Four Gravity Results[1]

Z. Dvořák[2], R. Škrekovski[3], T. Valla[2]

[2] Charles University, Faculty of Mathematics and Physics,
DIMATIA and Institute for Theoretical Computer Science (ITI)
Malostranské nám. 2/25, 118 00, Prague, Czech Republic
{rakdver, valla}@kam.mff.cuni.cz

[3] Department of Mathematics, University of Ljubljana,
Jadranska 19, 1111 Ljubljana, Slovenia
skreko@fmf.uni-lj.si

Abstract

The gravity of a graph H in a given family of graphs $\mathcal{H}$ is the greatest integer n with the property that for every integer m, there exists a supergraph $G \in \mathcal{H}$ of H such that each subgraph of G, which is isomorphic to H, contains at least n vertices of degree $\geq m$ in G. Madaras and Škrekovski introduced this concept and showed that the gravity of the path P_k on $k \geq 2$ vertices is $k-2$ for each $k \neq 5, 7, 8, 9$. They conjectured that for each of the four excluded cases the gravity is $k-3$. In this paper we show that this holds.

1 Introduction

Throughout the paper, we consider connected graphs without loops or multiple edges. Let $\mathcal{H}$ be a family of graphs, and let H be a connected graph such that infinitely many members of $\mathcal{H}$ contain a subgraph isomorphic to H. Let $\varphi(H, \mathcal{H})$ be the smallest integer with the property that each graph

[1]Supported in part by bilateral project SLO-CZ/04-05-002 between Slovenia and Czech Republic

[2]Supported in part by bilateral project MSMT-07-0405 between Slovenia and Czech Republic

[3]Supported in part by Research Project Z1-3129 of the Ministry of Science and Technology of Slovenia

$G \in \mathcal{H}$ which contains a subgraph isomorphic to H, contains also a subgraph $K \cong H$ such that, for every vertex $v \in K$,

$$\deg_G(v) \le \varphi(H, \mathcal{H}).$$

If such a finite $\varphi(H, \mathcal{H})$ does not exist, we write $\varphi(H, \mathcal{H}) = +\infty$. We say that the graph H is *light* in the family $\mathcal{H}$ if $\varphi(H, \mathcal{H}) < +\infty$, otherwise we call it *heavy*. Thus, H is heavy in $\mathcal{H}$ if, for every integer m, there is a graph $G \in \mathcal{H}$ such that each isomorphic copy of H in G contains a vertex of degree $\ge m$ in G.

It is well known that every plane graph contains a vertex of degree at most 5. Kotzig [9] stated that each 3-connected plane graph contains an edge of weight at most 13 and at most 11 in the case of absence of 3-vertices, and these bounds are sharp. This result was generalized in many directions; namely, it served as starting point for looking for subgraphs of small weight in plane graphs.

Borodin [2] extended Kotzing's theorem by showing that every simple planar graph with minimum degree ≥ 3 has also an edge of weight ≤ 13. This extension of Kotzing's theorem will be applied in few proofs in this paper. Fabrici and Jendrol' [3] proved that the only light graphs in the family of all 3-connected plane graphs are paths; this holds also for the family of all 3-connected plane graphs of minimum degree 4 (see [4]) and of minimum face size 4 (see [6]). In the family of plane graphs of minimum degree 5, there are light graphs other than paths [1, 5, 7, 13]. The lightness of paths and some other graphs in various families of planar graphs was studied also in [10, 12]. The survey of results on light graphs in various families of plane, projective plane, and general graphs can be found in the paper Jendrol' and Voss [8].

The *gravity* of a connected graph H in the family $\mathcal{H}$ of planar graphs is the greatest integer n with the property that for every integer m there exists a supergraph $G \in \mathcal{H}$ such that each subgraph of G, which is isomorphic to H, contains at least n vertices of degree $\ge m$ in G. Hence, a graph is light in a family of graphs if and only if its gravity is zero.

The concept of gravity was introduced by Madaras and Škrekovski [11]. They determined the gravity of stars in the class $\mathcal{P}_d$ of planar graphs of minimum degree $\ge d$ for $d \in \{1, \dots, 5\}$. In $\mathcal{P}_1$ the gravity of the path P_k is $k-1$ for each $k \ne 3, 5$ and it is $k-2$ for $k = 3, 5$. In $\mathcal{P}_2$ the gravity of the path P_k is at most $k-2$ for each $k \ge 2$ and the gravity reaches the bound of $k-2$ for each $k \ne 5, 7, 8, 9$. Škrekovski and Madaras conjectured that for each of the four excluded cases the gravity is $k-3$. In this paper we show that this holds. Our arguments are based on the fact that every planar graph of minimum degree three has an edge of weight ≤ 13 [2] and avoid direct use of the discharging metod.

For a fixed integer b, a vertex is called *b-big* if its degree is at least b, and *b-small* if the degree is less than b. If b is known from the context, we drop the b-prefix and call the vertex *big* or *small*, respectively. If x and y is a pair of adjacent b-small vertices in a graph, we call the edge xy *b-light*, or just *light* if b is known from context.

By P_k we denote the path on k vertices, we call it also a *k-path* in order to emphasize that it is a copy of P_k in some graph. Similarly, we define a *k-cycle* as a cycle of k vertices in some graph and a *k-vertex* as a vertex of degree k.

2 Long and Heavy Paths in $\mathcal{P}_2$

In this section we study the structure of the graphs from $\mathcal{P}_2$ which contain P_k as a subgraph and each such k-path contains at most two b-small vertices. Denote by $\mathcal{P}_2(b,k)$ this subclass of graphs of $\mathcal{P}_2$. Notice that if $a \geq b$, then $\mathcal{P}_2(a,k) \subseteq \mathcal{P}_2(b,k)$. Thus, if $b_1, b_2, b_3, \ldots$ is an increasing infinite sequence of integers and $\mathcal{P}_2(b_i,k) \neq \emptyset$ for each $i \geq 1$, then $\mathcal{P}_2(b,k) \neq \emptyset$ for each integer $b \geq 1$.

Lemma 1 *Let k and b be two integers such that $b \geq k \geq 3$. Suppose that G is a graph from $\mathcal{P}_2(b,k)$. Then G contains a $(k+1)$-path. Furthermore, if G contains a k-path P with two b-small vertices s_1 and s_2 such that the distance between s_1 and s_2 in P is at most $k-2$, then G also contains a $(k+1)$-path P' such that s_1 and s_2 belong to P', their distance in P' is the same as in P and one of vertices s_1 and s_2 is an end vertex of P'.*

Proof. Let $P = x_1x_2\cdots x_k$ be a k-path in G. If x_1 or x_k is big, then it has a neighbour outside of P, so we can extend P to a path on $k+1$ vertices. The same argument holds if x_1, x_k are small and some of them has a neighbour outside of the path. So we may assume that both x_1 and x_k are small and all their neighbours are on the path. Let x_i be one of the neighbours of x_1 distinct from x_2. Such a vertex exist since $G \in \mathcal{P}_2$. Then x_{i-1} must be big, otherwise P would contain three small vertices. Therefore it has a neighbour w outside of P. Then $x_kx_{k-1}\cdots x_ix_1x_2\cdots x_{i-1}w$ is a $(k+1)$-path in G.

Now consider the case that P contains two small vertices x_i and x_j in distance at most $k-2$. We may assume x_k is big and $i<j$. We first construct a k-path $P' = x'_1 x'_2 \cdots x'_k$ such that $x'_1 = x_i$ and $x'_{j-i+1} = x_j$. If $i=1$, we set $P' = P$. Otherwise let v be a neighbour of the big vertex x_k such that v does not belong to P. We consider a k-path $P'' = x_2 \cdots x_k v$ instead of P. The distance of x_i and x_j in P'' is unchanged and x_i is closer to start of P, so after a finite number of repetitions of this construction we construct the path P' as desired.

Vertex x'_k is big and therefore it has a neighbour v outside of P. The k-path P' can then be extended to a $(k+1)$-path $x'_1 x'_2 \cdots x'_k v$ that satisfies the conditions of the lemma. □

For an integer b, let $l_b(G)$ be the number of b-light edges in G. We define the following ordering: $G_1 \prec_b G_2$ if and only if either $l_b(G_1) < l_b(G_2)$, or $l_b(G_1) = l_b(G_2)$ and $|E(G_1)| < |E(G_2)|$. We denote by $\mathcal{P}_2^*(b,k)$ the set of all minimal graphs of $\mathcal{P}_2(b,k)$ in this ordering. Obviously, $\mathcal{P}_2^*(b,k)$ is non-empty if and only if $\mathcal{P}_2(b,k)$ is non-empty. The following lemma shows that some configurations do not appear in the graphs of $\mathcal{P}_2^*(b,k)$.

Lemma 2 *Suppose that $G \in \mathcal{P}_2^*(b,k)$, with $b \geq k > 4$. Then G does not contain any of the following configurations as a subgraph:*

(C1) Two adjacent small vertices u and v such that the degree of each of them is at least 3.

(C2) A vertex v of degree 2 adjacent to two nonadjacent vertices x and y.

(C3) Two adjacent vertices x and y of degree 2 with a common small neighbour v of degree at least 4.

(C4) Two adjacent vertices x and y of degree 2 with a common small neighbour v of degree 3.

(C5) Vertices v_1 and v_2 of degree 2 with common adjacent neighbours x and y such that x is big and y is small.

Proof. Suppose that G contains some of these configurations. We construct a graph G' such that $G' \in \mathcal{P}_2(b,k)$ and $G' \prec_b G$. Thus G' will contradict the minimality of G. Fig. 2 illustrates the construction of G' from G for each of these configurations (dotted lines present that vertices are non-adjacent and labels B, S that corresponding vertices are big or small, respectively). Consider each of the above configurations separately:

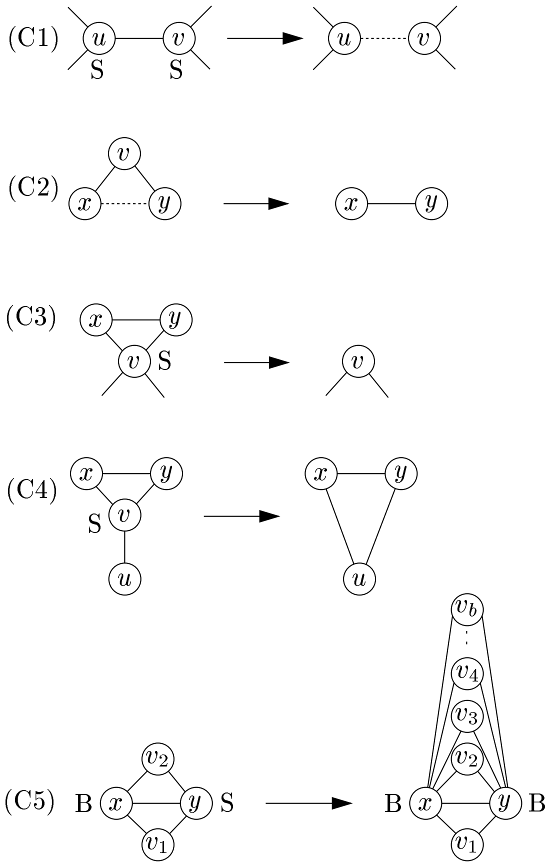

Figure 1: Forbidden configurations and their reductions.

(C1) In the first configuration we remove the edge $e = uv$. This produces the planar graph G' with minimum degree at least 2 and with smaller number of light edges than G. Thus $G' \prec_b G$.

We only have to show that $G' \in \mathcal{P}_2(b, k)$. A vertex is big in G' if and only if it is big in G, and since G' is a subgraph of G, the graph G' does not contain any k-path with more than two small vertices. The graph G must have at least one k-path, since if P is a k-path in G using the edge e, then according to Lemma 1, G also contains a $(k+1)$-path Q such that u and v are the first two vertices of Q. Say u is the first one. Then $Q - u$ is a k-path in G'. Therefore, G' belongs to $\mathcal{P}_2(b, k)$.

(C2) In this configuration, we remove v and add an edge xy in order to obtain G'. This could create a new light edge xy, but then both x and y are small and so we remove two light edges vx and vy. Therefore we do not increase the number of light edges and we always decrease by one the number of edges. Thus, $G' \prec_b G$.

We have to show that $G' \in \mathcal{P}_2(b, k)$. Obviously, G' has minimum degree at least 2. We claim that the graph G' contains a k-path, since according to Lemma 1, a k-path Q is a subgraph of G and the corresponding path in G' is of length k or $k+1$ (depending on whether v belongs to Q or not). We also could not create a new k-path containing more than 2 small vertices. Suppose P is such a path. Observe that P contains the edge xy. In G there is a corresponding path P' where the edge xy is replaced by path xvy. Thus, P' is a $(k+1)$-path with at least 4 small vertices, and so P' contains a subpath of length k with at least 3 small vertices.

(C3) In the third configuration we remove the vertices x and y to obtain G'. This removes three light edges, and so $G' \prec_b G$.

If P is a k-path in G that does not occur in G', then P contains v and one of x and y. We may assume that $P = xvu_1 \cdots u_{k-2}$. But then $P' = yxvu_1 \cdots u_{k-3}$ would be a k-path containing three small vertices in G, which is a contradiction. Therefore all k-paths in G also exist in G' and it is now easy to see that graph G' belongs to $\mathcal{P}_2(b, k)$.

(C4) Let u be the third neighbour of v. We remove v from G and add edges xu and yu in order to obtain G'. This operation reduces the number of light edges by at least one. Thus $G' \prec_b G$ and $G' \in \mathcal{P}_2$.

Similarly to the configuration (C3), we observe that no k-path of G uses a vertex from $\{v, x, y\}$, since otherwise we would find a k-path in G using three small vertices v, x and y. Therefore we preserve all k-paths of G by the described operation and so G' contains a k-path.

Note that we have increased the degree of u by one in this operation, so u is big in G'. Let $P = v_1 v_2 \cdots v_k$ be a k-path of G'. If P occurs in G, it obviously has at most two small vertices. Otherwise we may assume $v_1 = x$ and $v_2 \in \{y, u\}$. Then either $vuv_3 \cdots v_k$ or $xvuv_4 \cdots v_k$ is a k-path in G containing the same number of small vertices. Therefore $G' \in \mathcal{P}_2(b, k)$.

(C5) If $k = 5$, consider the path $v_1 y v_2 x$. It contains 3 small vertices and it can be extended to a 5-path, since x is big. This is a contradiction.

If $k \geq 6$, we add $b - 2$ new vertices $v_3, \ldots, v_b$ and connect them to both x and y to obtain G'. Let $S = \{v_1, v_2, \ldots, v_b\}$. Since y is big in G', the edges $v_1 y$ and $v_2 y$ are not light in G'. Thus the number of light edges is reduced by at least 2, and hence $G' \prec_b G$.

In the construction of G', we have not removed any path in G, so it is sufficient to argue that we could not create a k-path with at least three small vertices. Let P be a k-path of G'. Then P cannot use three of the vertices of S, since it would have length at most 5. If P uses at most two vertices of S, we may assume these vertices belong to $\{v_1, v_2\}$ and then P also exists in G. Since we have not decreased degree of any vertex, P contains at most two small vertices in G'. Thus, $G' \in \mathcal{P}_2(b, k)$.

□

If xyz is a 3-cycle in a graph, x and y have degree 2 and z is big, then we call the subgraph induced by these vertices a 3-*booster*. Similarly, if $xywz$ is a 4-cycle in a graph, x and y have degree 2, w and z are big, we call the subgraph induced by these four vertices a 4-*booster*.

Lemma 3 *Let $k > 4$ be an integer. Suppose that for each integer $b \geq k$ the class $\mathcal{P}_2^*(b, k)$ is nonempty. Then for each integer $b \geq k$ there exists a planar graph $G_1(b, k) \in \mathcal{P}_2(b, k)$ satisfying the property:*

(P1) If x and y are adjacent b-small vertices in $G_1(b, k)$, then x and y are of degree 2 and have a common b-big neighbour (i.e. x and y are a part of a 3-booster).

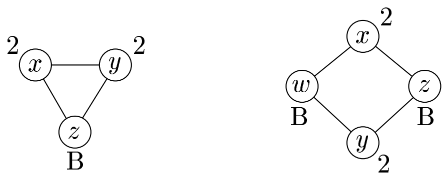

Figure 2: Illustration of a 3-booster and a 4-booster.

Proof. Let us fix one such b and let $G' \in \mathcal{P}_2^*(2b, k)$. Due to Lemma 2:

- Graph G' does not contain two adjacent $2b$-small vertices of degree at least three.
- If G' contains two adjacent vertices x and y of degree 2, then they have a common neighbour that is $2b$-big (otherwise, G' contains one of the configurations (C2), (C3) or (C4)).
- If x and y are adjacent, x has degree 2, y is a $2b$-small vertex of degree at least 3, then they have a common neighbour z that is $2b$-big. Moreover, there is no 2-vertex distinct from x, which is a commmon neighbour of both y and z.

We construct a graph $G = G_1(b, k)$ from G' in the following way: For each pair of vertices x and y such that x and y are adjacent in G', x is $2b$-big, $2 < \deg(y)$ and there exists a 2-vertex v adjacent to x and y but not belonging to a 4-booster, we remove the vertex v.

A vertex with degree d in G' has a degree at least $\lceil \frac{d}{2} \rceil$ in G. Otherwise if q is such a vertex and v a 2-vertex adjacent to q that is removed, the other neighbour r of v is adjacent to q and r and q have no other common neighbour of degree 2 due to the restrictions on the structure of G' described above. Consequently a vertex that has degree at least $2b$ in G' has a degree at least b in G.

In order to conclude the proof, let us check that graph G satisfies conditions of the lemma:

- *Graph G has a minimum degree at least* 2. The vertices incident to the removed edges had degree at least 3 in G', so their degrees are at least $\lceil \frac{3}{2} \rceil = 2$ in G.

- *Each k-path in G contains at most two b-small vertices.* Suppose P is a k-path in G with more than two b-small vertices. In G', the path P contains at most two $2b$-small vertices, therefore there must be a vertex q on P that has degree smaller than b in G, and degree at least $2b$ in G', which is a contradiction.

- *Graph G contains a k-path.* Let $S = V(G') \setminus V(G)$ be the set of removed vertices. Let us arbitrarily order the elements of S into a sequence $s_1, s_2, \ldots, s_m$. Thus we obtain a sequence of graphs $G' = G_0, G_1, \ldots, G_m = G$ where $G_{i+1} = G_i - s_{i+1}$. We prove by induction on i that each G_i belongs to $\mathcal{P}_2(b,k)$. By the above case we know that G_i does not contain k-path with more than two b-small vertices, therefore it suffices to prove that there is a k-path in G.

  We proceed by induction. For G_0 the statement is true. Let us assume that G_i contains a k-path. We will show that G_{i+1} contains a k-path as well. Note that the assumptions of Lemma 1 are preserved (for *big* vertices being those that have degree at least b). Therefore there also exists a $(k+1)$-path P in G_i. If s_{i+1} is an endvertex of P, the path $P - s_{i+1}$ is a k-path of G_{i+1}. If s_{i+1} is some of inner vertices of P, we know that the two adjacent neighbours x and y of s_{i+1} belong to P and therefore $P - s_{i+1}$ together with the edge xy is a k-path in G_{i+1}.

- *Graph G satisfies (P1).* Suppose x and y are adjacent b-small vertices in G. No vertex with degree at least $2b$ in G' has degree less than b in G. This means that both x and y are also $2b$-small in G'. Due to properties of G' one of x and y, say x, has degree 2 in G'. If degree of y in G' is as least three, then x and y have a common $2b$-big neighbour in G' and x should have been removed during construction of G. Otherwise degree of y in G' is 2 as well and x and y have a common neighbour z that is $2b$-big in G', and consequently z is b-big in G. Therefore vertices x, y and z induce a 3-booster in G.

□

Lemma 4 *Let $k > 4$ be an integer. Suppose that for each $b \geq k$ the class $\mathcal{P}_2^*(b,k)$ is nonempty. Then for each integer $b \geq k$ there exists a planar graph $G_2(b,k) \in \mathcal{P}_2(b,k)$ satisfying the property (P1) of Lemma 3 and the property*

(P2) If x is a vertex of degree two in G, then it is part of either a 3-booster or a 4-booster.

Proof. Let us fix one such b. Let $G' = G_1(2b, k)$ be a graph obtained by Lemma 3.

We construct $G = G_2(b, k)$ in the following way: First we create a graph G'' by supressing one by one 2-vertices of G' whose neighbours are not adjacent. Then we construct a graph G by removing each 2-vertex of G'' which does not belong to a 4-booster and whose neighbours are $2b$-big.

Using the same argumentation as in the proof of the second case of Lemma 2, one can show that G'' belongs to $P_2(b, k)$ and that it has the property (P1). Additionally, any vertex of degree two in G'' is either part of a 3-booster or a 4-booster, or its neighbours are two adjacent big vertices.

Similarly as in the proof of Lemma 3, one can show that $G \in P_2(b, k)$ and that it has the property (P1). Let us show that G satisfies the property (P2) as well. This will finish the proof of the lemma.

Suppose x has degree 2 in G and it does not belong to a 3-booster. Then its two neighbours u and v are b-big. We need to show that x belongs to a 4-booster. Suppose this is not the case. The degree of x in G'' is 2, since we did not change degree of any vertex that was not $2b$-big in the construction of G. Due to suppressing the vertices of degree two in G', we know that u and v must be adjacent in G''. Since they are adjacent to a 2-vertex in G'', vertices u and v are $2b$-big in G''. Therefore x should have been removed during construction of G, which is a contradiction. $\square$

Lemma 5 *Let $k > 4$ be an integer. Suppose that for each $b \geq k$ the class $\mathcal{P}_2^*(b, k)$ is nonempty. Then for each integer $b \geq k$ there exists a planar graph $G_3(b, k) \in \mathcal{P}_2(b, k)$ satisfying the property (P1) of Lemma 3, the property (P2) of Lemma 4 and the property*

(P3) No vertex of G belongs to both a 3-booster and a 4-booster.

Proof. Let us fix one such b. By Lemma 4 we may assume $G' = G_2(b, k)$ exists. Only a big vertex of G' can belong to both a 3-booster and a 4-booster. Let us construct the graph $G = G_3(b, k)$ in the following way: For each vertex v in G' such that it belongs both to a 3-booster with vertices $\{x_1, x_2, v\}$ and to a 4-booster with vertices $\{v, w, y_1, y_2\}$ (where w is a big vertex), we perform the following operation: Let R_v be a set of all vertices of degree 2 adjacent to v and belonging to 3-boosters. Add a set S_v of $|R_v|$ new vertices to G, and remove the vertices of R_v from G. Join all vertices in S_v to both v and w.

Let $S'(v) = S(v) \cup \{y_1, y_2\}$. Let $f : V(G') \to V(G)$ be a bijection that maps each element of R_v to an element of S_v, and it is an identity on vertices of $V(G) \cap V(G')$.

The graph G obviously satisfies the properties (P1), (P2) and (P3), it is therefore sufficient to show that $G \in \mathcal{P}_2(b,k)$ in order to finish the proof.

If $k = 5$, the configuration described in the construction cannot occur, otherwise we obtain a 5-path $x_1x_2vy_1z$ with at least three b-small vertices. So $G = G'$, and therefore G belongs to $\mathcal{P}_2(b,k)$.

Now, we consider the case $k \geq 6$. Let $P = v_1v_2\cdots v_k$ be a k-path in G' that does not exist in G. We may assume that no inner vertex of the path belongs to a 3-booster: If v_1 and v_2 are both 2-vertices, we can use Lemma 1 to get a $(k+1)$-path Q starting with v_1v_2 and so $Q - v_1$ is a k-path in G such that all its inner vertices are big. Then only v_1 and v_k could have been removed during the construction of G and $f(v_1)f(v_2)\cdots f(v_k)$ is a k-path in G. Therefore G contains a k-path.

It remains to show that G does not contain a k-path with more than two small vertices. Let P be a k-path in G. For each vertex v the path P uses at most two vertices of S'_v, since $k \geq 6$. So we may assume that P only uses the vertices of $S'_v \setminus S_v$ and that therefore P also occurs in G'. Then P contains at most two small vertices in G' and since the construction does not decrease the degree of any vertex, P also contains at most two small vertices in G. $\square$

3 Configurations of boosters

In this section we present lemmata showing that boosters are in some sense both frequent and rare in the graphs in $\mathcal{P}_2(b,k)$ for $k \in \{7,8,9\}$. We use these results in the next section to determine the gravity of some paths.

Let v be a big vertex of a graph $G \in \mathcal{P}_2(b,k)$. We say that v is *boosted* if v either belongs to a booster, or if v has $k+1$ neighbours $v_0, \ldots, v_k$ such that for each $0 \leq i \leq k$ the vertex v_i belongs to a 3-booster (notice that each v_i must be big).

Madaras and Škrekovski [11] have proved that the gravity of k-paths in $\mathcal{P}_2$ is at most $k-2$. This means that for each k, there exists an integer $b'_0(k)$ such that for each $b \geq b'_0(k)$, any graph in $\mathcal{P}_2$ that contains a k-path also contains a k-path with at least two b-small vertices. Let $b_0(k) = \max(k, b'_0(k))$.

Lemma 6 *For $k \geq 5$ and $b \geq b_0(k)$, let $G = G_3(b,k)$ be the graph constructed in Lemma 5. Suppose that for each k-path P in G with exactly two*

b-small vertices s_1 and s_2, the distance between s_1 and s_2 in P is $k-2$ or $k-1$. Then G contains a b-big vertex that is not boosted.

Proof. Since $b \geq b'_0(k)$, the graph G contains a k-path $P = v_1 \cdots v_k$ with two small vertices s_1 and s_2. Let P be such that the distance between s_1 and s_2 in P is the smallest possible. Vertices s_1 and s_2 have distance at least $k-2$ in P due to assumptions of the lemma, so we may assume that $s_1 = v_1$ and $s_2 \in \{v_{k-1}, v_k\}$.

If for any $2 < i < k$ the vertex v_i is adjacent to s_1, then the graph G contains the k-path $v_{i-1}v_{i-2}\cdots v_1 v_i v_{i+1} \cdots v_k$ where the distance from s_1 to s_2 is smaller by at least one than in P. Similarly if s_1 is adjacent to v_k, then the distance from s_1 to s_2 in the path $v_1 v_k v_{k-1} \cdots v_2$ is at most two.

Therefore, the vertex s_1 cannot have a neighbour in P distinct from v_2. Since the degree of s_1 is at least two, s_1 has a neighbour v outside of P. The vertex v cannot be small, otherwise $vv_1 \cdots v_{k-1}$ is a k-path in G with two adjacent small vertices. Therefore v is big. We show that v cannot be boosted. If v belongs to a booster, then let x be the 2-vertex of the booster that is not in P. Then $xvv_1 \cdots v_{k-2}$ is a k-path where small vertices have distance two. Finally, if v has $k+1$ neighbours and each of them belongs to a 3-booster, then let x be one of them that does not belong to P and let x_1 and x_2 be the 2-vertices of the 3-booster of x. Then $x_1 x_2 x v v_1 \cdots v_{k-4}$ is a k-path in G with three small vertices, a contradiction. □

Lemma 7 *For $k \in \{7, 8, 9\}$ and $b \geq k$, let $G = G_3(b, k)$ be the graph constructed in Lemma 5. Suppose that G contains a k-path P in G with two b-small vertices s_1 and s_2 such that the distance between s_1 and s_2 in P is at most $k-3$. Then G contains a b-big vertex v that is not boosted.*

Proof. Let $P = v_1 \cdots v_k$. By Lemma 1, we may assume that $v_1 = s_1$. Then v_{k-1} is big. The vertex v_{k-1} does not belong to a booster, otherwise it has a neighbour x of degree two distinct from v_1 and so $v_1 \cdots v_{k-1}x$ is a k-path in G with three small vertices. Therefore, if v_{k-1} is boosted, it must have a neighbour distinct from $v_1, \ldots, v_{k-2}$ which belongs to a 3-booster. We may assume that v_k is such a neighbour and let v_{k+1} and v_{k+2} be the 2-vertices of its 3-booster. Then the remaining small vertex of P must be v_2 (otherwise the k-path $v_3 \cdots v_{k+2}$ contains three small vertices), and therefore $\{v_1, v_2, v_3\}$ must induce a 3-booster in G due to property (P1).

Now, let us argue that v_5 cannot be boosted. The vertex v_5 cannot belong to a 3-booster, since if x_1 and x_2 are the 2-vertices of such a 3-booster, then $x_1 x_2 v_5 v_6 \cdots v_{k+2}$ is a k-path with four small vertices. The vertex v_5 also cannot have a neighbour outside P belonging to a 3-booster, since if x is such a neighbour and y is a small vertex of the 3-booster, then $yxv_5v_6\cdots v_{k+2}$ is a k-path with three small vertices.

So if v_5 is boosted, it belongs to a 4-booster. Let x be the other big vertex and y is a 2-vertex of this 4-booster. By property (P3), $x \neq v_k$. Then $x \in \{v_6, \ldots, v_{k-1}\}$, otherwise $xyv_5v_6\cdots v_{k+2}$ is a k-path with three small vertices. But v_{k-1} does not belong to any booster as argued before, and if $x = v_i$ for $i \in \{6, 7\}$, then $v_1v_2v_3v_4v_5yv_i\cdots v_{k+i-7}$ is a k-path containing three small vertices. Since $k \leq 9$, we considered all possible values of i. $\square$

By putting Lemma 6 and Lemma 7 together, we obtain

Corollary 8 *For $k \in \{7, 8, 9\}$ and $b \geq b_0(k)$, let $G = G_3(b, k)$ be the graph constructed in Lemma 5. Then G contains a b-big vertex v that is not boosted.*

We further study the structure of the neighbourhood of a vertex v that is not boosted. We are especially interested in the following configurations, where z is a neighbour of the vertex v from Corollary 8:

(B1) Vertex z is big and belongs to a 3-booster.

(B2) Vertex z is big and belongs to a 4-booster.

(B3) Vertex z is big and has $k + 1$ big neighbours, where each belongs to some 3-booster.

(B4) Vertex z is small and adjacent to a big vertex w that belongs to some 3-booster.

(B5) Vertex z is small and adjacent to a big vertex w that belongs to some 4-booster.

(B6) Vertex z is small and adjacent to a big vertex w having $k + 1$ big neigbours, where each belongs to some 3-booster.

Notice that $w \neq v$ in configurations (B4) – (B6), since v is not boosted. Let us call the edge vz in each of these configurations *important*. Let us emphasize that in case of the configuration (B2), if v is also adjacent to the other big vertex of the 4-booster, we consider this as two separate instances of the configuration (B2), and that each of these instances has just one important edge.

Lemma 9 *For $k \in \{7,8,9\}$ and $b \geq k$, let $G = G_3(b,k)$ be the graph constructed in Lemma 5. Let v be a big vetex of G that is not boosted. Then v is incident with at most*

(a) k important edges of instances of configuration (B1);

(b) 4 important edges of instances of configuration (B2);

(c) 1 important edge of an instance of configuration (B3);

(d) 1 important edge of an instance of configuration (B4);

(e) 3 important edges of instances of configuration (B5);

(f) 3 important edges of instances of configuration (B6).

Proof. Note that such vertex v exists due to Corollary 8. Let us handle the configurations (B1)–(B6) one by one:

(B1) If v is incident with $k+1$ important edges of instances of configuration (B1), then v is boosted.

(B2) Suppose that v is incident with 5 important edges of instances of configuration (B2). Then no two 4-boosters in these instances are disjoint, otherwise G contains a k-path with at least three small vertices. Using the pigeonhole principle, we observe that among these 5 instances there must be three of them that share only the big vertex of the 4-boosters that is not incident to the important edge of each of the instances (see Figure 3). Note that this configuration contains a k-path with at least three vertices of degree two for each $k \in \{7,8,9\}$, which is a contradiction.

(B3) If $k = 7$, the configuration (B3) contains a 7-path with four small vertices. So we may assume $k = 8$ or 9.

Suppose that v is incident with 2 important edges of instances of configuration (B3). Let z_1 and z_2 be the vertices of these edges distinct from v. Then we can find two distinct big vertices x_1 and x_2 such that each of them is distinct from z_1 and z_2, both x_1 and x_2 belong to 3-boosters and x_1z_1 and x_2z_2 are edges of G. This configuration however contains a k-path with at least three small vertices.

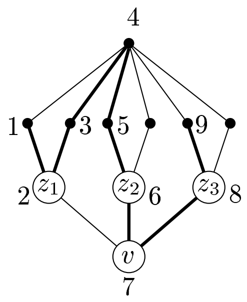

Figure 3: Forbidden path in case (B2)

(B4) Let v be incident with 2 important edges of instances of configuration (B4). Let z_1 and z_2 be the small vertices incident with the important edges of the configurations. By property (P1), vertices z_1 and z_2 are not adjacent. The two configurations corresponding to them cannot be disjoint, otherwise they contain a k-path with at least three small vertices. More precisely there is a big vertex w adjacent to both z_1 and z_2 such that w belongs to a 3-booster. Let w_1 and w_2 be the small vertices of that booster. Since v is not boosted, it is not adjacent to vertex of degree two, and therefore degree of z_2 must be at least three. Consequently the vertex z_2 has other big neighbour x distinct from v and w. Since x is big, it has a neighbour not in $\{z_1, z_2, v, w\}$, let this neighbour be y.

Then $Q = w_1 w_2 w z_1 v z_2 x y$ is an 8-path containing at least four small vertices, which cannot occur unless $k = 9$. If y would have a neighbour outside Q, we could extend Q to a 9-path. Consequently, y must be small. Besides x, the only neighbours of y could be v and w due to property (P1). If y is not adjacent to both v and w, vertex y has degree 2. Then the vertex y belongs to a 4-booster due to property (P2), and consequently v or w belongs to this 4-booster. But this cannot occur: the vertex v is not boosted, and by property (P3), the vertex w cannot belong to both a 3-booster and a 4-booster. Therefore y must be adjacent to both v and w.

Since y was an arbitrary neighbour of x distinct from $\{s_1, s_2, v, w\}$, we observe that any neighbour of x not in this set must be adjacent to both v and w. But due to planarity of G the vertex x can have at most two such neighbours, otherwise G contains $K_{3,3}$ as a subgraph. This means that the degree of x is at most 6, which is a contradiction with the fact that x is a big vertex.

(B5) Suppose that v is incident with 4 important edges of instances of configuration (B5). Let z_1, z_2, z_3 and z_4 be the small vertices incident with the important edges of the configurations. Since v is not boosted, property (P2) implies that the degree of each z_i is at least three. Let w be the big vertex adjacent to z_1 belonging to a 4-booster, w_1 and w_2 the small vertices of this booster and y the remaining big vertex of this booster. Then $Q_i = w_1 y w_2 w z_1 v z_i$ for $i \in \{2, 3, 4\}$ is a 7-path with four small vertices, so this cannot occur if $k = 7$. If z_i has a neighbour x distinct from w, y and v, then the vertex x must be big due to property (P1). Then x has other neighbour outside Q_i and together they extend Q_i to a 9-path. Therefore z_i has no such neighbour. Since degree of z_i is at least three, z_i must be adjacent to both w and y. But then $K_{3,3}$ is a subgraph of G, which contradicts the planarity of G.

(B6) If $k = 7$ the configuration (B6) contains a 7-path with four small vertices, so we may assume $k = 8$ or 9.

Let v be incident with 4 important edges of instances of configuration (B6). Let z_1, z_2, z_3 and z_4 be the vertices of these important edges distinct from v. Similarly as in the previous case we see that the degree of each z_i is at least three. Let w be the big vertex of the configuration adjacent to z_1 and let y be one of the big vertices adjacent to w that belongs to a 3-booster. Let y_1 and y_2 be the small vertices of this 3-booster.

The 7-path $Q_i = y_1 y_2 y w z_1 v z_i$ (for $i \in \{2, 3, 4\}$) contains four small vertices. If z_i has a neighbour x distinct from v, w and y, then vertex x must be big, and therefore x together with one of its neighbours would extend Q_i to a 9-path. Since degree of z_i is at least three, one can easily see that z_i must be adjacent to both w and y. But this is again a contradiction with the planarity of G.

□

4 Gravity of paths in $\mathcal{P}_2$

We are now ready to determine the gravity of k-paths in $\mathcal{P}_2$ for $k \in \{5, 7, 8, 9\}$.

Theorem 10 *Gravity of P_5 in the class of planar graphs with minimal degree 2 is 2.*

Proof. Due to [11], gravity of P_5 in $\mathcal{P}_2$ is either two or three. Suppose for the sake of contradiction that gravity of P_5 is three. This means for infinitely many integers b, the class $\mathcal{P}_2(b, 5)$ is non-empty. So by the remark at the begining of Section 2, one concludes that the class $\mathcal{P}_2(b, 5)$ is non-empty for all integers $b \geq 1$. Let us consider the graph $H = G_3(b, 5)$ constructed in Lemma 5, for sufficiently large b (at least 13).

Each big vertex v of H belongs to at most one booster, otherwise one can find a 5-path with three small vertices. By property (P1) a small vertex of degree at least three has only big neighbours.

We construct a graph H' by removing all vertices of degree 2 from H. Thus, big vertices lose at most two neighbours, the degree of remaining small vertices is unchanged. Therefore the graph H' has minimum degree three and each its edge contains at least one vertex of degree at least $b - 2 \geq 11$. But due to [2] each planar graph with minimum degree three contains an edge e such that sum of degrees of the vertices incident with e is at most 13, which is a contradiction. □

Theorem 11 *Gravity of P_k in the class of planar graphs with minimal degree 2 is $k - 3$ for $k \in \{7, 8, 9\}$.*

Proof. Let $k \in \{7, 8, 9\}$. Due to [11] gravity of k-paths in $\mathcal{P}_2$ is either $k - 2$ or $k - 3$. Suppose for the sake of contradiction that gravity of P_k is $k - 2$. Similarly as in the above theorem, one can conclude that $\mathcal{P}_2(b, k)$ is nonempty for all integers $b \geq 1$. Let us consider a graph $H = G_3(b, k)$ constructed in Lemma 5, for b sufficiently large (at least $\max(b_0(k), k+23)$). Now, we construct a graph H' by removing all boosted vertices and all small vertices that are adjacent to boosted vertices from H (note that by property (P2) every 2-vertex is removed).

By the choice of b, the graph H has a k-path which contains precisely two small vertices. So, Corollary 8 implies that H contains a big vertex that is not boosted, and therefore H' is nonempty. If v is a vertex of H' that is big in H, then Lemma 9 implies that vertex v has degree at least

$k + 23 - (k + 12) = 11$ in H'. If v is a vertex of H' that is small in H, then no neighbour of v was removed and v has degree at least three both in H and in H', since a small vertex of degree at least three has only big neighbours due to property (P1).

Therefore H' has minimum degree at least 3 and the sum of degrees of vertices of any of edges of H' is at least 14 (since there are no two small adjacent vertices of degree at least three in H). This however is a contradiction with results of [2]. □

References

[1] O. V. Borodin, *Solution of problems of Kotzig and Grünbaum concerning the isolation of cycles in planar graphs*, Mat. Zametki **46** (1989) 9–12.

[2] O. V. Borodin, *On the total colorings of planar graphs*, J. Reine Angew. Math. **394** (1989) 180–185.

[3] I. Fabrici and S. Jendrol', *Subgraphs with restricted degrees of their vertices in planar* 3*-connected graphs*, Graphs and Combin. **13** (1997) 245–250.

[4] I. Fabrici, E. Hexel, S. Jendrol', and H. Walther, *On vertex-degree restricted paths in polyhedral graphs*, Discrete Math. **212** (2000) 61–73.

[5] S. Jendrol' and T. Madaras, *On light subgraphs in plane graphs of minimum degree five*, Discuss. Math. Graph Theory **16** (1996) 207–217.

[6] J. Harant, S. Jendrol', and M. Tkáč, *On* 3*-Connected Plane Graphs without Triangular Faces*, J. Combin. Theory Ser. B **77** (1999) 150–161.

[7] S. Jendrol', T. Madaras, R. Soták, and Z. Tuza, *On light cycles in plane triangulations*, Discrete Math. **197/198** (1999) 453–467.

[8] S. Jendrol' and H.-J. Voss, *Light subgraphs of graphs embedded in plane and projective plane – a survey –*, Preprint Inst. of Algebra MATH–AL–XX–1999, TU Dresden.

[9] A. Kotzig, *Contribution to the theory of Eulerian polyhedra*, Mat. Čas. SAV (Math. Slovaca) **5** (1955) 111–113.

[10] T. Madaras and R. Škrekovski, *Heavy paths, light stars and big melons*, Discrete Math. (to be published)

[11] T. Madaras and R. Škrekovski, *Lightness, heaviness and gravity*, Discrete Math. (to be published)

[12] B. Mohar, R. Škrekovski, and H.-J. Voss, *Light subgraphs in planar graphs of minimum degree* 4 *and edge degree* 9, J. Graph Theory **44** (2003) 261–295.

[13] T. Madaras and R. Soták, *The* 10*-cycle* C_{10} *is light in the family of all plane triangulations with minimum degree five*, Tatra Mountains Math. Publ. **18** (1999) 35–56.