let $1 \leq b \leq u$ be a fixed integer. A star S_i is *allowed (in H)* if $i \in H$ and *forbidden (in H)* if i is a gap of H. A graph is said to be an *(t,s)-superstar* if it is constructed as follows: connect the center of a star S_t to the centers of s forbidden stars (see Fig. 1 for an example). Let $\mathcal{S}_{H,b}$ consist of all the allowed stars and of all the (t,s)-superstars for $0 \leq t+s \leq u$ and $1 \leq s \leq b$. This paper presents a polynomial algorithm for solving the $\mathcal{S}_{H,b}$-packing problem for all $1 \in H \subseteq \mathbb{N}$ with no two consecutive gaps and for all $1 \leq b \leq u$ where $u = \max H$. The presented algorithm is a generalization of the alternating forest matching algorithm of Edmonds. We accompany it by a Berge-type min-max theorem.

The sibling paper [9] shows another type of sets of superstars which can be added to the family $\{S_i : i \in H\}$ to maintain polynomiality. Here let $b \geq 1$ and let $\mathcal{C}_{H,b}$ consist of all the allowed stars and of all the (t,s)-superstars for $0 \leq t \leq u$ and $1 \leq s \leq b$. The algorithm of [9] solving a $\mathcal{C}_{H,b}$-packing problem for all $1 \in H \subseteq \mathbb{N}$ with no two consecutive gaps and



(1,1)-superstar (4,1)-superstar (3,2)-superstar (0,2)-superstar

Figure 1: Examples of superstars, $H = \{1, 3, 5\}$

for all $b \geq 1$ uses a reduction to the H-factor problem introduced by Lovász [15]. The approach of [9] implies also the fact that the $\mathcal{C}_{H,b}$-packing problem in *matroidal*, i.e. the vertex sets coverable by $\mathcal{C}_{H,b}$-packings form a matroid.

At the end of this paper, we will show a modification of our $\mathcal{S}_{H,b}$-packing algorithm, which solves the $\mathcal{C}_{H,b}$-packing problem.

Note that if $H = \{1, \ldots, u\}$ and $1 \leq b \leq u$ then $\mathcal{S}_{H,b} = \mathcal{C}_{H,b}$ consists of all the stars S_i, $1 \leq i \leq u$, and we get the star packing problem of Las Vergnas [10]. Hence both types of superstar packing problems contain the classical matching problem. Observe that in both types of problems there are several classes of graphs (one for each b), which can be added to the set of allowed stars to maintain polynomiality.

Loebl and Poljak [12, 13, 14] and Janata [8] studied $\mathcal{P}$-packing problems, where $\mathcal{P}$ contains K_2, some hypomatchable graphs and a set of graphs called propellers. A graph F is called *hypomatchable* if it has no perfect matching but for all $v \in V(F)$ the graph $F \setminus v$ (i.e. the graph constructed from F by deleting v) is perfectly matchable. A *k-propeller* ($k \geq 0$) is a graph with a specified vertex called *center*, whose deletion results in a graph consisting of $k+1$ hypomatchable components called *blades*, at least one of which is a single vertex. A graph constructed from a k-propeller P by deleting at most k of its blades is called a *subpropeller* of P. We denote by $P_1 + P_2$ a graph that arises from two propellers P_1, P_2 by glueing their centers c_1, c_2 into one new vertex c and arbitrarily selected neighbors r_1, r_2 of c_1, c_2 with degree one into one new vertex r. The $\mathcal{P}$-packing problem is matroidal if and only if $\mathcal{P}$ satisfies the heredity condition (every subpropeller of a $\mathcal{P}$-saturable propeller is $\mathcal{P}$-saturable) and the blade-exchange condition (if P, P' are $\mathcal{P}$-saturable propellers and P' is isomorphic to $P_1 + P_2$, where P_1 is a $\mathcal{P}$-saturable 1-propeller and P_2 is a subpropeller of P, then there exists a component B of $P \setminus P_2$ such that $(P \setminus B) + P_1$ is $\mathcal{P}$-saturable). The matroidal cases of the $\mathcal{P}$-packing problem are polynomially solvable. We call the reader's attention to similarity of the structure of polynomial cases of $\mathcal{P}$-packing problem and both types of the superstar-packing problem. In all families $\mathcal{C}_{H,b}$ and $\mathcal{S}_{H,b}$, an analogy of the heredity and blade-exchange conditions hold considering the sets of 'blades' consisting of K_1 and the forbidden stars. Even though we have dropped the condition that a blade has to be a hypomatchable graph, we have kept the crucial property: two 'blades' connected by an edge form a saturable graph.

Throughout the paper all graphs are finite and simple. K_n denotes the complete graph on n vertices. If G is a graph and $U \subseteq V(G)$ then *shrinking* the set U results in a graph G' obtained from G by deleting all the vertices of U (and their incident edges), and inserting a single vertex u adjacent to any vertex of $V(G) \setminus U$ that is adjacent to at least one vertex from U. (Note that this assures that there are no parallel edges in G'.) The new vertex u is called a *shrunk vertex.* If G is a graph and $\mathcal{Q}$ is its $\mathcal{F}$-packing then $V(\mathcal{Q}), E(\mathcal{Q})$ denote the sets of all vertices and edges of $\mathcal{Q}$, respectively.

2 Results

In this section, we state the Berge-type min-max theorem for the $\mathcal{S}_{H,b}$-packing problem. The proof of the theorem depends on the algorithm and will be presented in section 4. For the rest of the chapter, we assume H and b fixed.

A crucial technique used in the Edmonds alternating forest matching algorithm is shrinking of hypomatchable graphs. The important property of a hypomatchable graph is that joining it by an edge to another hypomatchable graph results in a perfectly matchable graph. In our algorithm, the role of hypomatchable graphs will be played by graphs that we call *hedgehogs.*

Definition 2.1. A connected graph W is called a *small hedgehog* if

(a) W is a single vertex or a non-$\mathcal{S}_{H,b}$-saturable odd cycle (the vertices of W are called *free*), or

(b) W is a tree containing $l \geq 1$ forbidden stars denoted by $L_1, \ldots, L_l$, such that $\bigcup_{i=1}^{l} E(L_i) = E(W)$, $\bigcup_{i=1}^{l} V(L_i) = V(W)$, $|V(L_i) \cap V(L_j)| \leq 1$ $(i \neq j)$ and if $V(L_i) \cap V(L_j) = \{x\}$ then x is a leaf in both L_i, L_j. The centers of L_i's are called *fixed* and the leaves of L_i's are called *free.* A fixed vertex with h free neighbors is called *h-fixed.*

Definition 2.2. A graph D is called a *hypomatchable union* of graphs $L_1, \ldots L_{2l+1}$ $(l \geq 0)$ if every L_i is a vertex-induced subgraph of D, $V(D) =$

$V(L_1)\dot{\cup}\dots\dot{\cup}V(L_{2l+1})$ and shrinking of every L_i results in a hypomatchable graph D' called the *skeleton* of D.

Let D be a hypomatchable union of small hedgehogs $W_1, \dots, W_{2l+1}$ ($l \geq 0$). The family $\mathcal{W} = \{W_1, \dots, W_{2l+1}\}$ is called the *decomposition of* D and the graphs W_i are called the *small hedgehogs of* $(D, \mathcal{W})$. A vertex $v \in V(D)$ is called *free* if there exists a decomposition $\mathcal{W}$ of D such that v is free in some small hedgehog of $(D, \mathcal{W})$. Otherwise, we say that v is *fixed* in D.

Definition 2.3. A hypomatchable union of small hedgehogs D is called a *hedgehog* if D is not $\mathcal{S}_{H,b}$-saturable.

Examples of small hedgehogs and hedgehogs can be seen in Figure 2. A polynomial recognition of $\mathcal{S}_{H,b}$-saturable hypomatchable unions of small hedgehogs will be provided in Section 3. We have the following Berge-type theorem:

Theorem 2.4. *The $\mathcal{S}_{H,b}$-deficiency of a graph G is*

$$\max |\mathcal{H}| - u|Y^{\mathcal{H}}_{free}| - b|Y^{\mathcal{H}}_{fixed}|,$$

where max *is taken over all sets $\mathcal{H}$ of induced disjoint non-adjacent hedgehog-subgraphs of G, $Y^{\mathcal{H}}_{fixed}$ are those neighbors of these hedgehogs which are adjacent only to fixed vertices and $Y^{\mathcal{H}}_{free}$ are the remaining neighbors.*

The proof of Theorem 2.4 together with the formulation of the algorithm will be introduced in Section 4. Before that, in Section 3, we will pay attention to the structure of hedgehogs and $\mathcal{S}_{H,b}$-packings.

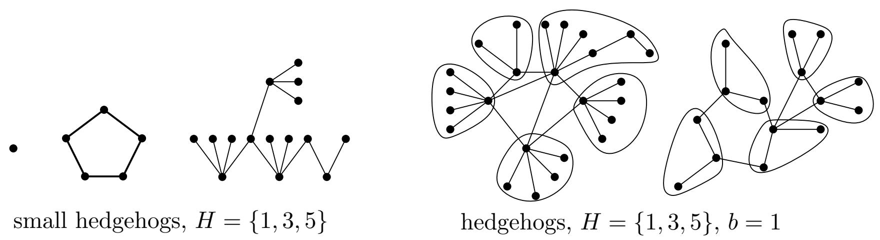


Figure 2: Examples of small hedgehogs and hedgehogs for $H = \{1, 3, 5\}$

3 Hedgehogs

In this section we introduce the crucial properties of small hedgehogs and hedgehogs with respect to $\mathcal{S}_{H,b}$-packings. For this purpose, j is called an *undergap* (in H) if $j \in H$ and $j+1$ is a gap in H. Similarly, j is an *overgap* (in H) if $j \in H$ and $j-1$ is a gap in H.

We start by defining a few more notions concerning superstars: We call a vertex c of a superstar L a (t,s)-*supercenter*, if $L \setminus c$ consists of $t+s$ components, s of which are forbidden stars and t of which are copies of K_1. Note that a superstar can have more supercenters: every $(t,1)$-superstar with $t \notin H$ has exactly two supercenters. From now, a star S_l can be viewed as a $(l,0)$-superstar with supercenter in the center or (if l is an overgap) as a $(0,1)$-superstar with supercenter in a leaf of S_l.

We call a 1-degree vertex of a superstar a *leaf* if it is not a supercenter. A vertex of a superstar that is neither supercenter nor leaf is called a *center*. A superstar L containing at least one center is called *underloaded*, otherwise L is called *loaded*. Observe that an underloaded superstar has a unique supercenter and that the degrees of all vertices of a loaded superstar and the degrees of all vertices of an underloaded star different from its supercenter are in H. We may observe that if $L \in \mathcal{S}_{H,b}$, v is a leaf of L and X_c is a subgraph of L induced by its center c and the neighboring leaves, then the graphs $L \setminus v$ and $L \setminus X_c$ are $\mathcal{S}_{H,b}$-saturable.

Let us continue by studying the properties of small hedgehogs. Recall that all neighbors of a fixed vertex of a small hedgehog are free and their number is a gap in H. We may easily find out that a small hedgehog is not $\mathcal{S}_{H,b}$-saturable. On the other hand, the following holds:

Proposition 3.1. *Let W be small hedgehog. The following graphs are $\mathcal{S}_{H,b}$-saturable:*

- *$W \setminus w$, where w is a free vertex of W*
- *$W \setminus L$, where L is a subgraph of W induced by a fixed vertex of W and all of its neighbors.*

Proof. If W is a single vertex or an odd cycle then the result is trivial. If W consists of l glued-together forbidden stars then the perfect $\mathcal{S}_{H,b}$-packings of $W \setminus w$, $W \setminus L$ consist of l and $l-1$ undergap-stars, respectively. $\square$

Proposition 3.2. *Let W be a connected graph constructed from two disjoint small hedgehogs W_1, W_2 by connecting arbitrary vertices $w_1 \in V(W_1)$, $w_2 \in V(W_2)$ by a new edge e. Then W is $\mathcal{S}_{H,b}$-saturable.*

Proof. Let L_1, L_2 be subgraphs of W_1, W_2, respectively, defined as follows: If w_i is free in W_i then $L_i = \{w_i\}$. If w_i is fixed in W_i then L_i is the star induced by w_i and all of its neighbors. The perfect $\mathcal{S}_{H,b}$-packing of W consists of perfect $\mathcal{S}_{H,b}$-packings of $W_i \setminus L_i$ guaranteed by Proposition 3.1 and of the graph $L_1 \cup L_2 \cup e$, which is a copy of K_2, an overgap-star, or a $(t, 1)$-superstar where t is a gap in H. □

Proposition 3.3. *Let W be a small hedgehog and let $u \neq v \in V(W)$ be two non-adjacent vertices of W. Then $W \cup uv$ is $\mathcal{S}_{H,b}$-saturable or a hedgehog.*

Proof. If W is a non-$\mathcal{S}_{H,b}$-saturable odd cycle then 2 is a gap of H and $W \cup uv$ is $\mathcal{S}_{H,b}$-saturable using edges and exactly one $(1, 1)$-superstar.

If W is a tree then by adding the edge uv, we create exactly one cycle C. Every fixed vertex of W lies in at most two edges of C. Consider the graph W' constructed from W by deleting every pair of edges $e, f \in E(C) \setminus \{uv\}$ such that e, f meet in a fixed vertex. W' has an odd number of components each of which is either a small hedgehog or $\mathcal{S}_{H,b}$-saturable. Moreover, every graph induced by an edge of C and the two adjacent components of W' is also $\mathcal{S}_{H,b}$-saturable. If any of the components is $\mathcal{S}_{H,b}$-saturable, then the whole $W \cup uv$ is, otherwise $W \cup uv$ is a hypomatchable union of small hedgehogs, and so it is either $\mathcal{S}_{H,b}$-saturable or a hedgehog. □

In the following, we will introduce the crucial properties of hypomatchable unions of small hedgehogs.

Proposition 3.4. *Let D be a hypomatchable union of small hedgehogs with decomposition $\mathcal{W}$. If W is a small hedgehog of $(D, \mathcal{W})$ then there exists a perfect $\mathcal{S}_{H,b}$-packing $\mathcal{Q}_W$ of $D \setminus W$. Moreover, $\mathcal{Q}_W$ uses only loaded superstars.*

Proof. Let D' be the skeleton of D and let $v_W \in V(D')$ be the shrunk vertex associated to W. The perfect $\mathcal{S}_{H,b}$-packing $\mathcal{Q}_W$ of $D \setminus W$ is the union of perfect $\mathcal{S}_{H,b}$-packings of pairs of small hedgehogs (the pairs are induced by a perfect matching of $D' \setminus v_W$) guaranteed by Proposition 3.2. □

The following corollaries follow easily from Propositions 3.1, 3.2 and 3.4.

Corollary 3.5. *Let D be a connected graph constructed from two disjoint hypomatchable unions of small hedgehogs D_1, D_2 by connecting arbitrary vertices $v_1 \in V(D_1)$, $v_2 \in V(D_2)$ by a new edge e. Then D is $\mathcal{S}_{H,b}$-saturable.*

Corollary 3.6. *Let w be a free vertex of a hypomatchable union of small hedgehogs D. Then $D \setminus w$ is $\mathcal{S}_{H,b}$-saturable.*

We say that $\mathcal{Q}$ is a *near-perfect* packing of a hypomatchable union of small hedgehogs D if for some $w \in V(D)$, $\mathcal{Q}$ is a perfect $\mathcal{S}_{H,b}$-packing of $D \setminus w$ using only loaded superstars, or if for some $l \notin H$, $\mathcal{Q}$ is a perfect $(\mathcal{S}_{H,b} \cup \{S_l\})$-packing of D using loaded superstars and and exactly one copy of S_l. The unique vertex uncovered by $\mathcal{Q}$ or covered by a center of a forbidden star is called the *critical vertex* of $\mathcal{Q}$.

Lemma 3.7. *A hypomatchable union of small hedgehogs D is $\mathcal{S}_{H,b}$-saturable if and only if*

- *D admits a near-perfect packing $\mathcal{Q}$ skipping a vertex neighboring to a vertex of a superstar of $\mathcal{Q}$ which is not a supercenter or is a (t,s)-supercenter with $s \geq 1$ and $t+s < u$, or $s = 0$ and $t+1 \in H$, or*
- *D admits a near-perfect packing $\mathcal{Q}$ using exactly one forbidden star whose center is a neighbor of a vertex of a superstar of $\mathcal{Q}$ which is not a supercenter or is a (t,s)-supercenter with $s < b$ and $t+s < u$.*

Proof. The "if"-part of the Lemma is obvious. In all cases we may simply alter the packing $\mathcal{Q}$ to get a perfect $\mathcal{S}_{H,b}$-packing of D.

Let us concentrate on the "only if" part of the proof. Let D be a $\mathcal{S}_{H,b}$-saturable hypomatchable union of small hedgehogs, let $\mathcal{Q}_p$ be a perfect $\mathcal{S}_{H,b}$-packing of D and let $\mathcal{Q}$ be a near-perfect packing of D such that

(I) if $\mathcal{Q}$ uses exactly one forbidden star S with center c then every edge of S is in $E(\mathcal{Q}_p)$, every leaf of S has degree one in $\mathcal{Q}_p$ and c belongs to an edge of $E(\mathcal{Q}_p) \setminus E(S)$.

We define the *distance of $\mathcal{Q}$ from $\mathcal{Q}_p$* by $\mathrm{dist}(\mathcal{Q}, \mathcal{Q}_p) = 2u(|E(\mathcal{Q}) \setminus E(\mathcal{Q}_p)| + |E(\mathcal{Q}_p) \setminus E(\mathcal{Q})|) - \deg_{\mathcal{Q}_p}(v)$, where v is the critical vertex of $\mathcal{Q}$. Without loss of generality assume $\mathcal{Q}$ is a packing with minimum $\mathrm{dist}(\mathcal{Q}, \mathcal{Q}_p)$.

If $\mathcal{Q}$ skips v and $\mathcal{Q}$ does not satisfy the conditions of Lemma 3.7 then every neighbor of v is a (t,s)-supercenter of a superstar of $\mathcal{Q}$ with $t+s = u$, or $s = 0$ and $t+1 \notin H$. Let $vy \in E(\mathcal{Q}_p) \setminus E(\mathcal{Q})$ such that $\deg_{\mathcal{Q}_p}(y)$ is a minimum. If $\deg_{\mathcal{Q}}(y) = u$ then let $yz \in E(\mathcal{Q}) \setminus E(\mathcal{Q}_p)$ be an edge such that the degree of z in $\mathcal{Q}$ is a maximum. If there is an edge zz' such that $zz' \in E(\mathcal{Q}) \setminus E(\mathcal{Q}_p)$ or $zz' \in E(\mathcal{Q}) \cap E(\mathcal{Q}_p)$ and $\deg_{\mathcal{Q}_p}(z') > 1$ then consider the $\mathcal{S}_{H,b}$-packing $\mathcal{Q}' = \mathcal{Q}\Delta\{vy, yz, zz'\}$. Otherwise, let $\mathcal{Q}' = \mathcal{Q}\Delta\{vy, yz\}$ and observe that $\mathcal{Q}'$ satisfies (I). In both cases $\text{dist}(\mathcal{Q}', \mathcal{Q}_p) < dist(\mathcal{Q}, \mathcal{Q}_p)$ which contradicts the selection of $\mathcal{Q}$.

If $\deg_{\mathcal{Q}}(y) < u$ then y is a center of a star S_i with $i + 1 \notin H$ in $\mathcal{Q}$. If there exists an edge $yz \in E(\mathcal{Q}) \setminus E(\mathcal{Q}_p)$ then let $\mathcal{Q}' = \mathcal{Q}\Delta\{vy, yz\}$. If all edges of S_i are in $E(\mathcal{Q}')$ and there is a leaf y' of S_i with $\deg_{\mathcal{Q}_p}(y') > 1$ then let $\mathcal{Q}' = \mathcal{Q}\Delta\{vy, yy'\}$. If all leaves of S_i have degree one in $\mathcal{Q}_p$ then put $\mathcal{Q}' = \mathcal{Q}\Delta\{vy\}$: this time $\mathcal{Q}'$ contains a forbidden star with center y satisfying (I); otherwise $\deg_{\mathcal{Q}_p}(y)$ was not a minimum. In all cases, $\mathcal{Q}'$ is a near-perfect packing of D with $\text{dist}(\mathcal{Q}_p, \mathcal{Q}') < \text{dist}(\mathcal{Q}_p, \mathcal{Q})$, which gives a contradiction with the selection of $\mathcal{Q}$.

If $\mathcal{Q}$ covers v by a center of a forbidden star S then by the assumption, there is an edge $vw \in E(\mathcal{Q}_p) \setminus E(\mathcal{Q})$. If $\mathcal{Q}$ does not satisfy the conditions of Lemma 3.7 then w is a (t, s)-supercenter of a superstar of $\mathcal{Q}$ with $t+s = u$, or $s = b$. If w is a center of S_u in $\mathcal{Q}$ then there is an edge $ww' \in E(\mathcal{Q}) \setminus E(\mathcal{Q}_p)$ and we put $\mathcal{Q}' = \mathcal{Q}\Delta\{vw, ww'\}$. Otherwise, there are two consecutive edges $wz, zz' \in E(\mathcal{Q})$ with $\deg_{\mathcal{Q}}(z) - 1 \notin H$ and $\deg_{\mathcal{Q}}(z') = 1$ such that at least one of wz, zz' is not in $E(\mathcal{Q}_p)$. If there exists such pair wz, zz' with $zz' \notin E(\mathcal{Q}_p)$ or $\deg_{\mathcal{Q}_p}(z') > 1$ then put $\mathcal{Q}' = \mathcal{Q}\Delta N\{xw, wz, zz'\}$. Otherwise in every such pair $zz' \in E(\mathcal{Q}_p)$ and so $wz \notin E(\mathcal{Q}_p)$. We select $wz, zz' \in E(\mathcal{Q})$ arbitrarily and put $\mathcal{Q}' = \mathcal{Q}\Delta\{xw, wz\}$.

In all cases, we have constructed a near-perfect packing $\mathcal{Q}'$ of D with $\text{dist}(\mathcal{Q}_p, \mathcal{Q}') < \text{dist}(\mathcal{Q}_p, \mathcal{Q})$, which is a contradiction. Hence $\mathcal{Q}$ satisfies the conditions of Lemma 3.7. □

Lemma 3.8. *Given a hypomatchable union of small hedgehogs D and a decomposition $\mathcal{W}$ of D in which two free vertices v, v' of two distinct small hedgehogs are adjacent, we may find a perfect $\mathcal{S}_{H,b}$-packing of D or a decomposition of D into smaller number of small hedgehogs in polynomial time.*

Proof. Let $\mathcal{Q}$ be a perfect $\mathcal{S}_{H,b}$-packing of $D \setminus v$ guaranteed by Proposition 3.6. Observe that $\deg_{\mathcal{Q}}(v') = 1$. Hence if $\mathcal{Q}$ covers v' by a graph different from K_2 or if $2 \in H$, then $\mathcal{Q}$ satisfies the conditions of Lemma 3.7 and so D is $\mathcal{S}_{H,b}$-saturable.

If $2 \notin H$ and $\mathcal{Q}$ covers v' by a copy of K_2 then let $\mathcal{Q}'$ be a perfect $\mathcal{S}_{H,b}$-packing $D \setminus v'$ guaranteed by Proposition 3.6. Consider a path P of maximum length starting in v and containing alternately edges of $E(\mathcal{Q}') \setminus E(\mathcal{Q})$ and $E(\mathcal{Q}) \setminus E(\mathcal{Q}')$ leading to vertices of degree one in $\mathcal{Q}'$ and $\mathcal{Q}$, respectively.

If the number of edges on P is odd then the last vertex z of P is a leaf or a 1-degree supercenter of a superstar L in $\mathcal{Q}$. In both cases, $\mathcal{Q}$ may be augmented to an $\mathcal{S}_{H,b}$-packing covering more vertices by swapping edges and non-edges along P and replacing the newly constructed graphs by their perfect $\mathcal{S}_{H,b}$-packings.

If the number of edges on P is even then the last vertex z of P is either uncovered by $\mathcal{Q}'$ or a leaf or a 1-degree supercenter of a superstar L in $\mathcal{Q}'$. If z is covered by $\mathcal{Q}'$ then $\mathcal{Q}'$ may be augmented to a $\mathcal{S}_{H,b}$-packing covering more vertices by swapping edges and non-edges along $\{v'v\} \cup P$ and by replacing the newly constructed graphs by their perfect $\mathcal{S}_{H,b}$-packings.

If z is uncovered by $\mathcal{Q}'$ then $z = v'$ and $C = P \cup zv$ is an odd cycle. If every small hedgehog of $(D, \mathcal{W})$ intersecting C is a subgraph of C then by replacing these small hedgehogs by C in $\mathcal{W}$, we obtain a smaller decomposition. Otherwise, there exists a small hedgehog W intersecting C with $|V(W) \setminus V(C)| > 0$. If all graphs of $\mathcal{Q}'$ intersecting C are copies of K_2 then $W \cup C$ is $\mathcal{S}_{H,b}$-saturable and so D is $\mathcal{S}_{H,b}$-saturable. Otherwise let L be the union of graphs of $\mathcal{Q}'$ intersecting C. Let us observe that $L \cup C$ is $\mathcal{S}_{H,b}$-saturable and so D is $\mathcal{S}_{H,b}$-saturable. $\square$

A decomposition $\mathcal{W}$ of D with no two free vertices of two distinct small hedgehogs joined by an edge is called *standard*. Given a decomposition $\mathcal{W}'$ of D, we may in polynomial time find a standard decomposition $\mathcal{W}$ such that vertices free in $(D, \mathcal{W}')$ are free in $(D, \mathcal{W})$, or conclude that D is $\mathcal{S}_{H,b}$-saturable.

Corollary 3.9. *For every $\mathcal{S}_{H,b}$-saturable hypomatchable union of small hedgehogs D, there exists a perfect $\mathcal{S}_{H,b}$-packing $\mathcal{Q}$ using at most one underloaded superstar. If $\mathcal{Q}$ uses an underloaded superstar L then L is a (t,s)-superstar with $s \leq 2$. Hence for at most one $v \in V(D)$, $\deg_{\mathcal{Q}}(v) \notin H$.*

Moreover, if $\mathcal{W}$ is a standard decomposition of D, then all free vertices of non-odd-cycle small hedgehogs of $(D, \mathcal{W})$ that are not covered by L have degree one.

Proof. Let $\mathcal{N}$ be the near-perfect packing guaranteed by Lemma 3.7. $\mathcal{N}$ does not contain any underloaded superstar. When transforming $\mathcal{N}$ to a perfect $\mathcal{S}_{H,b}$-packing $\mathcal{Q}$, we maintain at most two graphs of $\mathcal{N}$ and create at most one underloaded superstar L which is a (t,s)-superstar with $s \leq 2$. Due to the properties of superstars, for at most one vertex $v \in V(H)$, $\deg_{\mathcal{Q}}(v) \notin H$ (if such v exists then v is the supercenter of L).

Let us recollect the proof of Lemma 3.7. Consider a standard decomposition $\mathcal{W}$ of an $\mathcal{S}_{H,b}$-saturable hypomatchable union of small hedgehogs D. Consider a free vertex w of $(D, \mathcal{W})$ and a perfect $\mathcal{S}_{H,b}$-packing $\mathcal{N}_0$ of $D \setminus w$ guaranteed by Corollary 3.6. $\mathcal{N}_0$ uses only loaded superstars and every free vertex of $(D, \mathcal{W})$ has degree one in $\mathcal{N}_0$. Let $\mathcal{Q}_p$ be a perfect $\mathcal{S}_{H,b}$-packing of the whole D and consider the sequence of near-perfect packings $(\mathcal{N}_0, \ldots, \mathcal{N}_n = \mathcal{N})$ constructed as in the proof of Lemma 3.7 by subsequently improving the distance from $\mathcal{Q}_p$. We may observe that in every $\mathcal{N}_i$, all free vertices of non-odd-cycle small hedgehogs of $(D, \mathcal{W})$ have degree one, except for at most one which is uncovered by $\mathcal{N}_i$. It follows that after the transformation of $\mathcal{N}$ into a perfect $\mathcal{S}_{H,b}$-packing of D, all free vertices covered by loaded superstars have degree one. $\square$

Lemma 3.10. *Let D be a hypomatchable union of small hedgehogs with standard decomposition $\mathcal{W}$. If D is not $\mathcal{S}_{H,b}$-saturable then for no fixed vertex v of $(D, \mathcal{W})$, the graph $D \setminus v$ is $\mathcal{S}_{H,b}$-saturable.*

Proof. Suppose that for a fixed vertex $v \in V(D)$, $D \setminus v$ has a perfect $\mathcal{S}_{H,b}$-packing $\mathcal{Q}$. Let v' be a free neighbor of v from the same small hedgehog of $(D, \mathcal{W})$. Let $\mathcal{Q}'$ be a perfect $\mathcal{S}_{H,b}$-packing of $D \setminus v'$. Consider a path P of maximum length starting in v and containing alternately edges of $E(\mathcal{Q}') \setminus E(\mathcal{Q})$ and $E(\mathcal{Q}) \setminus E(\mathcal{Q}')$ leading to vertices of degree one in $\mathcal{Q}'$ and $\mathcal{Q}$, respectively.

If the number of edges on P is odd then the last vertex z of P is a leaf or a 1-degree supercenter of a superstar L in $\mathcal{Q}$. In both cases, $\mathcal{Q}$ may be augmented to a $\mathcal{S}_{H,b}$-packing covering more vertices by swapping edges and non-edges along P and replacing the newly constructed graphs by their perfect $\mathcal{S}_{H,b}$-packings.

If the number of edges on P is even then the last vertex z of P is either uncovered by $\mathcal{Q}'$ or a leaf or a 1-degree supercenter of a superstar L in $\mathcal{Q}'$. If z is covered by $\mathcal{Q}'$ then $\mathcal{Q}'$ may be augmented to a $\mathcal{S}_{H,b}$-packing covering more vertices by swapping edges and non-edges along $\{v'v\} \cup P$ and by replacing the newly constructed graphs by their perfect $\mathcal{S}_{H,b}$-packings.

If z is uncovered by $\mathcal{Q}'$ then $z = v'$. Since $\mathcal{W}$ is a standard decomposition of $\mathcal{W}$, P must contain an edge of $E(\mathcal{Q}')$ connecting two fixed vertices of $(D, \mathcal{W})$ or an edge of $E(\mathcal{Q}) \setminus E(\mathcal{Q}')$ connecting two free vertices of $(D, \mathcal{W})$. Both of these situations lead to a contradiction with the construction of $\mathcal{Q}'$: if an edge of $\mathcal{Q}'$ connects two fixed vertices then their degree must be at least 3, and each odd cycle of $(D, \mathcal{W})$ is visited by at most one edge of $\mathcal{Q}'$. $\square$

In our polynomial recognition algorithm for $\mathcal{S}_{H,b}$-saturable hypomatchable unions of small hedgehogs, we will use a reduction to the *H-factor problem*, introduced by Lovász [15]. This is the following: Let G be an undirected graph and let $H_v \subseteq \mathbb{N}$ be a degree-prescription for all $v \in V(G)$. A subgraph $F \subseteq G$ is called an *H-factor* if $\deg_F(v) \in H_v$ for all vertices $v \in V(G)$. The *H-factor problem* is to decide if there exists an H-factor of G. The instances of the H-factor problem were studied extensively. Lovász [15, 16] developed a structure theory for the H-factor problem. He proved that the H-factor problem is NP-complete when gaps of length 2 are allowed and gave a polynomial algorithm for solving the Antifactor problem, where degree prescriptions exclude only one value at each vertex [17]. Cornuéjols [1] proved the conjecture of Lovász that the H-factor problem is polynomial in case H_v has no two consecutive gaps for all $v \in V(G)$. Cornuéjols also showed reductions of some instances of the H-factor problem to the edge-and-triangle partitioning problem by replacing each vertex with a 'gadget' representing its degree prescription. Later, Loebl [11] finished his work by proving that only five types of degree prescriptions can be represented by a gadget.

In the following, we describe the reduction of the polynomial recognition of $\mathcal{S}_{H,b}$-saturable hypomatchable unions of small hedgehogs to the H-factor problem:

Let D be a hypomatchable union of small hedgehogs with standard decomposition $\mathcal{W}$. Let $v \in V(D)$. We denote by $e(v)$ and $f(v)$ the number of edges starting in v and ending in free and fixed vertices of $(D, \mathcal{W})$, respectively.

Let $\overline{D}$ be the graph constructed from D by replacing every vertex $v \in V(D)$ by the gadget depicted in Figure 3, where edges $e_1, \ldots, e_{e(v)}$ connect v to free vertices of $(D, \mathcal{W})$ and edges $f_1, \ldots, f_{f(v)}$ connect v to fixed vertices of $(D, \mathcal{W})$.

A family of degree prescriptions $\{H_{\overline{v}}\}_{\overline{v} \in V(\overline{D})}$ is said to be *induced* by a family of tuples $\{P_v = (P_v^1, P_v^2, P_v^3, P_v^4, P_v^5)\}_{v \in V(D)}$ if for every $v \in V(D)$, $H_{v_0} = P_v^1$, $H_{v_0'} = P_v^2$, $H_{v_0''} = P_v^3$, $H_{v_F} = P_v^4$ and $H_{v_i'} = P_v^5$ for $1 \leq i \leq f(v)$. A family of degree prescriptions induced by a family of tuples $\mathcal{P}$ will be denoted by $\overline{\mathcal{P}}$. Consider the following families of tuples and the induced families of degree prescriptions:

Let
$$\mathcal{P} = \{P_v = (H, \{0, 2\}, \{0\}, \{f(v)\}, \{1\})\}_{v \in V(D)} \tag{1}$$

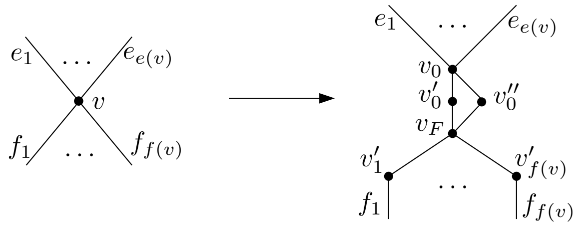


Figure 3: Gadget

For any $j \in V(D)$, let $\mathcal{P}^j = \{P_v^j\}_{v \in V(D)}$, , where

$$P_v^j = \begin{cases} P_v \text{ if } v \neq j \\ (\{1,\dots,u\},\{2\},\{0\},\{f(v)\},\{1\}) \text{ if } v = j \text{ and } b = 1 \\ (\{1,\dots,u\},\{2\},\{0,2\},\{f(v)\},\{1\}) \text{ if } v = j \text{ and } b \geq 2 \end{cases} \tag{2}$$

Lemma 3.11. *D is $\mathcal{S}_{H,b}$-saturable if and only if*

(i) $\overline{D}$ has a $\overline{\mathcal{P}}$-factor, or

(ii) $\overline{D}$ has a $\overline{\mathcal{P}^j}$-factor for some $j \in V(D)$.

Proof. If D is $\mathcal{S}_{H,b}$-saturable then let $\mathcal{Q}$ be the $\mathcal{S}_{H,b}$-packing of D from Corollary 3.9. If $\mathcal{Q}$ uses only loaded superstars then $deg_{\mathcal{Q}}(v) \in H$ for every $v \in V(D)$ and $\mathcal{Q}$ may be translated into a $\overline{\mathcal{P}}$-factor of $\overline{D}$. If $\mathcal{Q}$ uses exactly one underloaded superstar with supercenter $j \in V(D)$ then $\mathcal{Q}$ may be translated into a $\overline{\mathcal{P}^j}$-factor of $\overline{D}$.

Conversely, we may observe that every $\overline{\mathcal{P}}$-factor and every $\overline{\mathcal{P}^j}$-factor of $\overline{D}$ can be translated to a subgraph of D which is $\mathcal{S}_{H,b}$-saturable. $\square$

Lemma 3.11 has algorithmic implications. To check if a hypomatchable union of small hedgehogs D with a standard decomposition $\mathcal{W}$ is $\mathcal{S}_{H,b}$-saturable, at most $|V(D)| + 1$ factor problems have to be solved, namely those induced by $\mathcal{P}$ and every $\mathcal{P}^j$, $j \in V(D)$. Each of these factor problems consists of degree prescriptions with no two consecutive gaps an hence it is polynomially solvable. Given an arbitrary decomposition $\mathcal{W}'$, a standard decomposition may be obtained by a polynomial-time procedure (which may also end by finding a perfect $\mathcal{S}_{H,b}$-packing of D).

4 Algorithm for $\mathcal{S}_{H,b}$-packing

In this section we present the Edmonds-like algorithm for solving the $\mathcal{S}_{H,b}$-packing problem. Let G be a connected graph and let $\mathcal{Q}$ be a $\mathcal{S}_{H,b}$-packing of G. We say that (S, Y) is an *alternating structure* with respect to $\mathcal{Q}$, if S is a subgraph of G, Y is an independent set of vertices of S, called the *odd vertices* and the following properties are satisfied:

- Every connected component of $S \setminus Y$ is a hedgehog (such graphs are called the *hedgehogs of* S).
- The graph S' obtained from S by shrinking the hedgehogs of S is a forest.
- The odd vertices of S are of the following types
  - A *fixed odd vertex* lies on exactly $b+1$ edges of S leading to fixed vertices of distinct hedgehogs of S.
  - A *free odd vertex* lies on exactly $u+1$ edges of S leading to vertices of distinct hedgehogs of S at most b of which are fixed.
- Every edge $e \in E(\mathcal{Q})$ with $|e \cap V(S)| = 1$ starts in a fixed odd vertex of S and ends in a 1-degree vertex of $\mathcal{Q}$ called an *outside vertex of* S.

The graph induced by $V(S)$ and the edges of $\mathcal{Q}$ leading to outside vertices of S will be denoted by S^+.

Let T be a connected component of S. Due to the properties of hedgehogs, for any hedgehog W of T, the subgraph of G induced by $V(T^+ \setminus W)$ is $\mathcal{S}_{H,b}$-saturable.

The algorithm maintains a $\mathcal{S}_{H,b}$-packing $\mathcal{Q}$ of G, an alternating structure (S, Y) with respect to $\mathcal{Q}$, and standard decompositions of the hedgehogs of S. In each step, it grows S or finds a $\mathcal{S}_{H,b}$-packing of G covering more vertices than $\mathcal{Q}$ (which is called an *augmentation*). When stopping, it returns either a perfect $\mathcal{S}_{H,b}$-packing of G or a set $\mathcal{H}$ of components of $S \setminus Y$ which is a set of induced disjoint non-adjacent hedgehog-subgraphs of G with $|\mathcal{H}| - u|Y^{\mathcal{H}}_{free}| - b|Y^{\mathcal{H}}_{fixed}| > 0$, which shows that G is not $\mathcal{S}_{H,b}$-saturable.

Algorithm for the $\mathcal{S}_{H,b}$-packing problem

Step 0 (Initialization). Start with any $\mathcal{S}_{H,b}$-packing $\mathcal{Q}$ of G. Go to step 1.

Step 1 (Optimality test). If $\mathcal{Q}$ is perfect, stop. Otherwise, let $S = V(G) \setminus V(\mathcal{Q})$ and $Y = \varnothing$. Go to step 2.

Step 2 (Edge selection, and augmentation or growing). Look for an edge $xy \notin E(S)$ such that x is free in a hedgehog W of S and v is not a free odd vertex of S, or x is fixed in a hedgehog W of S and v is not an odd vertex of S. If no such edge exists, stop: we have a maximal $\mathcal{S}_{H,b}$-packing (this claim will be proved later). Otherwise, distinguish the following cases:

Case 1a. x is free in a hedgehog W of a component T of S, y is not in S and $\mathcal{Q}$ covers y by a leaf, a center or a (t, s)-supercenter of a superstar L with $s \geq 1$ and $s + t < u$, or $s = 0$ and $t + 1 \in H$. Augment $\mathcal{Q}$ by deleting the graphs intersecting $T^+ \cup L$ and adding perfect $\mathcal{S}_{H,b}$-packings of $T^+ \setminus W$ and $L \cup W \cup xy$. Go to step 1.

Case 1b. x is fixed in a hedgehog W of a component T of S, y is not in S and $\mathcal{Q}$ covers y by a leaf, a center, or a (t, s)-supercenter of a superstar L with $s < b$ and $s + t < u$. Augment $\mathcal{Q}$ by replacing the graphs intersecting $T^+ \cup L$ by the graphs contained in perfect $\mathcal{S}_{H,b}$-packings of $T^+ \setminus W$ and $L \cup W \cup xy$. Go to step 1.

Case 1c. x is fixed in a hedgehog W of S, y is not in S and $\mathcal{Q}$ covers y by a (t, s)-supercenter of a superstar L with $s = b$. Denote by L' the subgraph of L induced by y and the non-K_1 components of $L \setminus y$. Grow S by the following: $S = S \cup L' \cup xy$ and $Y = Y \cup \{y\}$ (y becomes a fixed odd vertex). Go to step 2.

Case 1d. y is not in S and $\mathcal{Q}$ covers y by a (t, s)-supercenter of a superstar L such that $t + s = u$, and $s < b$ or x is free in a hedgehog W of S. Grow S as follows: $S = S \cup L \cup xy$ and $Y = Y \cup \{y\}$ (y becomes a free odd vertex). Go to step 2.

Case 1e. x is free in a hedgehog W of a tree T of S, y is not in S and $\mathcal{Q}$ covers y by a center of an undergap-star L. Let $\mathcal{W}$ be the standard decomposition of W. If x is a vertex of an odd-cycle small hedgehog C of $(W, \mathcal{W})$ then augment $\mathcal{Q}$ by replacing the graphs intersecting $T \cup L$ by perfect $\mathcal{S}_{H,b}$-packings of $T \setminus W$, $W \setminus C$ and $C \cup L \cup xy$. Go to step 1.

Otherwise, grow S by adding $L \cup xy$. The standard decomposition $\mathcal{W}'$ of $W \cup L \cup xy$ is defined by $\mathcal{W}' = (\mathcal{W} \setminus \{P\}) \cup \{P \cup L \cup xy\}$ where P is the small hedgehog of $\mathcal{W}$ containing x. Go to step 2.

Case 2. x and y are in the same hedgehog W of a component T of S. Let $\mathcal{W}$ be the decomposition of W. If $W \cup xy$ is $\mathcal{S}_{H,b}$-saturable then augment

$\mathcal{Q}$ by replacing the graphs intersecting T by the graphs contained in perfect $\mathcal{S}_{H,b}$-packings of $W \cup xy$ and $T^+ \setminus W$. Go to step 1.

Otherwise, $W \cup xy$ is a hedgehog and there is a standard decomposition $\mathcal{W}'$ of $W \cup xy$ such that if v was free in $(W, \mathcal{W})$ then v is free in $(W \cup xy, \mathcal{W}')$. If a fixed odd vertex w_F of T is adjacent to a fixed vertex of W which is free in $(W \cup xy, \mathcal{W}')$ then distinguish the following cases: if w_F is adjacent to at most $u - b - 1$ outside vertices of T then $T^+ \cup xy$ is $\mathcal{S}_{H,b}$-saturable and we may augment $\mathcal{Q}$ and go to step 1. If w_F is adjacent to exactly $b - u$ outside vertices of T then grow S by the edges connecting w_F to the outside vertices (w_F becomes a free odd vertex). Go to step 2.

If no fixed odd vertex of T is adjacent to a fixed vertex of W which is free in $(W \cup xy, \mathcal{W}')$ then the types of odd vertices of T remain unchanged. We grow S by adding xy and continue by step 2.

Case 3a. x is in a hedgehog W of component T of S and y is in a hedgehog W_1 of a component $T_1 \neq T$ of S. Augment $\mathcal{Q}$ by replacing the graphs intersecting $T \cup T_1$ by the graphs contained in perfect $\mathcal{S}_{H,b}$-packings of $T \setminus W$, $T_1 \setminus W_1$ and $W \cup W_1 \cup xy$. Go to step 1.

Case 3b. x is in a hedgehog W of component T of S and y is in a hedgehog W' of the same component of S. In this case, adding the edge xy creates a cycle C in the shrunk graph T'. Let T_C be the subgraph of $T \cup xy$ consisting of the odd vertices contained in C, the hedgehogs intersected by C or adjacent to odd vertices of T_C, and the outside edges of T adjacent to odd vertices of C.

Consider the components of the graph obtained from T_C by deleting the edges of C. Every such component is a hedgehog or consists of an odd vertex z connected by edges to hedgehogs. If some z is connected to at least one fixed vertex or to exactly l free vertices of neighboring hedgehogs with $l \in H$ then the component containing z is $\mathcal{S}_{H,b}$-saturable, hence C and the whole T is $\mathcal{S}_{H,b}$-saturable and we may augment $\mathcal{Q}$ and go to step 1.

If every z is connected to free vertices of neighboring hedgehogs number of which is a gap in H, then T_C is a hypomatchable union of small hedgehogs. If T_C is $\mathcal{S}_{H,b}$-saturable then augment $\mathcal{Q}$ and go to step 1. Otherwise T_C is a hedgehog and we may obtain a standard decomposition $\mathcal{W}'$ of T_C such that the free vertices of hedgehogs of $T_C \setminus E(C)$ are free in $(T_C, \mathcal{W}')$. If a fixed odd vertex w_F of $T \setminus C$ is adjacent to a fixed vertex of a hedgehog of $T_C \setminus E(C)$ which is free in $(T_C, \mathcal{W}')$ then distinguish the following cases: if w_F is adjacent to at most $u - b - 1$ outside vertices of T then $T^+ \cup xy$ is $\mathcal{S}_{H,b}$-saturable and we may augment $\mathcal{Q}$ and go to step 1. If w_F is adjacent to exactly $b-u$ outside vertices of T then grow S by the edges connecting w_F to the outside vertices (w_F becomes a free odd vertex). Go to step 2.

If no fixed odd vertex of $T \setminus C$ is adjacent to a fixed vertex of a hedgehog of $T_C \setminus E(C)$ which is free in $(T_C, \mathcal{W}')$, then the types of odd vertices in $T \setminus C$ remain unchanged and we grow S by $S = S \cup xy$ and $Y = Y \setminus V(T_C)$ and go to step 2.

Case 4. x is a free vertex of a hedgehog W of component T of S and y is a fixed odd vertex of a component T_1 of S. Let L be the graph of $\mathcal{Q}$ covering y and let $X \subseteq E(L)$ be the set of edges connecting y to outside vertices of S. If $|X| < u-b$ then $T^+ \cup L \cup xy$ is $\mathcal{S}_{H,b}$-saturable and we may augment $\mathcal{Q}$ and go to step 1.

If $|X| = u-b$ and $T \neq T_1$ then let y' be a fixed neighbor of y in T. We put $S = (S \setminus yy') \cup xy \cup X$ (y becomes a free odd vertex of S) and go to step 2.

If $|X| = u-b$ and $T = T_1$ then by adding xy we create a cycle C in the shrunk graph T'. Let yy' be the edge of C different from xy. We put $S = (S \setminus yy') \cup xy \cup X$ (y becomes a free odd vertex of S) and go to step 2.

Proof. (Theorem 2.4) Let G be a graph such that every $\mathcal{S}_{H,b}$-packing skips at least d vertices. Let $\mathcal{Q}$ be a maximal $\mathcal{S}_{H,b}$-packing of G. The algorithm finds a family $\mathcal{H}$ of disjoint non-adjacent hedgehog-subgraphs of G with $|\mathcal{H}| - u|Y^{\mathcal{H}}_{free}| - b|Y^{\mathcal{H}}_{fixed}| = d$.

On the other hand, let G be the smallest graph in which we can find a family $\mathcal{H}$ of disjoint non-adjacent hedgehog-subgraphs of G with $|\mathcal{H}| - u|Y^{\mathcal{H}}_{free}| - b|Y^{\mathcal{H}}_{fixed}| > d$ and a maximal $\mathcal{S}_{H,b}$-packing $\mathcal{Q}$ skipping at most d vertices.

Let $Y = Y^{\mathcal{H}}_{free} \cup Y^{\mathcal{H}}_{fixed}$. Obviously, there is some $x \in Y$ contained in more than b edges of $\mathcal{Q}$ leading to fixed vertices of distinct hedgehogs of $\mathcal{H}$. Let L be the graph of $\mathcal{Q}$ covering x. Let xy be an edge of L such that y is fixed in $W \in \mathcal{H}$ and $deg_L(y) = 1$. W is not $\mathcal{S}_{H,b}$-saturable and due to Lemma 3.10, $W \setminus y$ is not $\mathcal{S}_{H,b}$-saturable.

Let $\mathcal{N}$ be a near-perfect packing of W using exactly one forbidden star X with center in y. Assume that the index i of X is a minimum (we know that $i \geq 2$). Let $\mathcal{N}'$ be the $\mathcal{S}_{H,b}$-packing of $W \setminus y$ constructed from $\mathcal{N}$ by deleting all vertices of X. $\mathcal{N}'$ is a maximal $\mathcal{S}_{H,b}$-packing of $H \setminus y$, otherwise H or $H \setminus y$ is $\mathcal{S}_{H,b}$-saturable. Hence the algorithm finds a family $\mathcal{H}'$ of disjoint non-adjacent hedgehog-subgraphs of $H \setminus y$ with $|\mathcal{H}'| - u|Y^{\mathcal{H}'}_{free}| - b|Y^{\mathcal{H}'}_{fixed}| \geq i \geq 2$. Note that the vertices that were fixed in W remain fixed in members of $\mathcal{H}'$, otherwise for some fixed vertex v of W, $W \setminus v$ is $\mathcal{S}_{H,b}$-saturable, which is a contradiction by Lemma 3.10. The vertices that were free in W have degree at most one in $\mathcal{N}'$ and it can be observed that during the run of the algorithm, none of them becomes fixed in a member of $\mathcal{H}'$.

This operation can be done for every hedgehog $W_i \in \mathcal{H}$ whose fixed vertex y_i is visited by a leaf-edge from x. Let $\mathcal{H}_i$ denote the hedgehog-set obtained by the algorithm in $W_i \setminus y_i$. We have a smaller graph - a union G' of components of $G \setminus L$ with $\mathcal{S}_{H,b}$-deficiency at most d, and a set $\mathcal{H}''$ of hedgehog-subgraphs of G' defined by

$$\mathcal{H}'' = (\mathcal{H} \setminus \{W_i \colon W_i \text{ is adjacent to } x\}) \cup \bigcup_i \mathcal{H}_i$$

with $|\mathcal{H}''| - u|Y^{\mathcal{H}''}_{free}| - b|Y^{\mathcal{H}''}_{fixed}| > d$. □

5 The $\mathcal{C}_{H,b}$-packing problem

The algorithm presented in Section 4 may be modified to solve the $\mathcal{C}_{H,b}$-packing problem. In this section, we will show the modification.

Let G be a graph and let $\mathcal{Q}$ be a $\mathcal{C}_{H,b}$-packing of G. Again (S, Y) is an alternating structure with respect to $\mathcal{Q}$ if S is a subgraph of G, Y is an independent set of vertices of S called the odd vertices and the components of $S \setminus Y$ are hedgehogs. Again, a hedgehog is defined as a non-$\mathcal{C}_{H,b}$-saturable hypomatchable union of small hedgehogs. The polynomial recognition of $\mathcal{C}_{H,b}$-saturable hypomatchable unions of small hedgehogs would be carried out similarly as in the $\mathcal{S}_{H,b}$-packing case. The only difference is that some (t, s)-superstars with $t + s \geq u$ have to be also admitted to the perfect $\mathcal{C}_{H,b}$-packings and so in the reduction,

$$P_v^j = \begin{cases} P_v \text{ if } v \neq j \\ (\{1, \ldots, u+1\}, \{2\}, \{0\}, \{f(v)\}, \{1\}) \text{ if } v = j \text{ and } b = 1 \\ (\{1, \ldots, u+1\}, \{2\}, \{0, 2\}, \{f(v)\}, \{1\}) \text{ if } v = j \text{ and } b \geq 2 \end{cases}$$

We do not go into details.

The odd vertices of the alternating structure neighbor either with $u + 1$ free vertices of distinct hedgehogs of S (the *free odd vertices*), or with $b + 1$ fixed vertices (the *fixed odd vertices*), or with b fixed vertices and $u + 1$ free vertices at least one of which belongs to an odd-cycle small hedgehog of the decomposition of the appropriate hedgehog (the *full odd vertices*).

But now, the trees of the alternating structure are not disjoint any more, even one tree may overlap itself: a vertex can be covered at most twice and if it is covered twice then it is a free odd vertex in one tree and fixed odd vertex in the other (these two trees may coincide). The vertices of hedgehogs of S are covered by only one tree of S. If a vertex is an odd vertex in only one tree, where it is free odd (fixed odd), then it may have at most l (u) outside stars (vertices), respectively. Twice covered odd vertices and full odd vertices have no outside things. In this way we can keep the property that deleting a hedgehog from a tree, the remaining of the tree can be perfectly packed. The $\mathcal{C}_{H,b}$-deficiency of the graph will be the number of the trees in the end of the algorithm.

As in Section 4, the algorithm maintains a $\mathcal{C}_{H,b}$-packing $\mathcal{Q}$ of G, an alternating structure (S, Y) with respect to $\mathcal{Q}$, and standard decompositions of the hedgehogs of S. In each step, it grows S or finds a $\mathcal{C}_{H,b}$-packing of G covering more vertices than $\mathcal{Q}$. When stopping, it returns either a perfect $\mathcal{C}_{H,b}$-packing of G or a set $\mathcal{H}$ of induced disjoint non-adjacent hedgehog-subgraphs of G with $|\mathcal{H}| - u|Z^{\mathcal{H}}_{free}| - b|Z^{\mathcal{H}}_{fixed}| > 0$, where $Z^{\mathcal{H}}_{free}$ is the set of neighbors of graphs from $\mathcal{H}$ which have at least one free neighbor and $Z^{\mathcal{H}}_{fixed}$ is the set of neighbors of graphs from $\mathcal{H}$ which have at least one fixed neighbor or a neighbor that is free in an odd-cycle small hedgehog of a standard decomposition of a graph from $\mathcal{H}$.

Algorithm for the $\mathcal{C}_{H,b}$-packing problem

Step 0 (Initialization). Start with any $\mathcal{C}_{H,b}$-packing $\mathcal{Q}$ of G. Go to step 1.

Step 1 (Optimality test). If $\mathcal{Q}$ is perfect, stop. Otherwise, let $S = V(G) \setminus V(\mathcal{Q})$ and $Y = \varnothing$. Go to step 2.

Step 2 (Edge selection, and augmentation or growing). Look for an edge $xy \notin E(S)$ such that x is free in a hedgehog W of S and v is not an free odd vertex of S, or x is fixed in a hedgehog W of S and v is not a fixed odd vertex of S. If no such edge exists, stop: we have a maximal $\mathcal{C}_{H,b}$-packing (this claim will be proved later). Otherwise, distinguish the following cases:

Case 1a. x is free in a hedgehog W of a tree T of S, y is not in S and $\mathcal{Q}$ covers y by a leaf, a center or a (t, s)-supercenter of a superstar L with $s \geq 1$ and $t < u$, or $s = 0$ and $t + 1 \in H$. Augment $\mathcal{Q}$ by deleting the graphs intersecting $T^+ \cup L$ and adding perfect $\mathcal{S}_{H,b}$-packings of $T^+ \setminus W$ and $L \cup W \cup xy$. Go to step 1.

Case 1b. x is fixed in a hedgehog W of a tree T of S, y is not in S

and $\mathcal{Q}$ covers y by a leaf, a center, or a (t,s)-supercenter of a superstar L with $s < b$. Augment $\mathcal{Q}$ by replacing the graphs intersecting $T^+ \cup L$ by the graphs contained in perfect $\mathcal{S}_{H,b}$-packings of $T^+ \setminus W$ and $L \cup W \cup xy$. Go to step 1.

Case 1c. x is fixed in a hedgehog W of S, y is not in S and $\mathcal{Q}$ covers y by a (t,s)-supercenter of a superstar L with $s = b$. Denote by L' the subgraph of L induced by y and the non-K_1 components of $L \setminus y$. Grow S by the following: $S = S \cup L' \cup xy$ and $Y = Y \cup \{y\}$ (y becomes a fixed odd vertex). Go to step 2.

Case 1d. x is free in a hedgehog W of S, y is not in S and $\mathcal{Q}$ covers y by a (t,s)-supercenter of a superstar L such that $t = u$. If x is a member of an odd-cycle small hedgehog of a standard decomposition of W and $s < b$ then augment $\mathcal{Q}$ and go to step 1. If x is a member of an odd-cycle small hedgehog of a stadard decomposition of W and $s = b$ then grow S by $S = S \cup L \cup xy$ and $Y = Y \cup \{y\}$ (y becomes a full odd vertex) and go to step 2.

If x is not a member of an odd-cycle small hedgehog of a stadard decomposition of W then denote by L' the subgraph of L induced by y and the K_1 components of $L \setminus y$ and grow S as follows: $S = S \cup L' \cup xy$ and $Y = Y \cup \{y\}$ (y becomes a free odd vertex). Go to step 2.

Case 1e. x is free in a hedgehog W of a tree T of S, y is not in S and $\mathcal{Q}$ covers y by a center of an undergap-star L. Let $\mathcal{W}$ be the standard decomposition of W. If x is a vertex of an odd-cycle small hedgehog C of $(W, \mathcal{W})$ then augment $\mathcal{Q}$ by replacing the graphs intersecting $T \cup L$ by perfect $\mathcal{S}_{H,b}$-packings of $T \setminus W$, $W \setminus C$ and $C \cup L \cup xy$. Go to step 1.

Otherwise, grow S by adding $L \cup xy$. The standard decomposition $\mathcal{W}'$ of $W \cup L \cup xy$ is defined by $\mathcal{W}' = (\mathcal{W} \setminus \{P\}) \cup \{P \cup L \cup xy\}$ where P is the small hedgehog of $\mathcal{W}$ containing x. Go to step 2.

Case 2. x and y are in the same hedgehog W of a tree T of S. Let $\mathcal{W}$ be the decomposition of W. If $W \cup xy$ is $\mathcal{S}_{H,b}$-saturable then augment $\mathcal{Q}$ by replacing the graphs intersecting T by the graphs contained in perfect $\mathcal{S}_{H,b}$-packings of $W \cup xy$ and $T^+ \setminus W$. Go to step 1.

Otherwise, $W \cup xy$ is a hedgehog and there is a standard decomposition $\mathcal{W}'$ of $W \cup xy$ such that if v was free in $(W, \mathcal{W})$ then v is free in $(W \cup xy, \mathcal{W}')$. If a fixed odd vertex w_F of T is adjacent to a fixed vertex of W which is free in $(W \cup xy, \mathcal{W}')$ (and hence a member of an odd-cycle small hedgehog of $\mathcal{W}'$) then distinguish the following cases: if w_F is adjacent to at most $u - 1$ outside vertices of T then $T^+ \cup xy$ is $\mathcal{S}_{H,b}$-saturable and we may augment

$\mathcal{Q}$ and go to step 1. If w_F is adjacent to exactly u outside vertices of T then grow S by the edges connecting w_F to the outside vertices (w_F becomes a full odd vertex). Go to step 2.

If a free odd vertex w_f of T is adjacent to a free vertex of W which is a member of an odd-cycle small hedgehog of $(W \cup xy, \mathcal{W}')$ then distinguish the following cases: if w_f is adjacent to at most $b-1$ outside stars of T then $T^+ \cup xy$ is $\mathcal{S}_{H,b}$-saturable and we may augment $\mathcal{Q}$ and go to step 1. If w_f is adjacent to exactly u outside stars of T then grow S by the outside stars and the edges connecting w_f to the outside stars (w_f becomes a full odd vertex). Go to step 2.

If none of the above occurs then the types of odd vertices of T remain unchanged. We grow S by adding xy and continue by step 2.

Case 3a. x is in a hedgehog W of tree T of S and y is in a hedgehog W_1 of a tree $T_1 \neq T$ of S. Augment $\mathcal{Q}$ by replacing the graphs intersecting $T \cup T_1$ by the graphs contained in perfect $\mathcal{S}_{H,b}$-packings of $T \setminus W$, $T_1 \setminus W_1$ and $W \cup W_1 \cup xy$. Go to step 1.

Case 3b. x is in a hedgehog W of tree T of S and y is in a hedgehog W' of the same tree of S. In this case, adding the edge xy creates a cycle C in the shrunk graph T'. Let T_C be the subgraph of $T \cup xy$ consisting of the odd vertices contained in C, of the hedgehogs intersected by C or adjacent to odd vertices of T_C and of the outside vertices and stars of T adjacent to odd vertices of C.

Consider the components of the graph obtained from T_C by deleting the edges of C. Every such component is a hedgehog or consists of an odd vertex z connected by edges to hedgehogs. If some z is connected to at least one fixed or odd-cycle free vertex, or to exactly l free vertices of neighboring hedgehogs with $l \in H$ then the component containing z is $\mathcal{S}_{H,b}$-saburable, hence C and the whole T is $\mathcal{S}_{H,b}$-saturable and we may augment $\mathcal{Q}$ and go to step 1.

If every such z is connected only to non-odd cycle free vertices of neighboring hedgehogs, number of which is a gap in H, then T_C is a hypomatchable union of small hedgehogs. If T_C is $\mathcal{S}_{H,b}$-saturable then augment $\mathcal{Q}$ and go to step 1. Otherwise T_C is a hedgehog and we may obtain a standard decomposition $\mathcal{W}'$ of T_C such that the free vertices of hedgehogs of $T_C \setminus E(C)$ are free in $(T_C, \mathcal{W}')$. If a fixed odd vertex w_F of $T \setminus C$ is adjacent to a fixed vertex of a hedgehog of $T_C \setminus E(C)$ which is free in $(T_C, \mathcal{W}')$ (and hence a member of an odd-cycle small hedgehog of $\mathcal{W}'$) then distinguish the following cases: if w_F is adjacent to at most $u-1$ outside vertices of T

then $T^+ \cup xy$ is $\mathcal{S}_{H,b}$-saturable and we may augment $\mathcal{Q}$ and go to step 1. If w_F is adjacent to exactly u outside vertices of T then grow S by the edges connecting w_F to the outside vertices (w_F becomes a full odd vertex). Go to step 2.

If a free odd vertex w_f of $T \setminus C$ is adjacent to a free vertex of a hedgehog of $T_C \setminus E(C)$ which is a member of an odd-cycle small hedgehog of $(T_C, \mathcal{W}')$ then distinguish the following cases: if w_f is adjacent to at most $b-1$ outside stars of T then $T^+ \cup xy$ is $\mathcal{S}_{H,b}$-saturable and we may augment $\mathcal{Q}$ and go to step 1. If w_f is adjacent to exactly u outside stars of T then grow S by the outside stars and the edges connecting w_f to the outside stars (w_f becomes a full odd vertex). Go to step 2.

If none of the above occurs then the types of odd vertices in $T \setminus C$ remain unchanged and we grow S by $S = S \cup xy$ and $Y = Y \setminus V(T_C)$ and go to step 2.

Case 4a. x is a free vertex of a hedgehog W of a tree T of S and y is a fixed odd vertex of a tree T_1 of S (T and T_1 may be the same). Let X be the set of outside edges of S containing y. If $|X| < u$ then $T^+ \cup T_1^+ \cup xy$ is $\mathcal{S}_{H,b}$-saturable and we may augment $\mathcal{Q}$ and go to step 1.

If $|X| = u$ then grow S by adding xy and the edges of X (y becomes a twice-covered odd vertex of S). Go to step 2.

Case 4b. x is a fixed vertex of a hedgehog W of a tree T of S and y is a free odd vertex of a tree T_1 of S (T and T_1 may be the same). Let L be the graph of $\mathcal{Q}$ covering y and let X be the set of outside stars of S connected by an edge of $\mathcal{Q}$ to y. If $|X| < b$ then $T^+ \cup W \cup xy$ is $\mathcal{S}_{H,b}$-saturable and we may augment $\mathcal{Q}$ and go to step 1.

If $|X| = b$ then grow S by adding xy, all graphs of X and the edges connecting y to X (y becomes a twice-covered odd vertex of S). Go to step 2.

We conclude this section by a Berge-type theorem for $\mathcal{C}_{H,b}$-packing.

Theorem 5.1. *The $\mathcal{C}_{H,b}$-deficiency of a graph G is*

$$\max |\mathcal{H}| - u|Z^{\mathcal{H}}_{free}| - b|Z^{\mathcal{H}}_{fixed}|,$$

where max *is taken over all sets $\mathcal{H}$ of induced disjoint non-adjacent hedgehog-subgraphs of G, $Z^{\mathcal{H}}_{free}$ is the set of neighbors of graphs from $\mathcal{H}$ which have at least one free neighbor and $Z^{\mathcal{H}}_{fixed}$ is the set of neighbors of graphs from $\mathcal{H}$ which have at least one fixed neighbor or a neighbor that is free in an odd-cycle of a decomposition of a graph from $\mathcal{H}$.*

Proof. Let G be a graph such that every $\mathcal{C}_{H,b}$-packing skips at least d vertices. Let $\mathcal{Q}$ be a maximal $\mathcal{C}_{H,b}$-packing of G. The algorithm finds a family $\mathcal{H}$ of disjoint non-adjacent hedgehog-subgraphs of G with $|\mathcal{H}| - u|Z^{\mathcal{H}}_{free}| - b|Z^{\mathcal{H}}_{fixed}| = d$.

On the other hand, let G be the smallest graph in which we can find a family $\mathcal{H}$ of disjoint non-adjacent hedgehog-subgraphs of G with $|\mathcal{H}| - u|Z^{\mathcal{H}}_{free}| - b|Z^{\mathcal{H}}_{fixed}| > d$ and a maximal $\mathcal{C}_{H,b}$-packing $\mathcal{Q}$ skipping at most d vertices.

Obviously, there exists $x \in V(G)$ such that $x \in Z^{\mathcal{H}}_{fixed} \setminus Z^{\mathcal{H}}_{free}$ an x is contained in more than b edges of $\mathcal{Q}$ leading to fixed vertices of distinct hedgehogs of $\mathcal{H}$, or $x \in Z^{\mathcal{H}}_{free} \setminus Z^{\mathcal{H}}_{fixed}$ and x is contained in more than u edges of $\mathcal{Q}$ leading to free vertices not contained in odd-cycle small hedgehogs of the standard decompositions of the appropriate hedgehogs.

If $x \in Z^{\mathcal{H}}_{fixed} \setminus Z^{\mathcal{H}}_{free}$ then similarly as in the proof of Theorem 2.4 at the end of Section 4, we may find a smaller graph contradicting the selection of G.

If $x \in Z^{\mathcal{H}}_{free} \setminus Z^{\mathcal{H}}_{fixed}$ then let L be the graph of $\mathcal{Q}$ covering x. Let xy be an edge of L such that y is free in a non-odd cycle small hedgehog of a standard decomposition of $W \in \mathcal{H}$ and $deg_{\mathcal{Q}}(y) > 1$. W is not $\mathcal{C}_{H,b}$-saturable and due to Lemma 3.10, $W \setminus L$ is not $\mathcal{C}_{H,b}$-saturable.

Let $\mathcal{N}$ be a near-perfect packing of W skipping y. Note that $\mathcal{N}$ covers all neighbors of y by supercenters of superstars. Assume that the sum of degrees of neighbors of y is a minimum (we know that it is at least 2). Let $\mathcal{N}'$ be the $\mathcal{C}_{H,b}$-packing of $W \setminus y$ constructed from $\mathcal{N}$ by deleting the graphs covering the neighbors of y. $\mathcal{N}'$ is a maximal $\mathcal{C}_{H,b}$-packing of $H \setminus L$, otherwise H or $H \setminus L$ is $\mathcal{C}_{H,b}$-saturable. Hence the algorithm finds a family $\mathcal{H}'$ of disjoint non-adjacent hedgehog-subgraphs of $(H \setminus L)$ with $|\mathcal{H}'| - u|Z^{\mathcal{H}'}_{free}| - b|Z^{\mathcal{H}'}_{fixed}| \geq 2$. Note that the vertices that were fixed in W remain fixed in members of $\mathcal{H}'$, otherwise for some fixed vertex v of W, $W \setminus v$ is $\mathcal{C}_{H,b}$-saturable, which is a contradiction by Lemma 3.10. The vertices that were free in W have degree at most one in $\mathcal{N}'$ and it can be observed that during the run of the algorithm, none of them becomes fixed in a member of $\mathcal{H}'$.

This operation can be done for every hedgehog $W_i \in \mathcal{H}$ whose non-odd-cycle free vertex y_i is visited by an edge from x and has degree more than one. Let $\mathcal{H}_i$ denote the hedgehog-set obtained by the algorithm in $W_i \setminus L$. We have a smaller graph - a union G' of components of $G \setminus L$ with $\mathcal{S}_{H,b}$-deficiency at most d, and a set $\mathcal{H}''$ of hedgehog-subgraphs of G' defined by

$$\mathcal{H}'' = (\mathcal{H} \setminus \{W_i \colon W_i \text{ is adjacent to } x\}) \cup \bigcup_i \mathcal{H}_i$$

with $|\mathcal{H}''| - u|Z^{\mathcal{H}''}_{free}| - b|Z^{\mathcal{H}''}_{fixed}| > d$. $\square$

6 Conclusion

We have introduced two types of graph sets that can be added to a family $\mathcal{S}_H = \{S_i,\, i \in H\}$ of stars with no two consecutive gaps in H such that the packing problem by the resulting family is polynomially solvable. For both types of the families $\mathcal{S}_{H,b}$ and $\mathcal{C}_{H,b}$, an Edmonds-type algorithm and a Berge-type theorem were presented. The classical Edmonds matching forest covers the vertices of the graph at most once. In the Edmonds-type algorithm for $\mathcal{C}_{H,b}$-packing, we had to allow 2-fold overlapping trees.

Generalizing our approach, another interesting question is which superstars from $\mathcal{S}_{H,b}$ and $\mathcal{C}_{H,b}$ (or propellers from the Loebl-Poljak's propeller packing families) can be replaced by other graphs to maintain polynomiality.

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