X. Midsummer Combinatorial Workshop

(M. Bálek, ed.)

Prague

July 28 — August 1, 2003

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Preface

The Tenth Prague Midsummer Combinatorial Workshop was held from July 28 to August 1, 2003 in a romantic Nosticovo divadlo (Nostic Theatre; our faculty building was at that time closed for reconstruction). The workshop was organized by the Department of Applied Mathematics (KAM) of Charles University jointly with the DIMATIA centre. Only a small but distinguished group of mathematicians was invited and we were particularly happy to have Endré Szemerédi among the participants. The list of participants is included in this booklet.

As it already became a tradition, the workshop benefited from participation of young researchers and PhD students, and as in the last two years within a (joint DIMATIA-DIMACS) program International REU (supported jointly by NSF and Czech ministery of education) 5 undergraduate students from the USA and 3 underargraduate students from Charles University took part in the workshop, together with their mentors Ondřej Pangrác and Clifford Smyth.

The workshop followed an informal daily routine with morning and early afternoon discussions and presentations. This report reflects some of the presentations during the workshop. Perhaps you can digest from these proceedings some of the atmosphere at the workshop and you can also see that the fruitful exchange of ideas led directly to some new results and papers.

This volume was edited by Martin Bálek. Most of the problems described here were supplied by the authors in electronic form; in a few cases, slight typographical changes were necessary. We apologize for any possible inaccuracies which might have occurred in the editing process.

The Tenth Midsummer Combinatorial Workshop was supported by the Kontakt CS-US Grants and by our institute ITI (financed by the Ministry of Education of the Czech Republic as project LN00A056). DIMATIA was the main organizer.

Based on our past experience and being encouraged by several participants, we hope to organize the Eleventh Prague Combinatorial Workshop in the summer of 2004. We hope to meet you there!

Jaroslav Nešetřil

Workshop programme

Monday 28.7., chairman J. Nešetřil

Martin Škoviera: Factorization of Irreducible Snarks

Andrzej Ruciński: Dirac Theorem for Hypergraphs

Jørgen Bang-Jensen: Problems/conjectures on Connectivity of Tournaments

Tuesday 29.7., chairman J. Kratochvíl

Daniel Král': Hamiltonian prisms

Jarek Grytczuk: Thue type problems for graphs, points and numbers

Jiří Fiala: BBC (backbone coloring)

Roman Nedela: H–W problem

Jan Kára: Triangle packings in chordal graphs

Nicholas J. Cavenagh: 3-homogenous Latin Trades

Wednesday 30.7., chairman P. Valtr

Endré Szemerédi: Three Combinatorial Problems

Jaroslav Nešetřil: An antiszemerédi problem

Trip to the Karlštejn castle

Thursday 31.7., chairman J. Matoušek

Jan Vondrák: Covering MTSs of random subgraphs

Martin Klazar: Some problems on counting permutations

Tomáš Kaiser: An odd graph question

Helmut Alt: Geometric Comparison of Patterns and Shapes

Attila Pór: Partitioning sets on the line into separated ones

Friday 1.8., chairman M. Klazar

Kryzstof Prezslawski: On System of boxes

Jan Kratochvíl: Intersection Graphs

Jaroslav Nešetřil: MAC

Riste Škrekowski: A Map Colour Theorem for the Union of Graphs

Patrice Ossona de Mendez: Balanced bicoloration, Homomorphism and Graph Dimension

Yared Nigussie: NP-completeness Problem

Małgorzata Bednarska: Games on Complete Graphs


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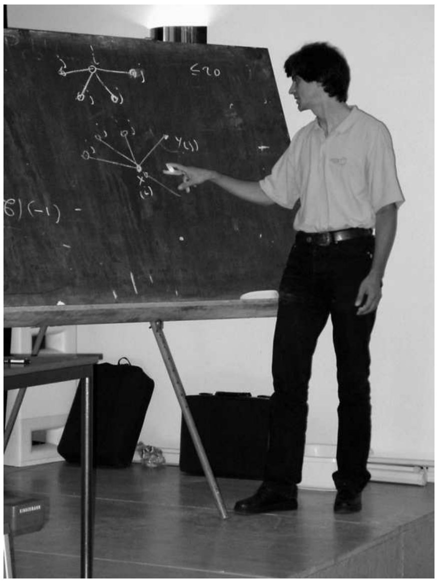
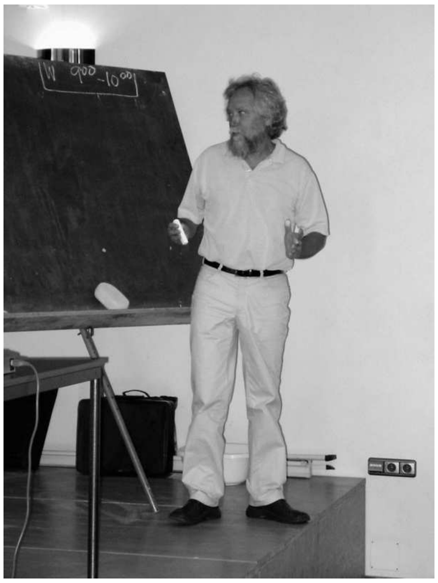
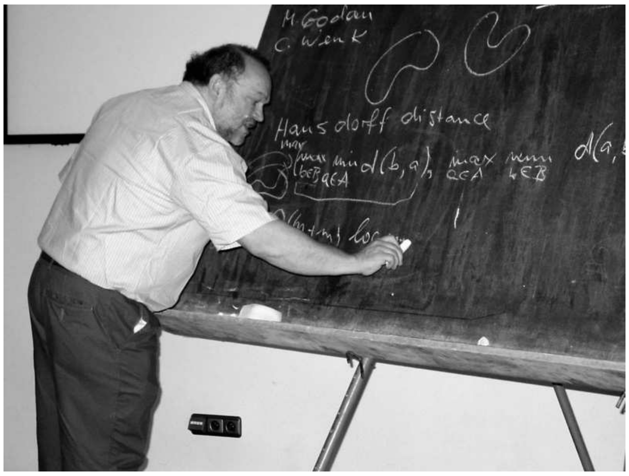
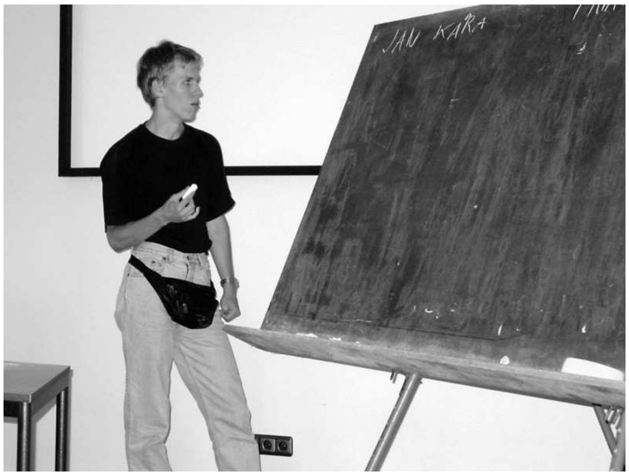
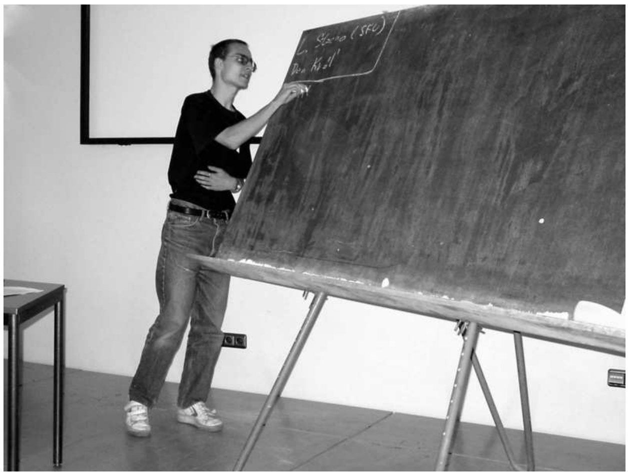





ANTI SZEMERÉDI PROBLEM


M. Godau
C. Wenk
Hausdorff distance


JAN KÁRA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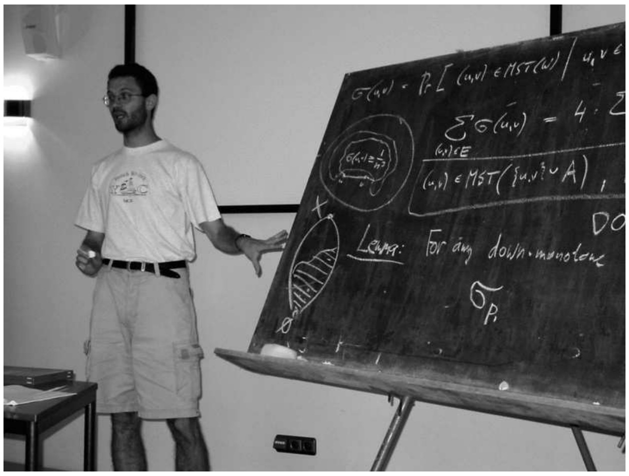
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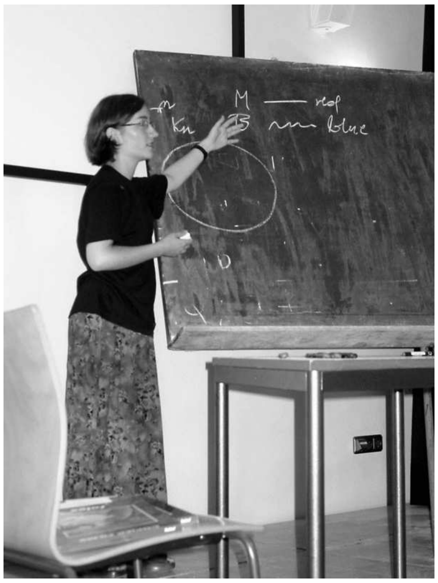


All photos from the workshop can be found at
`http://kam.mff.cuni.cz/photos/MidSummer03/`

Geometric Comparison of Patterns and Shapes

Helmut Alt

The comparison of geometric patterns and shapes is of importance in various application areas, in particular in computer vision and pattern recognition, but also in other disciplines concerned with the form of objects such as cartography, molecular biology, and computer animation.

The general situation is that we are given two objects A, B and want to know how much they *resemble* each other. Usually one of the objects may undergo certain transformations like translations, rotations or scalings in order to be *matched* with the other as well as possible. Variants of this problem include partial matching, i.e. when A resembles only some part of B, and a data structures version where, for a given object A, the most similar one in a fixed preprocessed set of objects has to be found, e.g. in character or traffic sign recognition.

First it is necessary to formally define the notions of objects, resemblance, matching, and transformations.

Objects are usually finite sets of points ("point patterns") or "shapes" given in two dimensions by polygons or polygonal chains. Generalizations to, for example, polyhedral surfaces in three and higher dimensions are possible, but most of the work has concentrated on two or three dimensions.

In order to measure "resemblance" various distance functions have been used, in particular much work has been based on the so-called *Hausdorff distance*.

For two compact subsets A, B of the d-dimensional space $\mathbb{R}^d$, we define the *one-sided* Hausdorff distance from A to B as

$$\tilde{\delta}_H(A,B) = \max_{a \in A} \min_{b \in B} ||a - b|| \,,$$

where $||\cdot||$ is the Euclidean distance in $\mathbb{R}^d$ (if not explicitly stated otherwise). The (bidirectional) Hausdorff distance between A and B then is defined as

$$\delta_H(A,B) = \max\left(\tilde{\delta}_H(A,B), \tilde{\delta}_H(B,A)\right) .$$

The Hausdorff distance simply assigns to each point of one set the distance to its closest point on the other and takes the maximum over all these values.

It performs reasonably well in practice but may fail if there is noise in the images.

What kind of geometric *transformations* are allowed to match objects A and B depends on the application. The most simple kind are certainly *translations*. The matching problem usually becomes much more difficult if we allow rotations and translations (*rigid motions, Euclidean transformations*). In most cases *reflections* can be included as well without any further difficulty.

In [ABB91] it was also observed that the *matching* problem under translations or rigid motions can be solved in polynomial time. These results are based on the fact that if the transformation has k degrees of freedom (e.g. $k = 3$ for rigid motions) then in the optimal position the Hausdorff-distance essentially must occur in at least $k+1$ different places.In some applications, the simplicity of the Hausdorff-distance can be a disadvantage. In fact, when the distance between *curves* is measured the Hausdorff-distance may give a wrong picture. Figure 1 shows an example where two curves have a small Hausdorff-distance although they have no resemblance at all.

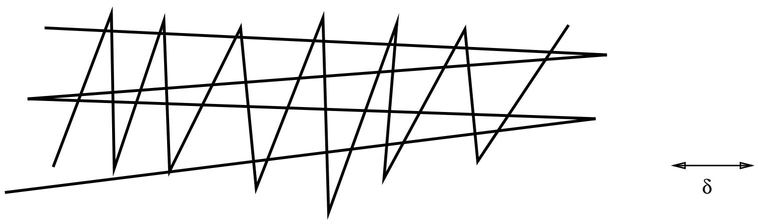

Figure 1: Two curves with small Hausdorff-distance δ.

The reason for this problem is that the Hausdorff-distance is only concerned with the point sets but not with the course of the curves. A distance considering the curves' courses can informally be illustrated as follows: Suppose a man is walking his dog, he is walking on one curve, the dog on the other. Both are allowed to control their speed but not to go backward. What is the shortest length of a leash that is possible? Formally, this distance measure between two curves in d-dimensional space can be described as follows

$$\delta_F(f,g) = \inf_{\alpha,\beta} \max_{t\in[0,1]} ||f(\alpha(t)) - g(\beta(t))||$$

where $f, g : [0,1] \to I\!R^d$ are *parameterizations* of the two curves and α, β :

$[0,1] \to [0,1]$ range over all continuous and monotone increasing functions. This distance measure is known under the name *Fréchet-distance.*

The Fréchet-distance seems considerably more difficult to handle than the Hausdorff-distance and no matching algorithms have been developed yet. The following algorithm for measuring the Fréchet-distance between two polygonal chains has been given by Alt and Godau [AG92, AG95].

Let P and Q be the given polygonal chains consisting of n and m line segments respectively. First we consider the decision problem, so in addition to P and Q some $\varepsilon > 0$ is given and we want to decide whether $\delta_F(P,Q) \leq \varepsilon$. We first consider the $m \times n$-diagram $D_\varepsilon(P,Q)$ shown in Figure 2 which indicates by the white area for which points $p \in P$, $q \in Q$ $||p-q|| \leq \varepsilon$. The horizontal direction of the diagram corresponds to the natural parameterization of P and the vertical one to that of Q. One square cell of the diagram corresponds to a pair of edges one from P and one from Q and can easily be computed since it is the intersection of the bounding square with an ellipse.

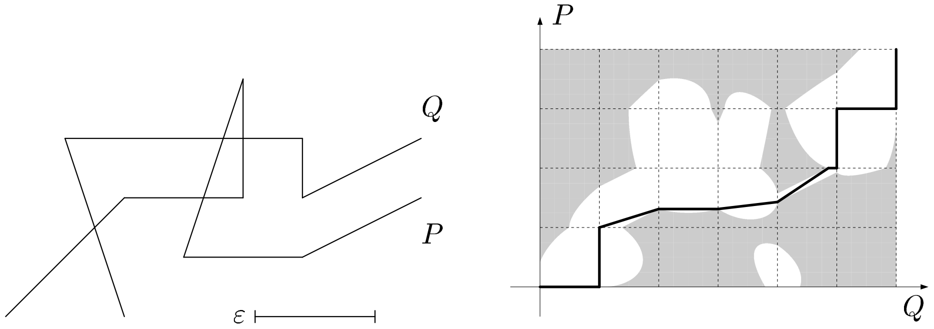

Figure 2: P, Q, ε and the diagram $D_\varepsilon(P,Q)$.

Now it follows from the definition that $\delta_F(P,Q) \leq \varepsilon$ exactly if there is a monotone increasing curve from the lower left to the upper right corner of the diagram. These considerations lead to an algorithm of running time $O(mn)$ for the decision problem. Then Cole's variant of *parametric search* [Col87] can be used to obtain an algorithm of running time $O(mn\log(mn))$ to compute the Fréchet-distance between P and Q. In practice, it seems more reasonable to determine $\delta_F(P,Q)$ bit by bit using binary search where in each step the algorithm for the decision problem is applied.

A matching algorithm for Fréchet distance is given in [AKW01], but it has a very high runtime, so that it makes sense to look for approximation algorithms for this problem.

References

[ABB91] H. Alt, B. Behrends, and J. Blömer. Approximate matching of polygonal shapes. In *Proc. 7th Annu. ACM Sympos. Comput. Geom.*, pages 186–193, 1991.

[AG92] H. Alt and M. Godau. Measuring the resemblance of polygonal curves. In *Proc. 8th Annu. ACM Sympos. Comput. Geom.*, pages 102–109, 1992.

[AG95] H. Alt and M. Godau. Computing the Fréchet distance between two polygonal curves. *International Journal of Computational Geometry and Applications*, 5:75–91, 1995.

[AKW01] Helmut Alt, Christian Knauer, and Carola Wenk. Matching polygonal curves with respect to the Fréchet distance. In *Proceedings 18th International Symposium on Theoretical Aspects of Computer Science*, pages 63–74, 2001.

[Col87] R. Cole. Slowing down sorting networks to obtain faster sorting algorithms. *J. ACM*, 34(1):200–208, 1987.

Problems and conjectures concerning connectivity of Tournaments

Jørgen Bang-Jensen

A digraph D is k-arc-strong, for some natural number k, if we must remove at least k arcs from D in order to obtain a digraph which is not strongly connected. We denote by $\lambda(D)$ the maximum k for which D is k-arc-strong.

The so-called Kelly conjecture[1] states that every regular tournament on $2k+1$ vertices has a decomposition into k-arc-disjoint hamiltonian cycles.

Conjecture 1 (Kelly, 1968) *Every k-regular tournament contains k arc-disjoint hamiltonian cycles (see [5]).*

In [1] a generalization of the Kelly conjecture is given

Conjecture 2 *[1] Every k-arc-strong tournament decomposes into k spanning strong digraphs.*

The paper [1] contains several results which support the conjecture

- If $D=(V,A)$ is a 2-arc-strong semicomplete digraph then it contains 2 arc-disjoint spanning strong subdigraphs except for one digraph on 4 vertices.
- The conjecture is true for every tournament (in fact semicomplete digraphs) which has a non-trivial cut (both sides of size at least 2) with precisely k arcs in one direction.
- Every k-arc-strong tournament with minimum in- and out-degree at least $37k$ contains k arc-disjoint spanning subdigraphs $H_1, H_2, \ldots, H_k$ such that each H_i is strongly connected.

Various weakenings on Conjecture 2 are listed below.

[1] A proof of the Kelly conjecture for large k has been announced by R. Häggkvist at several conferences and in [3] but to this date no proof has been published.

Conjecture 3 *Let k, s and t be natural numbers such that $k = s + t$. Then every k-arc-strong tournament contains arc-disjoint spanning strong subdigraphs D_1, D_2 such that D_1 is s-arc-strong and D_2 is t-arc-strong.*

Conjecture 4 *Every k-arc-strong tournament contains a spanning strong subdigraph H such that $T - A(H)$ is $(k-1)$-arc-strong.*

Thomassen proved [6, Theorem 4.2] that every 2-arc-strong tournament T contains a hamiltonian path P such that $T - A(P)$ is strong. It is interesting to note that we cannot replace hamiltonian path by hamiltonian cycle above, as shown by the infinite class of 2-arc-strong tournaments in Figure 1.

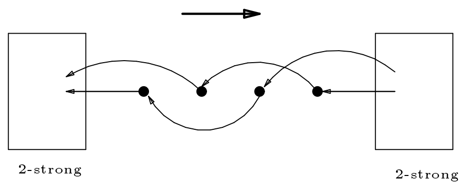

Figure 1: An infinite family of 2-arc-strong tournaments such that the deletion of the arcs of any hamiltonian cycle leaves a non-strong digraph. The first and the last box symbolizes arbitrary 2-arc-strong tournaments and the fat arc indicates that except for the 6 arcs shown from right to left all other arcs to from left to right.

Conjecture 5 *Except for finitely many exceptions for each k, every k-arc-strong semicomplete digraph can be decomposed in k arc-disjoint spanning strong subdigraphs.*

As pointed out by Thomassen in [6] there is no degree r of arc-strong connectivity which guaranties that every r-arc-strong tournament contains two arc-disjoint hamiltonian cycles. In fact, a tournament may have arbitrarily high arc-strong connectivity and still deleting a single arc may destroy all hamiltonian cycles. See Figure 2. Thomassen also mentions a construction of Jackson showing that a tournament may have arbitrary high arc-strong connectivity without having 4 arc-disjoint hamiltonian paths. On the other hand Thomassen conjectures that there is some $\alpha(k)$ such that every $\alpha(k)$-strong tournament contains k arc-disjoint hamiltonian cycles. He shows that $\alpha(2) > 2$ and conjectures that every 3-strong tournament contains 2 arc-disjoint hamiltonian cycles.

Conjecture 6 (Thomassen, 1982) *Every 3-strong tournament contains two arc-disjoint hamiltonian cycles.*

By a result of Fraisse and Thomassen [4] (saying that every k-strong tournament has a hamiltonian cycle which avoids any prescribed set of $k-1$ arcs), this conjecture would follow from the following conjecture.

Conjecture 7 *Every tournament T either contains two arc-disjoint hamiltonian cycles or a set A' of at most two arcs such that $T - A'$ has no hamiltonian cycle.*

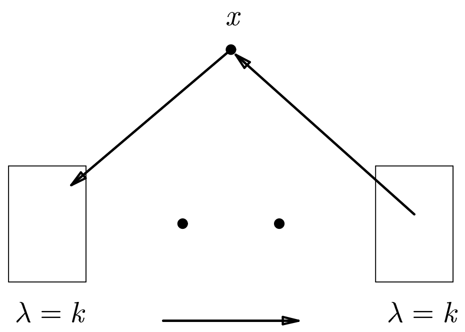

Figure 2: An Infinite family of k-arc-strong tournaments with no two arc-disjoint hamiltonian cycles. Each box symbolizes an arbitrary k-arc-strong tournament and the two dots in between are strong components of size one when x is deleted.

Problem 1 (Bang-Jensen, Huang and Yeo, 2001) *Which tournaments T contain a hamiltonian cycle C such that $\lambda(T - C) \geq \lambda(T) - 1$?*

Conjecture 8 *There exists a polynomial algorithm for deciding whether a given tournament contains two arc-disjoint hamiltonian cycles.*

An tournament is **almost transitive** if it can be obtained from a transitive tournament by reversing the arc from the vertex of maximum out-degree to the vertex of maximum in-degree. Thomassen [6] proved that a tournament T contains two arc-disjoint hamiltonian paths unless it has a strong component which is an almost transitive tournament of odd order or has two consecutive strong components of size 1.

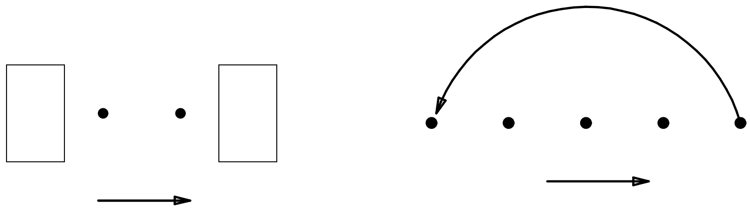

Figure 3: Examples of tournaments with no two arc-disjoint hamiltonian paths. The one to the right is an almost transitive tournament.

Problem 2 *Characterize those tournaments which contain two arc-disjoint hamiltonian paths with prescribed start vertices.*

By inspection of Figure 2 we see that no arc-strong connectivity suffices.

The following is a possible extension of the notion of hamiltonian-connectivity. Note that $f(1) = 4$ by Thomassen's theorem on hamiltonian connected tournaments and examples by Thomassen of 3-strong tournament with no (x, y)-hamiltonian path.

Problem 3 *Does there exists a function $f(k)$ such that every $f(k)$-strong tournament contains k arc-disjoint hamiltonian paths from x to y for every choice of x and y?*

Theorem 1 *[2] For any $n \geq 3$ and $k \geq 1$, every k-arc-strong tournament T on n vertices contains a spanning k-arc-strong subdigraph D' with at most $nk + 136k^2$ arcs. Furthermore, such a spanning subdigraph can be found in polynomial time.*

For any tournament T we denote by $h(k, T)$ the minimum number of arcs in a spanning subdigraph D of T in which has $\delta(D) \geq k$. If $\delta(T) < k$ we let $h(k, T) = \infty$.

The following result can be shown using flows. We leave the details to the interested reader.

Proposition 2 *[2] For every tournament with $\delta(T) \geq k$ we have $h(k, T) \leq nk + k(k+1)/2$ and this is sharp. Furthermore, if T is k-arc-strong $h(k, T) \leq nk + k(k-1)/2$.*

For any tournament T we denote by $i(k, T)$ the minimum number of arcs in a spanning k-arc-strong subdigraph D of T. If T is not k-arc-strong then $i(k, T) = \infty$.

Conjecture 9 *[2] For every natural number k and every k-arc-strong tournament T we have $i(k,T) = h(k,T)$.*

Problem 4 *Does there exist a function $g = g(k)$ such that every k-strong tournament contain a spanning k-strong subdigraph with at most $kn + g(k)$ arcs?*

References

[1] J. Bang-Jensen and A. Yeo, Decomposing k-arc-strong tournaments into strong spanning subdigraphs, *Combinatorica*, to appear.

[2] J. Bang-Jensen, J. Huang and A. Yeo, Finding small k-arc-strong spanning subdigraphs in k-arc-strong tournaments, *J. Graph Theory*, to appear.

[3] B. Bollobàs and R. Häggkvist, Powers of Hamilton cycles in tournaments, *J. Combinatorial Theory* **50** (1990) 309-318.

[4] P. Fraisse and C. Thomassen, Hamiltonian dicycles avoiding prescribed arcs in tournaments, *Graphs and Combinatorics* **3** (1987) 239-250.

[5] J.W.Moon, Topics on tournaments, Holt. Rinehart and Winston, New York 1968.

[6] C. Thomassen, Edge-disjoint hamiltonian paths and cycles in tournaments, *Proc. London Math. Soc.* **45** (1982) 151-168.

Density games on K_n

Małgorzata Bednarska

Suppose we are given natural numbers n, q and a family $\mathcal{F}$ of subgraphs of the complete graph K_n. Two players, Maker and Breaker, take turns (Maker starts the game) and select respectively 1 and q previously unclaimed edges of K_n. Maker wins if and only if he claims all edges of some graph from $\mathcal{F}$.

We are interested in finding $q_{\mathcal{F}}$, the minimum q such that Breaker has a winning strategy (i.e. he can prevent Maker from building some $F \in \mathcal{F}$). We refer to $q_{\mathcal{F}}$ as to *the threshold* of the game. The problem of finding the threshold for the positional games on graphs and hypergraphs originates from papers by Chvátal and Erdős [2] and Beck [1], and was extensively explored by the latter author afterwards.

For given $a \geq 1$, let us propose *a density game* $\mathfrak{D}_a(K_n, 1, q)$, in which Maker's aim is to claim any graph H of density $e(H)/v(H) \geq a$. For example, for $a = 1$ we obtain *the cycle game*, i.e. the game in which $\mathcal{F}$ contains all cycles of K_n. Let us denote the threshold for the density game $\mathfrak{D}_a(K_n, 1, q)$ by q_a.

We can show that for every integer $a \geq 1$ the threshold is

$$q_a = \left\lceil \frac{\binom{n}{2}}{an-1} \right\rceil - 1. \qquad (1)$$

The estimation $q_a \geq \left\lceil \frac{\binom{n}{2}}{an-1} \right\rceil - 1$ is easy to obtain: if $q + 1 < \frac{\binom{n}{2}}{an-1}$ then at the end of the game $\mathfrak{D}_a(K_n, 1, q)$ Maker's graph consists of at least an edges. That means every strategy of Maker is a wining strategy.

Thus, in order to obtain the desired threshold, we have to show that if $q + 1 \geq \frac{\binom{n}{2}}{an-1}$ then Breaker can win the game. In [3] we describe a winning strategy for Breaker, based on the decomposition of K_n into graphs G such that $\max_{H \subseteq G} \frac{e(G)}{v(G)} < a$. Unfortunately, the approach from [3] breaks down if a is not an integer.

Nevertheless, we conjecture that the threshold of any density game is roughly $n/(2a)$.

Conjecture 1 *For any real $a \geq 1$, the threshold of the density game $\mathfrak{D}_a(K_n, 1, q)$ is*

$$q_a = \frac{n}{2a} + o(n).$$

By (1), the conjecture holds for every integer a.

References

[1] J. Beck, *Remarks on positional games. I*, Acta Math. Acad. Sci. Hung. **40** (1982), 65–71.

[2] V. Chvátal and P. Erdős, *Biased positional games*, Ann. Discrete Math. **2** (1978), 221–229.

[3] M. Bednarska, O. Pikhurko *On biased positional games on matroids*, http://www.math.cmu.edu/~pikhurko/Papers/

3-homogeneous latin trades

Nicholas J. Cavenagh

1 Definitions

Let $N = N(n) = \{0, 1, 2, \ldots, n-1\}$. A *partial latin square* P of order n is a set of ordered triples of the form (i, j, k), where $i, j, k \in N$ with the following properties:

- if $(i, j, k) \in P$ and $(i, j, k') \in P$ then $k = k'$,
- if $(i, j, k) \in P$ and $(i, j', k) \in P$ then $j = j'$ and
- if $(i, j, k) \in P$ and $(i', j, k) \in P$ then $i = i'$.

We may also represent a partial latin square P as an $n \times n$ array with entries chosen from the set N such that if $(i, j, k) \in P$, the *entry* k occurs in cell (i, j). We sometimes refer to the entry k in cell (i, j) of a partial latin square P as $(i, j)_P$.

A partial latin square has the property that each entry occurs at most once in each row and at most once in each column. If all the cells of the array are filled then the partial latin square is termed a latin square. That is, a *latin square* L of order n is an $n \times n$ array with entries chosen from the set $N = N(n) = \{0, 1, 2, \ldots, n-1\}$ in such a way that each element of N occurs precisely once in each row and precisely once in each column of the array.

A partial latin square T of order n is said to be a *latin trade* (or *latin interchange*) if there exist two latin squares L_1 and L_2 of order n, with $L_1 \neq L_2$, such that $T = L_1 \setminus L_2$. The partial latin square $L_2 \setminus L_1$ is said to be the *disjoint mate* of T.

A latin trade T of order n is said to be k-homogeneous if it has either 0 or k entries in each row, either 0 or k entries in each column, and each entry occurs either 0 or k times within T. Clearly if T is k-homogeneous, its size is equal to km for some integer m, where $m \geq k$.

A *critical set* in a latin square L (of order n) is a partial latin square $P \subseteq L$, such that

(1) L is the only latin square of order n which has element k in cell (i, j) for each $(i, j, k) \in P$; and

(2) no proper subset of P satisfies (1).

Let T be a partial latin square that is a subset of a latin square L. Observe that T is a latin trade if and only if there exists a disjoint mate T', with $T' \cap T = \emptyset$, such that $(L \setminus T) \cup T'$ is a latin square. It follows that a critical set P in a latin square L must intersect every latin trade in L; and is minimal with respect to this property.

2 Known Results

Results on lower bounds for the sizes of critical sets often rely on constructions of latin trades. In [2], it is shown that if a critical set of order n has an empty row, its size must be at least $2n-4$. The proof in [2] relies on latin trades that occur within sets of three rows of a latin square. A more general result is given in [4], where it is shown that the size of a critical set in a latin square of order n is at least $\lfloor (4n-8)/3 \rfloor$. Their proof relies on the construction of latin trades in the union of three rows and three columns. In [3], it is shown that a critical set in the back circulant latin square B_n must have size at least $O(n^{4/3})$.

We are still a long way from the conjectured lower bound on the size of a critical set in a latin square of order n: $\lfloor n^2/4 \rfloor$ [1]. One way to go forward is to learn more about latin trades that intersect a *large* number of rows, a *large* number of columns and a *large* number of entries (*large* meaning with respect to n), yet are still relatively small in size.

One way to find such latin trades is to look at k-homogeneous latin trades.

3 New results, open problems

In the talk we presented a construction for 3-homogeneous latin trades based on the hexagonal packing of circles in the plane. This gave 3-homogeneous latin trades of size $3m$, for each $m \geq 3$. For details of this construction please email the author for a copy of the full paper (contact details above).

We pose the following open problems.

Open Problem 1 *For what values of k and m do there exist a k-homogeneous trade of size km?*

We must have $k \geq 2$, since each row, column and entry in a latin trade occurs at least twice. Also $m \geq k$, as each row must have k distinct entries. If

$k = 2$ then we must have $m = 2$, with the unique (minimal) 2-homogeneous trade being a 2×2 latin square. If $m = k - 1$ and $m \notin \{2, 6\}$, we can use a pair of mutually orthogonal latin squares of order m. The authors have made progress on the above open problem in the case where $k = 4$; we are currently in the process of writing up this result.

Next, we ask:

Open Problem 2 *Does the construction given in the talk give every possible 3-homogeneous latin trade?*

Finally, using k-homogeneous latin trades or otherwise,

Open Problem 3 *Improve the following bounds:* $\lfloor (4n - 8)/3 \rfloor \leq$ *the size of the smallest critical set of order* $n \leq \lfloor n^2/4 \rfloor$.

References

[1] J.A. Bate and G.H.J. van Rees, Minimal and near-minimal critical sets in back circulant latin squares, Australasian J. Combinatorics (to appear).

[2] N.J. Cavenagh, Latin trade algorithms and the smallest critical set in a latin square, J. Automata, Languages and Combinatorics (to appear).

[3] N.J. Cavenagh, The size of the smallest critical set in the back circulant latin square, submitted.

[4] P. Horák, R. E. L. Aldred and H. Fleischner, Completing Latin squares: critical sets, J. Combinatorial Designs (to appear).

Backbone coloring of planar graphs

Jiří Fiala

Given a graph G and its spanning tree T, the *backbone coloring* of G is an assignment of integers to vertices of G that is a valid coloring, and moreover whenever two vertices are adjacent in T their labels differ by at least two. The mimimum possible span of the set of used labels is denoted as the *bbc* number of G.

This notion stems from channel assignment area, where for a *backbone* of a possible network higher separation (i.e. less possible interfernce of the associated signals) is required.

A simple example shows that there exist planar graphs with *bbc* number at least 6. On the other hand the famous Four-color theorem implies that the *bbc* number of planar graphs is at most 7.

Then the two questions follows:

1. What is the right upper bound on the *bbc* number of planar graphs: 6 or 7?
2. Can you prove the same bound without use of the Four-color theorem?

Thue type problems for graphs, points, and numbers

Jarosław Grytczuk

1 Introduction

A *repetition* is a sequence of the form $x_1x_2...x_nx_1x_2...x_n$, where n is an arbitrary positive integer. A *block* in a sequence S is any subsequence of consecutive terms of S. A finite sequence is *nonrepetitive* if none of its blocks looks like a repetition. About 100 years ago Axel Thue [8] discovered that there are arbitrarily long nonrepetitive sequences over the set $\{0, 1, 2\}$. This remarkable fact inspired many further investigations leading to a variety of *pattern avoidance* problems and a range of new types of nonrepetitiveness. Recently a number of new challenges appeared in this area connecting it more closely to classical combinatorics. Below we present four of them.

2 Graphs

A coloring of the vertices of a graph G is *nonrepetitive* if the sequence of colors on any simple path in G is nonrepetitive. The minimum number of colors needed is denoted by $\pi(G)$ and is called the *Thue chromatic number* of a graph G. Thus, the theorem of Thue asserts that $\pi(P_n) = 3$ for all $n \geq 3$, where P_n denotes a path with n edges. It follows immediately that $\pi(C_n) \leq 4$ for all cycles. Actually, $\pi(C_n) = 3$ for all $n \geq 3$ except $n = 5, 7, 9, 10, 14, 17$, as it as proved by Currie [2]. For graphs with larger maximum degree the situation is less clear. Applying the probabilistic method it can been proved that $\pi(G) \leq 100\Delta^2$ for any graph G with maximum degree at most Δ (see [1]). However, the following question remains open.

Problem 1 *Does there exist an absolute constant C such that $\pi(G) \leq C$ for any planar graph G?*

The answer is not known even for outerplanar or bipartite planar graphs. But it is known for trees, namely $\pi(T) \leq 4$ holds for any tree T. There are also some constructions showing that $C \geq 10$ if it exists at all.

3 Points

Let X be a discrete set of points in the plane. A coloring of X is *nonrepetitive* if no straight line segment looks like a repetition. Denote by $\pi(X)$ the related *Thue number* of a set X defined as the minimum number of colors needed. Again, Thue's theorem asserts that $\pi(\mathbb{Z}) = 3$.

Problem 2 *(Currie-Simpson, 2002) Is $\pi(\mathbb{Z}^2)$ finite?*

Currie and Simpson [3] proved that there is a 5-coloring of $\mathbb{Z}^2$ in which no line of one of the four directions $\pm 1, 0, \infty$ is colored repetitively.

4 Numbers

A stronger form of nonrepetitiveness is defined as follows. An *anagram repetition* is a sequence of the form $r_1 r_2 ... r_n r_{\sigma(1)} r_{\sigma(2)} ... r_{\sigma(n)}$, where σ is any permutation of n symbols. A sequence S is said to be *strongly nonrepetitive* if no block of S looks like an anagram repetition. It is easy to check that no analog of Thue's theorem is possible over the set $\{0, 1, 2\}$. In 1961 Erdős [4] asked whether four symbols will do and it was open until Keränen [7] found a computer aided confirmation. This result may be interpreted in arithmetic terms as follows: there is a function $F : \mathbb{N} \to \{2, 3, 5, 7\}$ such that $\prod_{i \in I} F(i) \neq \prod_{j \in J} F(j)$, for any two adjacent segments I and J. But what happens if we try to avoid equal *sums* instead of equal products in adjacent segments?

Problem 3 *(Halbeisen-Hungerbühler, 2000) Is there a natural number N and a function $F : \mathbb{N} \to \{1, 2, ..., N\}$ such that $\sum_{i \in I} F(i) \neq \sum_{j \in J} F(j)$ for any two adjacent segments I and J, with $|I| = |J|$?*

Van der Waerden's theorem on arithmetic progressions shows easily that without the last assumption the answer is in the negative. In fact, for any function mapping $\mathbb{N}$ into a finite subset of itself there will be arbitrarily many consecutive segments with equal sums (see [5]).

Similar problems may be investigated for other combinatorial structures as well. Consider for instance a coloring of the edges of a graph G in which the multisets of colors around adjacent vertices are never the same. A recent result of Karoński at al. [6] shows that there is a finite number K such that any connected graph G (with at least two edges) may be colored that way, using only K colors. What is the minimal value of K remains however a mystery. An additive version of this problem is even more difficult.

Problem 4 *(Karoński-Łuczak-Thomason, 2002) Is there a natural number M such that every connected graph $G = (V, E)$, with at least two edges, has an edge labelling $f : E \to \{1, 2, ..., M\}$ satisfying $\sum_{x \in N(u)} f(ux) \neq \sum_{y \in N(v)} f(vy)$, for every pair of adjacent vertices $u, v \in V$?*

Actually, they conjecture that $M = 3$ should suffice.

References

[1] N. Alon, J. Grytczuk, M. Hałuszczak, O. Riordan, Non-repetitive colorings of graphs, Random Struct. Alg. 21 (2002), 336-346.

[2] J.D. Currie, There are ternary circular square-free words of length n for $n \geq 18$, Electron. J. Combin. 9 (2002) #N10, 7pp.

[3] J.D. Currie, J. Simpson, Non-repetitive tilings, The Electr. J. Comb. **9** (2002).

[4] P. Erdős, Some unsolved problems, Magyar Tud. Akad. Mat. Kutato. Int. Kozl. **6** (1961), 221-254.

[5] L. Halbeisen, N. Hungerbühler, An application of van der Waerden's theorem in additive number theory, Integers: Electr. J. Comb. Num. Th. **0** (2000).

[6] M. Karoński, T, Łuczak, A. Thomason, Edge weights and vertex colours, manuscript.

[7] V. Keränen, Abelian squares are avoidable on 4 letters, Automata, Languages and Programming: Lecture notes in Computer science **623** (1992) Springer-Verlag 4152.

[8] A. Thue, Über unendliche Zeichenreichen, Norske Vid. Selsk. Skr., I Mat. Nat. Kl., Christiania, **7** (1906), 1-22.

An odd graph question

Tomáš Kaiser

Abstract

We propose a problem related to the Odd graph conjecture of N. L. Biggs.

Given integers m and n, the *Kneser graph* $K(m, n)$ is defined as follows. Its vertices are the n-element subsets of $[m] = \{1, \dots, m\}$, and two such subsets are joined by an edge if they are disjoint. Although the chromatic number of Kneser graphs has been known for a long time (see [3]), less information seems to be available on their chromatic index (edge-chromatic number).

In particular, an appealing long-standing conjecture due to N. L. Biggs concerns the chromatic index of *odd graphs* $O_n = K(2n-1, n-1)$. Since O_n is an n-regular graph, Vizing's theorem [5] implies that its chromatic index $\chi'(O_n)$ is either n or $n+1$. The Odd graph conjecture of Biggs [1] states that the latter is the case.

Conjecture 1 (Odd graph conjecture) *For all $n \geq 2$,*

$$\chi'(O_n) = n + 1.$$

(As observed by Naserasr [4], the account of this conjecture in the — otherwise excellent — book [2] is erroneous; it is quoted there as saying essentially the opposite.) Note that O_3, the Petersen graph, is a notorious example of cubic graph whose chromatic index is 4.

We first reformulate Conjecture 1 as a problem about systems of paths. Let $\mathcal{P}$ be a set of *directed* paths in the symmetric orientation K of the complete graph K_{2n-1}. We shall call $\mathcal{P}$ *admissible* if it has the following properties:

(1) all paths in $\mathcal{P}$ have $n-1$ vertices, and for each $(n-1)$-element subset $A \subset V(K)$, there is exactly one path $P_A \in \mathcal{P}$ with $V(P_A) = A$,

(2) for $P_A, P_B \in \mathcal{P}$, if $|A \cap B| = 1$, then the vertex $x \in A \cap B$ is at the same position in both paths (i.e., the distance of x from the beginning of each path is the same).

It is not difficult to establish a connection between the Odd graph conjecture and admissible systems of paths.

Proposition 2 *The Odd graph conjecture is equivalent to the following claim:*

> *There is no admissible system of directed paths in the symmetric orientation of K_{2n-1}, where $n \geq 2$.*

Consider now the undirected analogue of an admissible system. More precisely, define a system of *undirected* paths in K_{2n-1} to be *admissible* if it has property (1) (with K replaced by K_{2n-1}) and the obvious modification of property (2) for undirected paths. Are there any n such that K_{2n-1} has an admissible system? Indeed there are; those in K_3 and K_5 are given in Fig. 4. (In these two cases, they are unique up to an automorphism of the complete graph.)

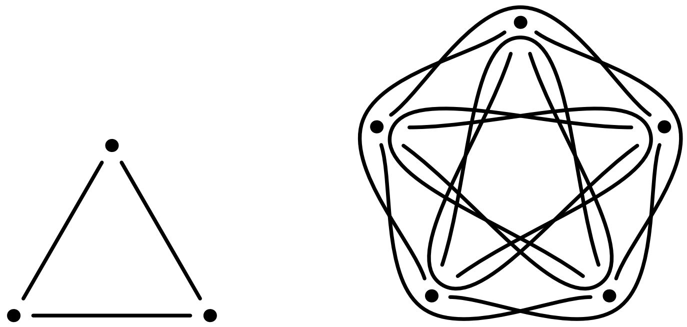

Figure 4: Admissible systems of undirected paths in K_3 and K_5.

It turns out that admissible systems of undirected paths also exist in K_7 and K_9. This suggests a surprising difference from the oriented case, and leads us to pose the following problem.

Problem 3 *Does K_{2n-1} have an admissible system of undirected paths for all $n \geq 2$?*

Acknowlegment. I thank Reza Naserasr for an inspiring conversation on the subject.

References

[1] N. L. Biggs, An edge-colouring problem. *Amer. Math. Monthly* **79** (1972), 1018–1020.

[2] T. R. Jensen and B. Toft, *Graph Coloring Problems.* J. Wiley and Sons, New York, 1995.

[3] L. Lovász, Kneser's conjecture, chromatic number, and homotopy. *J. Combin. Theory Ser. A* **25** (1978), 319–324.

[4] R. Naserasr, personal communication, 2003.

[5] V. G. Vizing, On an estimate of the chromatic class of a p-graph (in Russian). *Diskret. Analiz* **3** (1964), 23–30.

Triangle packings in chordal graphs

Jan Kára

The problem of covering a graph with graphs from some class is really broad and deeply studied topic. Common problem of this type is for example the cycle packing problem (that is covering a graph with disjoint cycles). Computing cycle packing with maximum number of cycles is NP-complete for general graphs and even for planar graphs because it contains a partition into triangles as a subproblem.

When we require all the cycles in the packing to be triangles we get the TRIANGLE PACKING problem. This problem is also known to be NP-hard. It is also known that it remains NP-hard also in the case of planar graphs and is APX-hard for graphs with maximum degree four. In chordal graphs the maximum cycle packing problem and the maximum triangle packing problem are identical since in any cycle packing in a chordal graph we can replace any cycle of size greater than three by a triangle. It was shown by Jan Kratochvíl that the maximum triangle packing problem is NP-hard for chordal graphs. We show that checking whether there exists a perfect triangle packing (a triangle packing covering all vertices of a graph) in a chordal graph is polynomial.

In the following text n will denote the number of vertices and m the number of edges of a graph $G = (V, E)$. A *triangle packing* of a graph $G = (V, E)$ is a set of triples of vertices of G $\{\{v_1^1, v_2^1, v_3^1\}, \{v_1^2, v_2^2, v_3^2\}, \ldots, \{v_1^k, v_2^k, v_3^k\}\}$ such that for each $i \in \{1, \ldots, k\}$ $\{v_1^i, v_2^i\}, \{v_2^i, v_3^i\}, \{v_1^i, v_3^i\} \in E$ and for each $i, j \in \{1, \ldots, k\}, i \neq j$ holds that $\{v_1^i, v_2^i, v_3^i\} \cap \{v_1^j, v_2^j, v_3^j\} = \emptyset$. A *perfect triangle packing* is such a triangle packing which covers all vertices of a graph — i.e., $\bigcup_{i=1}^k \{v_1^i, v_2^i, v_3^i\} = V$.

It is a well known fact that a chordal graph can be represented as an intersection graph of subtrees in a tree. To be more precise for each chordal graph $G = (V, E)$ there exists a tree T and trees $T_1, T_2, \ldots, T_n$, $T_i \subseteq T$ for each $1 \leq i \leq n$, such that $\{v_i, v_j\} \in E$ if and only if $T_i \cap T_j \neq \emptyset$. In the following text it will be more comfortable for us to consider T as a tree rooted in some of its leaves and with edges oriented from the root. The orientation of the edges of T naturally defines an orientation of its subtrees $T_1, T_2, \ldots, T_n$ and their root. We will call the ordered set $(T, r, T_1, T_2, \ldots, T_n)$ the *tree representation* of the graph G. A tree T will be called a *base tree*, trees $T_1, T_2, \ldots, T_n$ *vertex trees* and a vertex $r \in T$ *root of a tree representation.* Actually we can assume that a tree representation has the properties as written in the following lemma:

Lemma 1 *For each chordal graph $G = (V, E)$ there exists a tree representation $R = (T, r, T_1, T_2, \dots, T_n)$ such that:*

- *A root of each vertex tree is its leaf.*
- *No vertex tree has a root in a vertex of degree three or more of the base tree.*
- *No vertex in the base tree is a root of more than one vertex tree.*

Moreover this representation R will have size $O(n^2)$ and can be computed in time $O(n^2)$.

Now we introduce an algorithm which reduces the problem of finding a perfect triangle packing to the problem of maximum flow in the created network.

Let us consider a chordal graph $G = (V, E)$, n is divisible by three (otherwise the graph trivially does not have a perfect triangle packing). From Lemma 1 we know that there exists a tree representation $R = (T, r, T_1, \dots, T_n)$ of G with special properties. Now we create a *basic network* from this representation. We will number the vertices in the preorder numbering of the depth first search from the root of the representation (i.e., the numbering in which vertices receive mutually distinct numbers from one to $|V(T)|$ such that parent has always greater number than any of its children). For each vertex tree T_j on k vertices $v_{i_1}, v_{i_2}, \dots, v_{i_k}$ we add the new vertices $v_{i_1}^{T_j}, v_{i_2}^{T_j}, \dots, v_{i_k}^{T_j}$ and appropriate edges between those vertices (copied from T_i) to the basic network. We call the vertices $v_i^{T_j}, j \in \{1, \dots, n\}$ the vertices *derived from the vertex v_i*. Finally we add the source and the sink to the basic network and connect the source to all the roots (the only vertices with an input degree zero) of the created trees — for the following text we will identify the vertex trees T_j from the representation of the graph with the trees in the network induced on the vertices $v_{i_1}^{T_j}, \dots, v_{i_k}^{T_j}$. All the edges in the basic network have capacity one.

A *final network* is a network created from a basic network by choosing some vertices $u_1, \dots, u_{n/3}$ of T (we call these vertices *gathering vertices*) which are roots of some vertex trees and for each $i \in \{1, \dots, n/3\}$ connect all the vertices derived from u_i to the new vertex s_{u_i} by edges with capacity one and then connect all the vertices s_{u_i} to the sink by edges with capacity three. Sometimes we will say that the final network is *induced by the gathering vertices $u_1, \dots, u_{n/3}$*. We will show that the problem of the perfect triangle packing of the original graph G is equivalent to the existence of a flow of size at least n in some final network. First we can make an easy observation:

Lemma 2 *If a chordal graph $G = (V, E)$ has a perfect triangle packing T then there exists a final network N with a flow of size n.*

Now we will show how we can construct the set of gathering vertices (and hence the final network): We will go through the vertices of T in the increasing order (we use the numbering described above). Let v denote the actual vertex and N the network created so far. N_v is then the induced subgraph of N on the vertices $\{s, s'\} \cup \{s_{u_i} : u_i \text{ is a gathering vertex in the subtree of } v\}$ plus the vertices derived from any vertex of the subtree of v (e.g., if $v_1, v_2, \ldots, v_k$ were the vertices in the subtree of v then the derived vertices would be $v_i^{T_j}, i \in \{1, \ldots, k\}, j \in \{l : v_i \in T_l\}$).

If there is no root of any vertex tree in v we simply do nothing. If there is root of some vertex tree T_i in v (actually it is the only vertex tree which has a root in this vertex due to the third condition in Lemma 1) we will do the following: We take the network N_v, connect the source to all the roots of the trees in N_v (if they are not already connected to it) by edges with capacity one, connect all the vertices derived from v except for the root of the tree T_i to a new vertex s_0 by edges with capacity one and connect this vertex to the sink by an edge with capacity $t_v - 3 \cdot g_v$ where t_v is the number of trees in N_v and g_v is the number of gathering vertices in the subtree of v. Let N'_v denote the constructed network. If there exists a flow of size at least t_v in N'_v we do nothing and continue by processing of the next vertex. Otherwise we add the vertex v to the set of gathering vertices and check that $t_v \geq 3 \cdot g_v$ (we count also the freshly added vertex v). If the inequality holds we continue by processing of the next vertex otherwise we answer that a perfect triangle packing does not exist. If we manage to process all the vertices of T we have just constructed a final network.

Lemma 3 *Let $G = (V, E)$ be a chordal graph and N the final network created by the algorithm. Then the graph G has a perfect triangle packing.*

Theorem 1 *There exists an algorithm running in time $O(n^3)$ deciding whether a chordal graph $G = (V, E)$ has a perfect triangle packing.*

It can be shown that a maximum triangle packing in a split graph can be computed in polynomial time. Still the following interesting problem remains unsolved:

Problem 1 *Can be a maximum triangle packing in an interval graph computed in polynomial time?*

Some problems on counting permutations

Martin Klazar

Recall that a permutation $\pi = a_1a_2\dots a_m$ of $[m] = \{1, 2, \dots, m\}$ is *contained* in a permutation $\rho = b_1b_2\dots b_n$ of $[n]$, in symbols $\pi \prec \rho$, if ρ has a subsequence $b_{k_1}b_{k_2}\dots b_{k_m}$ (thus, in particular, $m \le n$) such that for every two indices i, j we have $a_i < a_j$ if and only if $b_{k_i} < b_{k_j}$. Let $S(A)$, for a set A of permutations (which may have distinct lengths and A may be infinite), be the set of all permutations ρ such that $\rho \not\succ \pi$ for every $\pi \in A$. Let $S_n(A)$ be the $\rho \in S(A)$ with length n.

Problem 1. (Stanley–Wilf conjecture, 1997). Prove that for any $A \neq \emptyset$ one has the bound $|S_n(A)| < c^n$ for all $n = 1, 2, \dots$ ($c > 1$ is a constant depending only on A).

Problem 2. Find the exact asymptotics for the numbers $|S_n(\{1324\})|$. Is there a formula or recurrence for them?

Problem 3. (I. Gessel's problem, 1990). Suppose A is finite. Is the sequence $(|S_n(A)|)_{n\ge1}$ always P-recursive?

Remarks.
P1. Clearly, it suffices to look only at the case $|A| = 1$. An almost exponential bound was proved in [Alon & Friedgut]. For further partial results see [Bóna1] and [Bóna2]. Remark added on November 19, 2003: The conjecture was proved by A. Marcus and G. Tardos, see [Marcus & Tardos].
P2. For more background see [Marinov & Radoičič] and [Bousquet-Mélou].
P3. This problem was posed, although not very explicitly, in [Gessel]. P-recursiveness of a sequence of numbers $a = (a_n)_{n\ge1} \subset \mathbf{C}$ means that there are polynomials $p_0(x), \dots, p_k(x) \in \mathbf{C}[x]$, not all zero, such that for every $n = 1, 2, \dots$ one has $p_0(n)a_n + p_1(n)a_{n+1} + \dots + p_k(n)a_{n+k} = 0$. An algebraic argument shows that if a consists of integers then also all $p_i(x)$ may be taken with integral coefficients. If A may be infinite, then an application of Cantor's diagonalization shows that the set of all counting sequences $(|S_n(A)|)_{n\ge1}$ is uncountable, and hence (for infinite A's) almost all of them are not P-recursive. (This was observed by myself in July 2003.) But probably even for finite A's there are examples of non-P-recursive counting sequences but so far nobody could present one.

References

[Alon & Friedgut] N. Alon and E. Friedgut, On the number of permutations avoiding a given pattern, *J. Comb. Theory, Ser. A* **89** (2000), 133–140.

[Bóna1] M. Bóna, Permutations avoiding certain patterns; The case of length 4 and generalizations, *Discrete Math.* **175** (1997), 55–67.

[Bóna2] M. Bóna, The solution of a conjecture of Wilf and Stanley for all layered patterns, *J. Comb. Theory, Ser. A* **85** (1999), 96–104.

[Bousquet-Mélou] M. Bousquet-Mélou, Four classes of pattern-avoiding permutations under one roof: generating trees with two labels, *Electr. J. of Combinatorics* **R19** (2003), 31 pages.

[Gessel] I. Gessel, Symmetric functions and P-recursiveness, *J. Comb. Theory, Ser. A* **53** (1990), 257–285.

[Marcus & Tardos] A. Marcus and G. Tardos, A linear bound on excluded permutation matrices, www.renyi.hu/˜amarcus or www.renyi.hu/˜tardos

[Marinov & Radoičič] D. Marinov and R. Radoičič, Counting 1324-avoiding permutations, *Electr. J. of Combinatorics* **R13** (2003), 9 pages.

Closure for the Property of Having a Hamiltonian Prism

Daniel Král'
(joint work with Ladislav Stacho)

A graph has a *Hamilton cycle* if it contains a spanning cycle. A couple of generalizations of the concept of a Hamilton cycle were introduced, among them, so-called k-walks. A *k-walk* is a closed spanning walk which visits each vertex at most k times (thus a Hamilton cycle is a 1-walk) and a *k-tree* is a spanning tree with maximum degree k. It is not hard to show that a graph which has a k-tree has also a k-walk and a graph which has a k-walk has a $(k+1)$-tree. Hence, the properties of "having a k-walk" and "having a k-tree" are interlaced in the following sense:

$$\text{1-walk} \Rightarrow \text{2-tree} \Rightarrow \text{2-walk} \Rightarrow \text{3-tree} \Rightarrow \text{3-walk} \ldots$$

Recently, another property sandwiched between "having a 2-tree", i.e., a Hamilton path, and "having a 2-walk" has attracted attention of researchers. This property is that the prism of a graph is hamiltonian. The *prism* of a graph G is the graph obtained from two copies of G by connecting all the pairs of images of the same vertex by an edge.

One of concepts which does not obviously translate to the case of hamiltonian prisms is the concept of graph closures. A *k-closure* of a graph G, denoted by $\mathrm{Cl}_k(G)$, is the unique graph obtained from G by recursively joining pairs of non-adjacent vertices whose degree sum is at least k until no such pair remains. A graph property is called k-stable if G has the property if and only if $\mathrm{Cl}_k(G)$ has. The motivation for this concept comes from the original closure of Bondy and Chvátal developed for Hamilton cycles: A graph G of order n is hamiltonian if and only if $\mathrm{Cl}_n(G)$ is hamiltonian and it is known that this cannot be weaken to $\mathrm{Cl}_{n-1}(G)$, i.e., the property of "having a Hamilton cycle" is n-stable but not $(n-1)$-stable. It is also known that a property of "having a k-walk" for $k \geq 2$ is $(n-1)$-stable but not $(n-2)$-stable.

Our main result is that the prism of a graph G of order n is hamiltonian if and only if the prism of $\mathrm{Cl}_k(G)$ is hamiltonian for $k = 4n/3 - 4/3$. A double counting argument (formulated using the discharging method) is used to show that the property is k-stable with $k = 4n/3 - 1$ and this is then improved to $k = 4n/3 - 4/3$ by a little technical case analysis. We strongly believe that this can be further improved by more tedious case analysis. On the other hand, the property of "having a hamiltonian prism" is not k-stable for $k = 4n/3 - 16/3$. We pose the following conjecture, mainly to encourage the research to close the (quite small) gap between the upper and the lower bound, that the lower bound is tight:

Conjecture 1 *The property of "having a hamiltonian prism" is k-stable with $k = 4n/3 - 5$ for graphs of order n and this cannot be further improved.*

The Hamilton-Waterloo problem

Roman Nedela

(joint work with P. Horák and A. Rosa)

Problem. Is there a decomposition of K_m into 2-factors such that exactly one of the factors is a Hamiltonian cycle and all the other $(m-3)/2$ factors are triangle factors?

Note that if there is a solution for an integer m then $m = 6k + 3$ for some integer k. It is known [FR] that for $m = 9$ there is no solution. For small values of $m = 6k+3 \geq 15$ the solution was shown to exists for $m = 15$ in [FMR], for $m = 21$ in [M], and for $m = 39$ and 57 in [HNR]. Applying a triplicating lemma proved in [HNR] one can construct solutions for integers of the form $m = b \cdot 3^a$, where $b \in \{5, 7, 13, 19\}$ and $a > 0$ is an integer. Hence the least number $m \geq 15$ congruent to 3 modulo 6 for which the existence of a solution was not confirmed is $m = 27$.

History and background. The Oberwolfach problem first formulated by Ringel at a meeting in Oberwolfach asks for 2-factorization of the complete graph K_{2n+1} into 2-factors each of which is isomorphic to a given 2-factor Q. If Q is a disjoint union of cycles of lengths $c_1, c_2, \ldots, c_s$ with $2n+1 = \sum_{i=1}^{s} c_i$ then the corresponding instance of the Oberwolfach problem is denoted by $OP(2n+1; c_1, c_2, \ldots, c_s)$. It has been conjectured that a solution to $OP(c_1, c_2, \ldots, c_s)$ exists always except $OP(9; 4, 5)$ and $OP(11; 3, 3, 5)$ when the solution is known not to exist. The prescribed 2-factor Q will be called *uniform* if all components of Q have the same (necessarily odd) length. The Oberwolfach problem has been completely settled in the case of uniform Q, a solution has been shown to exist in all such cases [ASSW] (see also [A]). However, the Oberwolfach problem remains open in general.

The Hamilton-Waterloo problem is an extension of the Oberwolfach problem. It asks for a 2-factorization of the complete graph K_{2n+1} such that r of the factors are isomorphic to a given 2-factor R and q of the 2-factors are isomorphic to a given 2-factor Q, with $r + q = n$. For $q = 0$ or $r = 0$ we get the Oberwolfach problem. The corresponding instance of the Hamilton-Waterloo problem problem will be denoted $HW(q, r; Q, R)$. A special subcase when R is a triangle factor and Q is a Hamiltonian cycle we denote by $HW^*(q, r)$. Note that $HW^*(0, n)$ is the classical question on the existence of a Kirkmann triple system on n-points (see [CR]) while

$HW^*(n,0) = OP(2n+1; 2n+1, \ldots, 2n+1)$ (see [ASSW] for a solution). Hence we can assume $r > 0$ implying that the number of vertices m is congruent to 3 modulo 6. Given $m = 6k+3$ we set $HW^*(m)$ to be the set of integers r, $0 < r \leq (m-1)/2$ such that $HW^*(r, (m-1)/2-r)$ has a solution. Denote by $I(m)$ the integer interval $[1, \ldots, (m-1)/2]$. It is conjectured in [HNR] that $HW^*(m) = I(m)$ for all $m = 6k+3 > 9$. It is proved there that if $k = 1 \bmod 3$ then $I(n) \setminus \{1\} \subseteq I(n)$. Hence to verify the conjecture for m congruent to 9 modulo 18 it is sufficient to find the decomposition of K_m into one Hamiltonian cycle and $(m-3)/2$ triangle factors.

References

[ABBE] P.Adams, E.J.Billington, D.E.Bryant, S.I.El-Zanati, On the Hamilton-Waterloo problem, Graphs Combin. **18** (2002), 31–51

[A] B.Alspach The Oberwolfach problem The CRC Handbook of Combinatorial Designs (Ed.C.J.Colbourn, J.H.Dinitz), CRC Press, 1996.

[ASSW] B.Alspach, P.J.Schellenberg, D.R.Stinson, D.Wagner The Oberwolfach problem and factors of uniform odd length cycles J. Combin. Theory (A) **52**, (1989), 20–43.

[BGG] G.K.Bennett, M.J.Grannell, T.S.Griggs Bi-embeddings of the projective space PG(3,2) J. Statist. Plan. Infer. **86** (2000), 321–329.

[B] J. Bosák, Decompositions of Graphs, Kluwer, 1990.

[CR] C.J.Colbourn, A.Rosa, Triple Systems, Oxford Univ. Press, 1999.

[FMR] F.Franek, R.Mathon, A.Rosa Maximal sets of triangle-factors in K_{15} J. Combin. Math. Combin. Comput. **17** (1995), 111–124.

[FR] F.Franek, A.Rosa Two-factorizations of small complete graphs J. Statist. Plan. Infer. **86** (2000), 435–442.

[G] P.Gvozdjak, On the Oberwolfach problem for complete multigraphs, Discrete Math. **173** (1997), 61–69.

[HNR] P.Horák, R.Nedela, A. Rosa, The Hamilton-Waterloo problem: The case of Hamilton cycles and triangle-factors, Submitted.

[HKR] C.Huang, A.Kotzig, A.Rosa, On a variation of the Oberwolfach problem, Discrete Math. **27** (1979), 261–277.

[M] M.Meszka, personal information.

[P] W.Piotrowski, The solution of the bipartite analogue of the Oberwolfach problem, Discrete Math. **97** (1991), 339–356.

[RS] R.Rees, D.R.Stinson, On the existence of Kirkman triple systems containing Kirkman subsystems, Ars Combin. **26** (1988), 3–16.

An AntiSzemerédi Problem

Jaroslav Nešetřil

Is it true that for every k there exists $\varepsilon_k > 0$ and a set $X^{(k)}$ of natural numbers such that

(1) For every finite partition $X^{(k)} = \mathcal{A}_1 \cup \cdots \cup \mathcal{A}_t$ one of the classes $\mathcal{A}_i$ contains an arithmetic progression with k-terms;

(2) For every finite set $Y \subset X^{(k)}$ there exists a subset $Z \subset Y$ such that

  (i) $|Z| \geq \varepsilon_k |Y|$

  (ii) Z contains no arithmetic progression with k-terms.

Comments

(a) By celebrated Szemerédi theorem $X^{(k)}$ has density 0. It is easy to construct a Van der Waerden set X (i.e. set with (1)) which has density 0. But (2) asks for "hereditary density".

(b) The problem seems to be first formulated by Erdős, Nešetřil and Rödl in [1], [2] in the context of *Pisier-type theorems*. Famous Pisier problem [3] asks a similar problem for sets with all subsetsums distinct. See [1, 2] for other Pisier-type statements.

References

[1] P. Erdős, J. Nešetřil, V. Rödl: On Pisier type Problems and Results (Combinatorial applications to number theory). In: Mathematics of Ramsey theory (J. Nešetřil, V. Rödl, eds.), Springer Verlag (1990), 214–231.

[2] P. Erdős, J. Nešetřil, V. Rödl: On coloring and independent sets (Pisier type theorems), KAM-DIMATIA Series 96-320 (to appear in Combinatorica)

[3] G. Pisier: Arithmetic characterization of Sidon sets, Bull. Amer. Mat. Soc. 8(1983), 87–89.

MAC

Jaroslav Nešetřil

Graphs (oriented or undirected) G_1, G_2 are called (big) MAC if the following holds:

(1) There are no homomorphisms $G_1 \to G_2$ and $G_2 \to G_1$ (i.e. $\{G_1, G_2\}$ is an antichain in the homomorphism order)

(2) For every G there exist i such that either $G \to G_i$ or $G_i \to G$ (i.e. $\{G_1, G_2\}$ is maximal antichain in the homomorphism order).

For finite graphs holds (with trivial exceptions):

Theorem [1]: *$\{G_1, G_2\}$ is MAC if and only if (G_1, G_2) is a dual pair.*
(Here dual pair in the sense of homomorphism duality, see [1], [2]).

For countable graphs there are also examples of MAC, for example $\{K_3, U_3\}$ where U_3 is the universal triangle free (Henson) graph.
Homomorphism order for countable graphs was studied e.g. in [3]. This motivates the following

Problem: *Let $\{G_1, G_2\}$ be a MAC, G_1, G_2 countable graphs. Then one of the graphs G_i is finite.*

This is open even in the case of countable oriented graphs.

References

[1] J. Nešetřil, C. Tardif: On maximal finite antichains in the homomorphism order of directed graphs, Disscussiciones Math. 23,2(2003), 325–333.

[2] J. Nešetřil, C. Tardif: Duality theorems for finite structures (characterizing gaps and good characterizations), J. Comb. Th. B, 80(2000), 80–97.

[3] J. Nešetřil, S. Shelah: On the order of countable graphs, European J. Comb. 24(2003), 649–663.

NP-completeness Problem

Yared Nigussie

Abstract

The complexity of languages for membership of a graph G in a topological-minor closed set $\mathcal{I}$ can be stated in a simplified manner using finite sequences of 0's and 1's. For a fixed $\mathcal{I}$, determining membership of a given graph is known to be polynomial time solvable, as long as sufficient information about $\mathcal{I}$, (for instance the forbidden minors of $\mathcal{I}$) is given as input. However, when the set $\mathcal{I}$ varies with the input, the complexity of the general problem is not quite clear. Formally, we consider a language $L = \{< G, \mathcal{I}_n >: G \in \mathcal{I}_n, n \geq 1\}$, where the sequence $\mathcal{I}_1, \mathcal{I}_2, \ldots$ is defined in a certain way. We prove a language called LS1 to be NP-complete and present a slightly different language LS2 in NP. Deciding the class of LS2 is remarkable for reasons we explain later. We conjecture LS2 to be NP-hard.

The simplified languages LS_1 and LS_2 are stated as follows:

Definition Let $s = [u_1, u_2, \ldots, u_p]$, $s' = [u'_1, u'_2, \ldots, u'_{p'}]$, where $u_i, u'_j \in \{0,1\}, 1 \leq i \leq p, 1 \leq j \leq p'$. We write $s \leq_M s'$ if there exists a subsequence s'' of s', $s'' = [u'_{i_1}, u'_{i_2}, \ldots, u'_{i_p}], 1 \leq i_1 < i_2 < \cdots < i_p \leq p'$, such that $u_j \leq u'_{i_j}$ for all $j = 1, 2, \ldots, p$, under the usual integer inequality $\leq$. Let $S = [s_1, s_2, \ldots s_n]$ and $S' = [s'_1, s'_2, \ldots, s'_{n'}]$ be sequences of $\{0,1\}$-sequences. We write $S \leq_{1-1} S'$ if there exists an injective map $f : S \to S'$ such that $s_i \leq_M f(s_i), \forall i, 1 \leq i \leq n$.

$LS1 = \{< \mathcal{S}_m, \mathcal{S}^q >: \mathcal{S}_m = [S_1, S_2, \ldots, S_m]$, S_i is a sequence of $\{0,1\}$-sequences, $\mathcal{S}^q = \{S^1, S^2, \ldots, S^q\}$, where $S^i = \{x_i, y_i\}$, x_i and y_i are $\{0,1\}$-sequences, and there exist a choice $z_1, z_2, \ldots, z_q$, $z_i \in \{x_i, y_i\}$ such that $[z_1, z_2, \ldots, z_q] \not\leq_{1-1} S_i$, for all $i, 1 \leq i \leq m\}$.

$LS2 = \{< \mathcal{S}_m, \mathcal{S}^q >: \mathcal{S}_m = [S_1, S_2, \ldots, S_m]$, S_i is a sequence of $\{0,1\}$-sequences, $\mathcal{S}^q = \{S^1, S^2, \ldots, S^q\}$, where $S^i = \{x_i, y_i\}$, x_i and y_i are $\{0,1\}$-sequences, and there exist a choice $z_1, z_2, \ldots, z_q$, $z_i \in \{x_i, y_i\}$ such that $[z_1, z_2, \ldots, z_q] \not\leq_{1-1} [s_1, s_2, \ldots, s_m]$, for any m-tuple such that $s_i \in S_i, 1 \leq i \leq m\}$.

We prove next that $LS1$ is NP-complete.

Theorem 1 $LS1 \in NP$.

proof Let an instance $I =< \mathcal{S}_m, \mathcal{S}^q >$ be given. If $I \in LS1$, then there exists a choice $\{z_1, \dots, z_q\}, z_i \in \{x_i, y_i\}$, such that $[z_1, z_2, \dots, z_q] \not\leq_M S_i$ for all $i, i = 1, 2, \dots, m$. A guess gives as a certificate $c = [z_1, \dots, z_q]$. A verifier needs to check if $z_i \in \{x_i, y_i\}$, for $i = 1, 2, \dots, q$, and check if $c \not\leq_{1-1} S_i$ for $i = 1, 2, \dots, m$, which can be done in polynomial time in the size $|\mathcal{S}_m| + |\mathcal{S}^q|$. $\diamond$

Theorem 2 $LS1$ is NP-hard.

proof We show that 3-SAT can be reduced to LS1. Let Φ be a formula with k clauses $C = \{c_1, \dots, c_k\}$ and n variables $X = \{x_1, x_2, \dots, x_n\}$. Using C and X, we construct $2n$, $\{0,1\}-$sequences, $\{\{x_1, y_1\}, \dots, \{x_n, y_n\}\} = \mathcal{S}^n$ as follows: For $i = 1, \dots, n$, let x_i and y_i be initially sequences of 0's of length k. For $j = 1, 2, \dots, k$, replace 0 by 1 at the jth index of x_i if $x_i \in c_j$, and of y_i if $\overline{x_i} \in c_j$.

We next construct the elements $S_1, S_2, \dots, S_k$ of $\mathcal{S}_k$ as follows: For $i = 1, 2, \dots k$, let $S_i = [s_{i,1}, s_{i,2}, \dots, s_{i,n}]$ be initially sequence of 1's of length k. At the ith index of $s_{i,j}$ replace 1 by a 0. Note that $s_{1,1} = s_{1,2} = \cdots = s_{1,n} = [0, 1, 1, \dots, 1]$. , $s_{2,1} = s_{2,2} = \cdots = s_{2,n} = [1, 0, 1, 1, \dots, 1]$ and so on.

We have an instance $\mathcal{S}_k = [S_1, \dots, S_k]$ and $\mathcal{S}^n = \{\{x_1, y_1\}, \dots \{x_n, y_n\}\}$. We have constructed $2n$ sequences that are of size k, and $\mathcal{S}_k$ which has size $k^2 n$, with $O(k^2 n)$ total size. Hence the reduction is polynomial in k and n. We show $\Phi \in$ 3-SAT if and only if $< \mathcal{S}_k, \mathcal{S}^n > \in$ LS1.

Assume $\Phi \in$ 3-SAT. Then Φ has a satisfying assignment. Pick a satisfying assignment. For $i = 1, \dots, n$, if x_i is assigned true, let $z_i = x_i$, and if x_i is assigned false, let $z_i = y_i$. Let $Z = [z_1, \dots, z_n]$. We show $Z \not\leq_{1-1} S_i$, for all $i, i = 1, 2, \dots, k$.

We claim that for each $i, 1 \leq i \leq k$, Z has at least one sequence with ith entry 1. Fix some $j, 1 \leq j \leq k$, and consider the clause c_j. Then some x_i or $\bar{x_i}$ in c_j is valued TRUE, since Φ is satisfied. If it is x_i, then x_i is assigned TRUE and by our choice $z_i = x_i$, and there is a 1 at the jth entry, by construction of x_i. If it is $\bar{x_i}$ that is TRUE, then x_i is assigned FALSE and so $z_i = y_i$, by our choice, and so y_i has a 1 at the jth entry. But then every sequence in S_i has all 0's at the ith entry. Hence the claim follows, and so $< \mathcal{S}_k, \mathcal{S}^n > \in$ LS1.

Conversely, assume there is a $Z = [z_1, \dots, z_n]$, such that $z_i \in \{x_i, y_i\}$ and $Z \not\leq_{1-1} S_i, i = 1, 2, \dots, k$. By the construction of $\mathcal{S}_k$, we deduce that Z has at least one 1 at some sequence jth entry, for all $1 \leq j \leq k$.

Now if $z_i = x_i$ assign x_i =TRUE. If $z_i = y_i$, assign x_i =FALSE. Take any clause c_j, $1 \leq j \leq k$. By the above observation, there is some z_i that has an entry 1 at the jth entry. If $z_i = x_i$, then x_i is in c_j and x_i is assigned TRUE and hence c_j is satisfied. If $z_i = y_i$ then $\bar{x_i}$ is in c_j and x_i is assigned FALSE and so c_j is satisfied. Hence $\Phi \in$ 3-SAT. $\diamond$

Theorem 3 $LS2 \in NP$.

proof This is obvious since one can use maximal matching algorithm from a given guess $c = [z_1, z_2, \ldots, z_q]$ to $[S_1, S_2, \ldots, S_k]$ under the relation $\leq_m$, to check if c is a valid certificate. If the maximal matching number $M = q$ rejects otherwise accepts. $\diamond$

Conjecture LS2 is NP-hard.

A result that determines if either LS2 is NP-complete or in P has an interesting implication. This is because, if $LS2$ is NP-complete, then unlike most other NP-complete problems, the complement of LS2 seems to have an easy to verify certificate. On the other hand, if one proves LS2 to be in P, it is by far more surprising because some computer generated tests of possible number of certificates, for example problems with small n, tend to grow faster than, with a very generous lower bound, double exponential functions.

Balanced bicoloration, Homomorphism and Graph Dimension

Patrice Ossona de Mendez
(joint work with Pierre Rosenstiehl)

Abstract

The dimension of a graph, that is the dimension of its incidence poset, became a major bridge between posets and graphs. Although allowing a nice characterization of planarity, this dimension badly behaves with respect to homomorphisms.

We introduce the *universal dimension* of a graph G as the maximum dimension of a graph having a homomorphism to G. The universal dimension, which is clearly homomorphism monotone, is related to the existence of some balanced bicoloration of the vertices with respect to some realizer.

Non trivial new results related to the original graph dimension are subsequently deduced from our study of universal dimension, including chromatic properties, extremal properties and a disproof of two conjectures of Felsner and Trotter.

Introduction

Half a century ago, from the concept of dimension of a partial order [9], Dushnik started an investigation of the poset of one- and k- element subsets of $[\,n\,] = \{1, \ldots, n\}$ [8], and introduced a construction which has a natural interpretation in terms of abstract simplicial complexes, and which has been used or reintroduced in several contexts, like mathematical economy [18], integer programming [2], or commutative algebra [3].

The dimension of the poset of one- and two-element subsets of $[\,n\,]$ (the complete graph incidence poset) has been particularly investigated by Spencer [20], Füredi, Hajnal, Rödl and Trotter [12] and, more recently, by Hoşten and Morris [13].

The study of the dimension of the incidence poset of graphs (other than complete ones) have gained a particular attention these last years, since it was proved by W. Schnyder that the incidence poset of a graph has dimension at most 3 if and only if the graph is planar [19]. This theorem has been generalized by Brightwell and Trotter [6][7] to a characterization of the dimension of the vertex, edges and faces posets of planar graphs, and to a generalization to multigraphs.

Although the generalization of Schnyder's result to the problem of the geometric realization of an abstract simplicial complex is quite natural [16] and may be viewed as the existence of embeddings of posets in Euclidean space with a general separation property [17], the property of the class of planar graphs to be minor closed disappears in dimension at least 4. The incidence poset dimension is not homomorphism monotone (any non planar graph with an homorphism to K_4 has dimension 4, although K_4 has dimension 3). In this paper, we introduce a new graph dimension, the *universal dimension* of a graph, which will make clear this connection. We will prove that this universal dimension differs by at most two from the incidence poset dimension, while being (by its definition) homomorphism monotone.

We will relate the universal dimension of a graph to the existence of realizers admitting a special bicoloration and will give some applications of our results, like the non-bounding of chromatic numbers of graphs of dimension 4, or extremal results about graphs of incidence poset dimension d, in the framework investigated by Bollobás and Thomason [5]. We also disprove two conjectures of Felsner and Trotter.

Definitions and Notations

In the following we will only consider finite simple loopless graphs with at least one edge.

Let $P = (X, \leq)$ be a finite poset. A *realizer* of P is a nonempty family $\mathcal{R} = (<_1, \ldots, <_t)$ of linear orders on X whose intersection $\bigcap_{i=1}^{t} <_i$ is P. The *dimension* of P is the minimum cardinality of a realizer of P [9]. *Homomorphisms* of graphs are adjacency preserving maps: a map $f : V(H) \to V(G)$ is a homomorphism of the graph H to the graph G if $\{f(x), f(y)\} \in E(G)$ whenever $\{x, y\} \in E(H)$. We will denote by $H \to G$ the existence of a homomorphism of H to G. The *chromatic number* $\chi(H)$ of a graph H is thus the smallest integer n, such that H has a homomorphism to K_n.

Definition 1 *Let G be a graph. The* incidence poset P_G of G *is the height 2 poset having $V(G) \cup E(G)$ as its ground set, such that $a < b$ in P_G if $a \in V(G)$, $b \in E(G)$ and b is incident to a.*

Definition 2 *Let G be a graph. A* graph realizer *of G of size t is a family $\mathcal{R} = (<_1, \ldots, <_t)$ of linear orders on $V(G)$, such that:*

$$\begin{aligned} \forall x \neq y \in V(G), &\quad \exists i \in [\,t\,], x <_i y \\ \forall \{x,y\} \in E(G), \forall z \in V(G) \setminus \{x,y\}, &\quad \exists i \in [\,t\,], x <_i z \text{ and } y <_i z \end{aligned}$$

The dimension $\dim G$ *is the smallest size of a graph realizer of G.*

Remark 1 *Notice that $\dim G = \dim P_G$. The reduction of the ground set to the vertex set of the graph already appeared in [8].*

Remark 2 *Let G be a graph and let H be a subgraph of G. Then, the restriction of the linear orders of a graph realizer of G to $V(H)$ defines a graph realizer of H:*

$$H \subseteq G \Longrightarrow \dim H \leq \dim G$$

The universal dimension of a graph

Definition 3 *Let G be a graph. The* universal dimension $\operatorname{udim} G$ *is the supremum of the dimensions of the graphs having a homomorphism to G:*

$$\operatorname{udim} G = \sup_{H \to G} \dim H \tag{2}$$

Remark 3

$$H \to G \Longrightarrow \operatorname{udim}(H) \leq \operatorname{udim}(G)$$

Definition 4 *Let $\mathcal{R} = \{<_1, \ldots, <_t\}$ be a graph realizer of a graph G. A* balanced bicoloration *of G with respect to $\mathcal{R}$ is a map $\Gamma : V(G) \times [\,t\,] \to \{0,1\}$ such that, for every vertex x and for every vertex y adjacent to x (in G), there exists $i, j \in [\,t\,]$ so that*

$$\begin{cases} y <_i x \text{ and } \Gamma(x,i) = 0 \\ y <_j x \text{ and } \Gamma(x,j) = 1 \end{cases} \tag{3}$$

When such a map exists, $\mathcal{R}$ is said to be bicolorable.

Definition 5 *Following the definition of [14], the* multiplication $G^{::}(W_1, \ldots, W_n)$ *of a graph G with vertex set $V(G) = \{v_1, \ldots, v_n\}$ is defined by:*

$$\begin{aligned} & V(G^{::}) = W_1 \cup \cdots \cup W_n, \\ \forall 1 \leq i \leq n, \quad & |\,W_i\,| \geq 1 \\ \forall 1 \leq i < j \leq n, \quad & W_i \cap W_j = \emptyset, \\ \forall 1 \leq i \leq j \leq n, \forall u \in W_i, \forall v \in W_j, \quad & \{u,v\} \in E(G^{::}) \text{ iff } \{v_i, v_j\} \in E(G) \end{aligned}$$

The sets $W_1, \ldots, W_n$ are called the multivertices *corresponding to vertices $v_1, \ldots, v_n$, respectively.*

Theorem 1 *Let G be a graph of order n, let $d > 3$ be an integer, and let $G^+ = G^{::}(W_1, \ldots, W_n)$ be the multiplication of G, where $\left| W_i \right| = (2i-1)^d+1$. Then, the following assertions are equivalent:*

1. *G has a bicolorable graph realizer of size d.*
2. *$\operatorname{udim} G \leq d$,*
3. *$\dim G^+ \leq d$,*

Thus, the universal dimension $\operatorname{udim} G$ of a graph G is the smallest size of a bicolorable graph realizer of G.

Corollary 1 *Let $1 < k < d$ be integers. Assume G has a graph realizer $(<_1, \ldots, <_d)$ such that no two adjacent vertices are comparable in $\prod_{i=1}^{k} <_i$ or $\prod_{i=k+1}^{d} <_i$. Then $(<_1, \ldots, <_d)$ is bicolorable and thus $\operatorname{udim} G \leq d$.*

Corollary 2 *For any graph G, $\dim G \leq \operatorname{udim} G \leq \dim G + 2$*

Theorem 2 *Let t be a non null integer. Then,*

$$\dim K_{2t+\operatorname{udim} K_t} \leq \operatorname{udim} K_t \leq \dim K_{t+1} + 1 \tag{4}$$

Remark 4 *Let $p \geq 3$. If $\dim K_{p+1} > \dim K_p$, then $\dim K_{2(p-1)} = \operatorname{udim} K_{p-1} = \dim K_p + 1$. Thus, Theorem 2 is optimal for an infinite set of graphs.*

Problem 1 *Let G be a graph with dimension at least 3. Is it true that $\operatorname{udim}(G) \leq \dim(G) + 1$?*

Chromatic Number

Theorem 3 ([15]) *For each $k > 0$, there exists a finite set X and and linear orderings $<_1, <_2$ on X such that the Hasse diagram H of $<_1<_2$ has chromatic number at least k.*

Theorem 4 *For any integer $k \geq 2$, there exists a graph G_k with chromatic number at least k, universal dimension 4, and having a realizer of the form $(<_1, <_2, \overline{<_1}, \overline{<_2})$.*

Lemma 1 *Let $P = (X, \leq)$ be a poset of dimension at least 2 and let $H(P)$ be its Hasse diagram. Then, $H(P)$ has a graph realizer of the form $(<_1, \ldots, <_{\dim P}, \overline{<_1}, \ldots, \overline{<_{\dim P}})$ and thus $\operatorname{udim}(H(P)) \leq 2 \dim P$.*

Extremal Graphs

The *speed* $\left|(\dim \le d)^n\right|$ of the property $(\dim \le d)$ is the number of graphs of order n having dimension at most d.

The *size* $e_{\dim\le d}(n)$ of the property $(\dim \le d)$ (resp. $(\operatorname{udim} \le d)$) is the maximum number of edges a graph G can have when it has n vertices and dimension at most d.

Theorem 5 (Bollobás, Thomason [5]) *Let $\mathcal{P}$ be a hereditary property of graphs and let $\mathcal{P}^n$ be the set of graphs in $\mathcal{P}$ with vertex set $[n]$. Then*

$$\left|\mathcal{P}^n\right| = 2^{(1-1/r+o(1))n^2/2}$$

where $r = r(\mathcal{P})$ is the coloring number of $\mathcal{P}$.

Theorem 6 *Let $d \ge 4$ be an integer, and let t be the greatest integer such that* $\operatorname{udim} K_t = d$*. Then the coloring number of the property* $(\dim \le d)$ *is*

$$r(\dim \le d) = t$$

Thus, the number of labeled graphs of order n having dimension at most d is $2^{(1-1/t+o(1))n^2/2}$

Known results:

$$
\begin{aligned}
e_{\dim\le 2}(n)/n^2 &= n-1 && \text{(paths)}\\
e_{\dim\le 3}(n)/n^2 &= 3n-6 && \text{from Schnyder's theorem}\\
\lim_{p\to\infty} e_{\dim\le 4}(n)/n^2 &= 3/8 && [1]
\end{aligned}
$$

Theorem 7 *Let $d \ge 4$ be an integer, and let t be the greatest integer such that* $\operatorname{udim} K_t = d$*. Then:*

$$e_{\dim\le d}(n) = \left(1-\frac{1}{t}\right)\binom{n}{2} + O(n^{3/2})$$

From $\operatorname{udim} K_{30} = 5, \operatorname{udim} K_{1088} = 6$ and $\operatorname{udim} K_{583507} = 7$, we get:

$$1-\frac{1}{30} \le \lim_{p\to\infty}\frac{e_{\dim\le 5}(n)}{\binom{n}{2}} \le 1-\frac{1}{38}$$

$$1-\frac{1}{1088} \le \lim_{p\to\infty}\frac{e_{\dim\le 6}(n)}{\binom{n}{2}} \le 1-\frac{1}{1320}$$

$$1-\frac{1}{583507} \le \lim_{p\to\infty}\frac{e_{\dim\le 7}(n)}{\binom{n}{2}} \le 1-\frac{1}{712728}$$

$<_1$:	17	15	16	5	$\overline{\underline{1}}$	$\overline{\underline{2}}$	$\overline{\underline{3}}$	$\overline{\underline{4}}$	$\overline{\underline{7}}$	$\overline{\underline{6}}$	12	$\overline{\underline{8}}$	$\overline{\underline{9}}$	$\overline{\underline{10}}$	$\overline{\underline{11}}$	$\overline{\underline{14}}$	$\overline{\underline{13}}$	$\overline{\underline{21}}$	$\overline{\underline{22}}$	$\overline{\underline{18}}$	$\overline{\underline{19}}$	$\overline{\underline{20}}$	28	27	25	$\overline{\underline{23}}$	$\overline{\underline{24}}$	$\overline{\underline{26}}$	$\overline{\underline{29}}$	$\overline{\underline{30}}$
$<_2$:	20	18	19	$\overline{\underline{21}}$	$\overline{\underline{22}}$	6	$\overline{\underline{7}}$	2	1	4	3	$\overline{\underline{5}}$	13	$\overline{\underline{14}}$	9	8	11	10	$\overline{\underline{12}}$	$\overline{\underline{15}}$	$\overline{\underline{16}}$	$\overline{\underline{17}}$	30	29	26	24	23	$\overline{\underline{25}}$	$\overline{\underline{27}}$	$\overline{\underline{28}}$
$<_3$:	26	25	23	24	28	27	30	29	7	6	5	3	4	1	2	14	13	12	10	11	8	9	17	16	15	20	19	18	22	21
$<_4$:	29	30	27	28	23	24	25	26	21	22	18	19	20	15	16	17	8	9	10	11	12	13	14	1	2	3	4	5	6	7
$<_5$:	$\overline{\underline{30}}$	$\overline{\underline{29}}$	$\overline{\underline{28}}$	$\overline{\underline{27}}$	$\overline{\underline{26}}$	$\overline{\underline{25}}$	$\overline{\underline{24}}$	$\overline{\underline{23}}$	$\overline{\underline{22}}$	$\overline{\underline{21}}$	$\overline{\underline{20}}$	$\overline{\underline{19}}$	$\overline{\underline{18}}$	$\overline{\underline{17}}$	$\overline{\underline{16}}$	$\overline{\underline{15}}$	$\overline{\underline{14}}$	$\overline{\underline{13}}$	$\overline{\underline{12}}$	$\overline{\underline{11}}$	$\overline{\underline{10}}$	$\overline{\underline{9}}$	$\overline{\underline{8}}$	$\overline{\underline{7}}$	$\overline{\underline{6}}$	$\overline{\underline{5}}$	$\overline{\underline{4}}$	$\overline{\underline{3}}$	$\overline{\underline{2}}$	$\overline{\underline{1}}$

Figure 5: A bicolorable graph realizer of K_{30}

Problems

For any $k \geq 2$ and any $d \geq 4$, does there exists a finite k-chromatic graph $U_{d,k}$ of universal dimension d such that any k-chromatic graph of universal dimension d has a homomorphism to $U_{d,k}$?

Let G be an extremal graph of universal dimension d (i.e.: $\operatorname{udim} G = d$, but the addition of any edge to G increases the universal dimension). Then, is the complementary graph of G disconnected?

References

[1] G. Agnarsson, S. Felsner, and W.T. Trotter, *The maximum number of edges in a graph of bounded dimension, with application to ring theory*, Discrete Mathematics **201** (1999), 5–19.

[2] L. Barany, R. Howe, and H. Scarf, *The complex of maximal lattice free simplices*, Mathematical Programming **66 (Ser. A)** (1994), 273–281.

[3] D. Bayer, I. Peeva, and B. Sturmfels, *Monomial resolutions*, Math. Res. Lett. (1998), no. 5, 31–46.

[4] B. Bollobás, *Extremal graph theory*, Handbook of Combinatorics (R.L. Graham, M. Grötschel, and L. Lovász, eds.), vol. 2, Elsevier, 1995.

[5] B. Bollobás and A. Thomason, *Hereditary and monotone properties of graphs*, The mathematics of Paul Erdös II (R.L. Graham and J. Nešetřil, eds.), Algorithms and Combinatorics, vol. 14, Springer, Berlin, 1997, pp. 70–78.

[6] G. Brightwell and W.T. Trotter, *The order dimension of convex polytopes*, SIAM Journal of Discrete Mathematics **6** (1993), 230–245.

[7] ———, *The order dimension of planar maps*, SIAM jouranl on Discrete Mathematics **10** (1997), no. 4, 515–528.

[8] B. Dushnik, *Concerning a certain set of arrangements*, Proceedings of the AMS, vol. 1, 1950, pp. 788–796.

[9] B. Dushnik and E.W. Miller, *Partially ordered sets*, Amer. J. Math. **63** (1941), 600–610.

[10] P. Erdös and A.H. Stone, *On the structure of linear graphs*, Bulletin Amer. Math. Soc. **52** (1946), 1087–1091.

[11] S. Felsner and W.T. Trotter, *Posets and planar graphs*, submitted.

[12] Z. Füredi, P. Hajnal, V. Rödl, and W.T. Trotter, *Interval orders and shift graphs*, Colloq. Math. Soc. Janos Bolyai, vol. 60, 1991, pp. 297–313.

[13] S. Hoşten and W.D. Morris Jr., *The order dimension of the complete graph*, Discrete Math. **201** Issue 1-3 (1999), 133–139, Special issue on partial ordered sets.

[14] J. Kratochví l and P. Mihók, *Hom-properties are uniquely factorizable into irreducible factors*, Discrete Mathematics (2000), no. 213, 189–194.

[15] I. Kříž and J. Nešetřil, *Chromatic number of Hasse diagrams, eyebrows and dimension*, Order (1991), no. 8, 41–48.

[16] P. Ossona de Mendez, *Geometric Realization of Simplicial Complexes*, Graph Drawing (J. Kratochvil, ed.), Lecture Notes in Computer Science, vol. 1731, Springer, 1999, pp. 323–332.

[17] ———, *Realization of posets*, Journal of Graph Algorithms and Applications **6** (2002), no. 1, 149–153.

[18] H. Scarf, *The computation of economic equilibria*, Cowles foundation monograph, vol. 24, Yale University Press, 1973.

[19] W. Schnyder, *Planar graphs and poset dimension*, Order **5** (1989), 323–343.

[20] J. Spencer, *Minimal scrambling sets of simple orders*, Acta Math. Acad. Sci. Hung. **22** (1971), 349–353.

Partitioning sets on the line into separated ones

Attila Pór

1 Introduction

We call two subsets of the real numbers, $X, Y \subset \mathbb{R}$, *separated* if either the maximal element of X is smaller then the minimal element of Y or the minimal element of X is larger then the maximal element of Y or one of the sets is empty.

For any $k > 1$ let n_k denote the smallest integer for which the following holds: Let $A_1, \ldots, A_k$ be arbitrary finite pairwise disjoint subsets of $\mathbb{R}$ of the same cardinality. There exists a partition of each A_i into n_k parts (there might be empty sets in between)

$$A_i = A_{i,1} \cup \ldots \cup A_{i,n_k}$$

such that:

(*i*) $|A_{i_1,j}| = |A_{i_2,j}|$ for all $1 \leq j \leq n_k$ and $i_1 \neq i_2$.

(*ii*) For all j the sets $A_{1,j}, A_{2,j}, \ldots, A_{k,j}$ are pairwise separated.

Asymptotically tight bounds were shown by A. Pór[P], namely

Theorem 1.1

$$\left\lfloor \frac{k-1}{2} \right\rfloor \cdot \left\lfloor \frac{k+1}{2} \right\rfloor + 1 \leq n_k \leq (k-1)^2 + 1$$

Open questions:

- What is the truth, i.e. the exact value of n_k?
- We know that $4 \leq n_3 \leq 5$, but which one is the right answer?

We give a sketch of the proof for the Theorem, a complete proof can be found in [P].

Proof of the upper bound Let $A = \cup_i A_i$. Let $i_0, \ldots, i_k \notin A$ be real numbers such that

$$|A \cap [-\infty, i_l]| = l|A_1| \text{ for all } l.$$

Let $I_l = [i_{l-1}, i_l]$ be intervals for $1 \leq l \leq k$ and $X_{i,l} = A_i \cap I_l$. Let X be the $k \times k$ matrix with elements $x_{i,l} = |X_{i,l}|$. Observe that the sum of elements in each row and each column of X is equal to $|A_1|$. The idea is to have $A_{i,j}$ as a subset of $X_{i,l_{i,j}}$ and that $l_{i_1,j} \neq l_{i_2,j}$. This would imply the property (ii) of separation. There is a known result, for example in [MM], that X can be written as a positive integer linear combination of at most $(k-1)^2+1$ permutation matrices. This combination yields almost automatically the desired partitions. □

Proof of the lower bound Let suppose that $k = 2l+1$ is odd. We would like to construct k sets of equal cardinality such that any partition of the desired form has to have at least l^2+l+1 parts. The idea is to use each second set for separation, i.e. for $1 \leq i \leq l$ the set A_{2i} will have all of its elements almost equal to i. The other $l+1$ sets will have elements in the interval $[0, l+1]$ but not close to any integer. We again derive an $(l+1) \times (l+1)$ matrix X with elements $x_{i,j} = |A_{2i-1} \cap [j-1, j]|$. In this way we may construct any positive integer matrix X where in each row the sum of elements is equal. Let call these matrices *nice matrices.* Let call *special matrix* a matrix that has in each row an element equal to one and all other elements in this row are zero. A partition of the sets A_i would yield that X is a positive integer linear combination of at most n_k special matrices.

From the fact that any positive integer nice matrix can be written as positive integer linear combination of at most n_k special matrices one can prove that any positive rational nice matrix can be written as a positive rational linear combination of at most n_k special matrices.

From this you can prove that any positive nice matrix can be written as positive linear combination of at most n_k special matrices. Now you observe that the dimension of the space of nice matrices, l^2+l+1, is a lower bound for n_k. This, in fact, can be obtained if you consider the elements in the first l columns and the first element of the last column to be a subset of some Hummel base. □

References

[MM] M. Marcus, H. Minc, *Convexity and Matrices* in: "A survey of matrix theory and matrix inequalities" Ally and Bacon, Boston, pp. 121–138, 1964.

[P] A. Pór, *A partitioned version of the Erdős Szekeres theorem for quadrilaterals,* accepted in Discrete and Computational Geometry

On systems of boxes

Krzysztof Przesławski

Let $X_1, \dots, X_d$ be finite sets and let each of them contain at least two elements. A non-empty subset A of the cartesian product $X := X_1 \times \cdots \times X_d$ is said to be a *box* if there are subsets $A_i \subseteq X_i$, $i = 1, \dots d$, such that $A = A_1 \times \cdots \times A_d$. If $A_i \neq X_i$ for each i, then A is a *proper* box.

A box B is a *partial complement* of a box A if $B_i = X_i \setminus A_i$ for some $i \in [d]$.

Let $F \subseteq X$. We say that F is a pc set if there is a partition $\mathcal{P}$ of F, which consists of proper boxes, such that

(M) any box $A \in \mathcal{P}$ is a partial complement of the remaining members of $\mathcal{P}$.

It can be proved that for any pc set F and any partition $\mathcal{P}$ of F into proper boxes saying that $\mathcal{P}$ satisfies (M) is equivalent to saying that $\mathcal{P}$ has the minimum cardinality among all partitions of F into proper boxes. (This result is shown in [2] for the case $F = X$. The proof rests on an idea drawn from [1].)

The above equivalence suggests to call all partitions of F that satisfy (M) *minimal*.

We say that two boxes A and B contained in X are *coherent* if A is a partial complement of B and for all $i \in [d]$ but one $A_i = B_i$.

Let $\mathcal{P}$ and $\mathcal{R}$ be different minimal partitions of a pc set G. We say that they are adjacent if and only if there are coherent boxes A, B in $\mathcal{P}$, and C, D in $\mathcal{R}$ such that

$$A \cup B = C \cup D \qquad \text{and} \qquad \mathcal{P} \setminus \{A, B\} = \mathcal{R} \setminus \{C, D\}.$$

In this way, one may define a graph Γ on the family of all minimal partitions. **It is conjectured that Γ is connected.**

There are questions related to this conjecture. For example, A. Kisielewicz asks whether each minimal partition of X contains coherent boxes.

The connectedness of Γ would give a common explanation of numerous facts concerning minimal partitions. To give an example, let us fix a proper box A in X, and for each $\varepsilon \in \{-1, 1\}^d$ define the box A^ε by the formula

$$A_i^\varepsilon = \begin{cases} A_i & \text{if} \quad \varepsilon_i = 1, \\ X_i \setminus A_i & \text{if} \quad \varepsilon_i = -1. \end{cases}$$

Let $\pi(\varepsilon) = \varepsilon_1 \cdot \dots \cdot \varepsilon_d$. For each minimal partition $\mathcal{P}$ of G one can define a kind of degree:

$$\deg(A, \mathcal{P}) = \sum_{\varepsilon} \varepsilon 1_{\mathcal{P}}(A^{\varepsilon}).$$

It can be shown that the function $\mathcal{P} \mapsto \deg(A, \mathcal{P})$ is constant. On the other hand, the connectedness of Γ implies this fact immediately: It suffices to observe that $\deg(A, \mathcal{P}) = \deg(A, \mathcal{R})$, whenever $\mathcal{P}$ and $\mathcal{R}$ are adjacent.

References

[1] N. Alon, T. Bohman, R. Holzman and D.J. Kleitman, On partitions of discrete boxes, *Discrete Math.* 257(2002), 255–258.

[2] J. Grytczuk, A. Kisielewicz, K. Przesławski, Minimal partitions of a box into boxes. (submitted)

A Dirac-type theorem for 3-uniform hypergraphs

Andrzej Ruciński

(joint work with Vojtěch Rödl and Endré Szemerédi)

A substancial amount of research in graph theory continues to concentrate on the existence of hamiltonian cycles. A classic theorem of Dirac states that a sufficient condition for an n-vertex graph to be hamiltonian is that the minimum degree is at least $n/2$, and there is an obvious counterexamples showing that this is best possible. For 3-uniform hypergraphs a natural extension of Dirac's theorem has been conjectured in [3], where as a sufficient condition one demands that every pair of vertices belongs to at least $n/2$ triples. Here too there is an example showing that the above, if true, is best possible.

The study of hamiltonian cycles in hypergraphs was initiated in [1] where, however, a different definition than the one considered here was introduced. By a hamiltonian cycle in a 3-uniform hypergraph we mean a spanning subgraph whose vertices can be ordered $v_1, \ldots, v_n$ in such a way that for each $i = 1, \ldots, n$, the triple (v_i, v_{i+1}, v_{i+2}) is an edge, and so are the triples (v_{n-1}, v_n, v_1) and (v_n, v_1, v_2).

This notion and its generalizations has a potential to be applicable in many contexts which still need to be explored. As observed in [3], the square of a (graph) hamiltonian cycle naturally coincides with a hamiltonian cycle in a hypergraph built on top of the triangles of the graph.

Let us add here one other possible context in which, we believe, 3-uniform (and more generally, k-uniform) hamiltonian cycles may prove to be the right mathematical model.

Consider a robot walking through a tough terrain with the task to visit n designated locations and return to the base (one may view these locations as fuel providers). In order for the robot to move from one location to another, after reaching any one of them it has to be able to "see' the next one. To optimize, we do not want the robot to visit a location more than once. So far, this is just the standard travelling salesman problem, but suppose that in order to speed up the motion, or to smooth out the trajectory, we request that the robot must "see" the next two locations. Then our problem becomes that of finding a hamiltonian cycle in a 3-uniform hypergraph whose edges are those triplets of the n locations which can mutually "see" each other.

In [3] the authors found a sufficient condition for a hypergraph to have a hamiltonian cycle in terms of the pair degrees, which can be viewed as an analog of Dirac's degree condition for graphs (in fact, as they worked with k-uniform hypergraphs they introduced the notion of $(k-1)$-degree for arbitrary $k \geq 2$.) For $k = 3$ they proved that if every pair of vertices belongs to more than $\frac{5}{6}(n-1)+1$ edges, then the hypergraph contains a hamiltonian cycle. They also conjectured that, in fact, a much stronger result is true, namely that $\frac{5}{6}(n-1)+1$ can be replaced by $n/2$. The support for this conjecture stems from a construction of an edge-maximal, 3-uniform hypergraph with each pair degree at least $\lfloor n/2 \rfloor - 1$, not containing a hamiltonian cycle (see [3], Theorem 3). Recently, we have proven an asymptotic version of this conjecture.

We say that a 3-uniform hypergraph H is an (n, γ)-graph if H has n vertices and every pair of vertices belongs to at least $(1/2 + \gamma)n$ edges.

Theorem 1 *Let $\gamma > 0$. Then, for sufficiently large n, every (n, γ)-graph contains a hamiltonian cycle.*

Problem 1 *Eliminate γ from Theorem 1, that is prove that for sufficiently large n, every 3-uniform hypergraph on n vertices with every pair belonging to at least $n/2$ edges contains a hamiltonian cycle.*

Problem 2 *Prove analogous statement for k-uniform hypergraphs for $k \geq$ 4.*

References

[1] J. C. Bermond et al., Hypergraphes hamiltoniens, *Prob. Comb. Theorie Graph Orsay* 260 (1976) 39-43.

[2] Extremal problems on set systems, *Random Struct. Algorithms* 20, no. 2, (2002) 131-164.

[3] Gyula Y. Katona and H. A. Kierstead Hamiltonian chains in hypergraphs, *J. Graph Theory* 30 (1999) 205-212.

Three Combinatorial Problems

Endré Szeméredi

We discussed 3 problems:

Problem 1. G is a bipartite graph. Let $G = (V, E)$ be a bipartite graph on n vertices ($|V| = n$). $E(G) = \bigcup_{i=1}^{l} M_i$. M_i is a partial matching $|M_i| \geq \frac{n}{\log n}$, $l \geq \frac{n}{\log n}$. Prove that there are four vertices, say a, b, c, d, such that $(a, b), (c, d) \in M_i$ and $(b, c), (a, d) \in M_k$ for some i and k.

Problem 2. Let $A = \{a_1, a_2, \ldots a_n\}$ be a set of n integers. Prove that there exists $B \subset A$, $|B| \geq \frac{n}{3} + w(n)$ and for $x, y, z \in B$, $x + y \neq z$ (for $|B| \geq \frac{n}{3}$ it is not hard) $w(n) \mapsto \infty$ if $n \mapsto \infty$.

Problem 3. Let $A = \{a_1, a_2, \ldots a_n, \ldots\}$ be and infinite sequence of integers. $A(n)$ is the number of elements of A less than n. A has the following property: $a_i + a_j + a_k \neq a_t + a_q + a_s$ (The sum of any 3 is different). Prove that for any $\varepsilon > 0$ $A(n) < \varepsilon n^{1/3}$ for infinitely many n. (Notice: there is a set $B \subset [1, n]$, $|B| \geq n^{1/3}$ and the sum of any 3 elements is different.

Factorisation of irreducible snarks

Martin Škoviera

A *snark* is a "non-trivial" cubic graph which has no edge-colouring by three colours in such a way that adjacent edges receive distinct colours. The idea of what the loose term *non-trivial* means has been extensively discussed by many researchers.

It has been generally accepted that any "non-trivial" snark should be cyclically 4-edge-connected and have girth at least 5, for otherwise it could be obtained from one or two smaller snarks by a "simple" modification. In 1987, Cameron, Chetwynd and Watkins [J. Graph Theory **11** (1987), 13–19] showed that certain cyclically 4-edge-connected snarks of girth at least 5 can still be obtained from smaller snarks by certain well-defined operations such as the dot-product. Nedela and Škoviera [J. Graph Theory **22** (1996), 253–279] therefore suggested a structural and dynamical approach to the idea of non-triviality of a snark by introducing the concept of a k-reduction of a snark. A snark H is said to be a *k-reduction* of a snark G if G has a k-edge-cut S whose removal leaves a component of $G - S$ which is *not* 3-edge-colourable and which can be completed to H by adding some vertices and edges; moreover, H is a *proper* k-reduction of G if the order of H is strictly smaller than that of G. Roughly speaking, the essence of non-3-edge-colourability of G is contained in the reduced snark H.

We say that a snark G is *m-irreducible* if it has no proper k-reduction for any $k < m$. A snark which is m-irreducible for each $m \geq 1$ will be simply called *irreducible*. In the above mentioned paper, Nedela and Škoviera gave the following characterisation of m-irreducible snarks.

For $1 \leq m \leq 4$, a snark is m-irreducible if and only if it is cyclically m-edge connected. For $5 \leq m \leq 6$, a snark is m-irreducible if and only if it is critical. *For $m \geq 7$, a snark is m-irreducible if and only if it is* bicritical.

A snark is *critical* if the removal of any two adjacent vertices yields a 3-edge-colourable graph, and is *bicritical* if the removal of any two distinct vertices yields a 3-edge-colourable graph.

It follows from the above characterisation that the hierarchy of m-irreducible snarks has only six different levels, with the top level formed by 7-irreducible snarks which coincide with irreducible snarks, and which in turn are identical with bicritical snarks.

It should be mentioned that every irreducible snark is cyclically 4-edge-connected and has girth at least 5, but not vice versa. Thus irreducible snarks are "non-trivial" in the usual sense.

It was proved by Goldgerg [J. Combin. Theory Ser B **31** (1981), 282–291] and reproved by Cameron, Chetwynd and Watkins [loc. cit] that any snark which has a cycle-separating 4-edge-cut can be expressed as a dot-product of two smaller snarks. It turns out that if a snark is irreducible, then both smaller snarks are again irreducible (this fact is not immediate). Thus, by iterating this process as long as possible, we see that every snark can be decomposed into a sequence of cyclically 5-connected irreducible snarks. Note that the dot-product operation is not commutative and even not associative. Moreover, in a snark there may be many cycle separating 4-edge cuts, some of which may intersect, and by disconnecting one particular cut we can destroy some other cuts. Thus, in general a snark (in particular, an irreducible snark) may have many different decompositions into cyclically 5-connected snarks. However, Chladný and Škoviera have recently showed that up to isomorphism and ordering of the factors, any irreducible snark has only one decomposition into cyclically 5-connected snarks. This *Unique-Factorisation Theorem* for irreducible snarks shows that, in some sense, cyclically 5-connected irreducible snarks are the basic stones from which all irreducible snarks can be obtained by iterated dot-products.

We remark that the assumption of irreducibility is necessary for the textitUnique-Factorisation Theorem to be true since there exist infinitely many critical snarks with at least two non-isomorphic decompositions.

Nevertheless, Cameron, Chetwynd and Watkins [loc. cit] showed that there is a well-defined operation similar to the dot-product which from two cyclically 5-edge connected snarks can produce a new cyclically 5-edge connected snark. Simply, take a snark G_1 and a 5-cycle C_1 in G_1, and similarly take a snark G_2 and a 5-cycle C_2 in G_2. Remove the 5-cycles, thereby producing in both G_1 and G_2 five dangling edges, and connect the dangling edges in a Petersen-like manner. It can be shown that the resulting graph is a snark.

The question which I would like to pose today is: *Can one develop a similar (but perhaps more restricted) decomposition theory for cyclically* 5-*connected irreducible snarks with respect to the operation just described?*

A Map Colour Theorem for the Union of Graphs

Riste Škrekovski

(joint work with Michael Stiebitz)

In 1890 Heawood established an upper bound $H(g) = \left\lfloor \frac{7+\sqrt{24g+1}}{2} \right\rfloor$ for the chromatic number of a graph embedded on a surface of Euler genus $g \geq 1$. This upper bound becomes known as the Heawood number. Almost a century later, Ringel and Ringel & Youngs proved that the Heawood number $H(g)$ is in fact the maximum chromatic number as well as the maximum clique number of graphs embbeded on a surface of Euler genus $g \geq 1$ beside the Klein bottle.

In this talk we present a Heawood type formula for the edge disjoint union of two graphs that are embedded on a given surface Σ of Euler genus $g \geq 0$. Define the number $H_2 = H_2(\Sigma)$ as follows: If Σ is orientable and $g = 12(2q+1)^2$ for some integer $q \geq 0$, then $H_2 = 6 + \sqrt{12g} = 24q + 18$. If Σ is orientable and $g \equiv 2 \pmod 4$, then $H_2 = 7 + \lfloor \sqrt{12g} \rfloor$. In all other cases, $H_2 = 7 + \lfloor \sqrt{12g+1} \rfloor$. So, if a graph G embedded on Σ is the edge disjoint union of two graphs G_1 and G_2, then

$$\omega(G_1) + \omega(G_2) \leq \chi(G_1) + \chi(G_2) \leq H_2(\Sigma).$$

Similar to the results of Ringel and Ringel & Youngs, we show that this bound is sharp for all but at most one non-orientable surface Σ.

Covering Minimum Spanning Trees of Random Subgraphs

Jan Vondrák

(joint work with Michel X. Goemans)

We consider the problem of covering the minimum spanning tree (MST) of a random subgraph of G by a sparse set of edges, with high probability. The two random models that we consider are subgraphs induced by a random subset of vertices, each vertex included independently with probability p, and subgraphs generated as a random subset of edges, each edge with probability p.

Let n be the number of vertices in G. We show that in both cases, there is a covering set Q of cardinality $O(n \log n)$ and this is asymptotically optimal. More generally, we show a similar bound on the covering set in a matroid, which contains the minimum-weight basis of a random subset with high probability.

More precisely, we have the following theorems.

Theorem 1 *Let G be a weighted graph on n vertices, $1/n < p < 1$, and $c > 0$. Let $b = 1/(1-p)$. Then there exists a set $Q \subseteq E$ of size*

$$|Q| \leq en((c+1)\log_b n + \log_b(ep)) + O(n)$$

such that for a random $W = V(p)$,

$$Pr[MST(W) \subseteq Q] > 1 - \frac{1}{n^c}.$$

Theorem 2 *For any weighted matroid $(E, \mathcal{M}, w)$ of rank n, $0 < p < 1$, $c > 0$, and $b = 1/(1-p)$, there exists a set $Q \subseteq E$ of size*

$$|Q| \leq en((c+1)\log_b n + \log_b e) + O(n)$$

such that for a random $F = E(p)$,

$$Pr[MB(F) \subseteq Q] > 1 - \frac{1}{n^c}.$$

The proofs of both theorems use the fact that the event of an edge appearing in the minimum spanning tree of a random subgraph, conditioned on the edge being included in the subgraph, is a down-monotone event. We derive a "boosting lemma" which states how the probability of a down-monotone event depends on the sampling probability.

Lemma 3 *Let X be a finite set, $\mathcal{F}$ a down-monotone event on the space 2^X, $0 < q < p < 1$, and let $X(p)$ denote a random subset of X, where each element is sampled independently with probability p. If*

$$Pr[X(p) \in \mathcal{F}] \geq (1-p)^k$$

for some $k \in \mathbf{N}$, then

$$Pr[X(q) \in \mathcal{F}] \geq (1-q)^k.$$

This lemma is a key element in the proof, which shows that there cannot be too many edges (u, v) such that

$$Pr[(u,v) \in MST(W)] > \frac{1}{n^c}.$$

Thus we can include all edges with sufficiently large probability in Q, which yields a set of desired properties. We feel that this approach could be applied to other situations involving down-monotone events, however the matroid setting is the most general one we could find so far.

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