Equipartite polytopes and graphs

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Abstract

A graph G of even order is *weakly equipartite* if for any partition of its vertex set into subsets V_1 and V_2 of equal size the induced subgraphs $G[V_1]$ and $G[V_2]$ are isomorphic. A polytope P with $2n$ vertices is *equipartite* if for any partition of its vertex set into two equal-size sets V_1 and V_2, there is a symmetry of the polytope P that maps V_1 onto V_2.

A complete characterization of equipartite graphs is provided. This is then used to prove that an equipartite polytope in $\mathbb{R}^d$ can have at most $2d+2$ vertices. We prove that this bound is sharp.

1 Introduction

Classification of polytopes possessing a variety of symmetries has been extensively studied: centrally symmetric polytopes, vertex transitive polytopes, self-dual polytopes are few such examples. In this paper, we introduce a new kind of symmetry which we call *equipartiteness*. A d-polytope

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$P \subseteq \mathbb{R}^d$ with $2n$ vertices is *equipartite* if for every n-element set A of its vertices, there is a symmetry of P that maps A onto the complementary set of vertices of P. Examples of equipartite d-polytopes include rectangles and regular hexagons for $d = 2$, rectangular boxes, regular tetrahedra, regular octahedra and regular three-sided prisms for $d = 3$ and regular simplices for odd $d \geq 3$. A complete list of all equipartite polytopes in $\mathbb{R}^2$ and $\mathbb{R}^3$ is obtained in Section 7.

In Section 6, we prove that the number of vertices of an equipartite d-polytope is at most $2d+2$ and, later in Section 8, we show that the bound is tight by constructing equipartite d-polytopes with $2d+2$ vertices for every $d \geq 2$. When restricting the definition of equipartiteness of polytopes to its 1-skeletons, we are naturally led to the notion of equipartite graphs: A graph G of order $2n$ is *equipartite* if for every n-element subset A of its vertices, there is an automorphism of G mapping to the set A to the complementary set of the vertices. It is noteworthy that in our proof we use a weaker notion of equipartiteness defined as follows: A graph G of order $2n$ is *weakly equipartite* if for every n-element subset A of its vertices, the subgraph of G induced by A is isomorphic to the subgraph induced by rest of the vertices. As a consequence of our results, we prove that the isomorphism between the two parts of the graph can be induced by an automorphism of the entire graph. Hence, if a graph is weakly equipartite, it is also equipartite.

All equipartite graphs of order six and eight are depicted in Figures 1 and 2. In Sections 3–5, we provide a full characterization of equipartite graphs.

Equipartite 2- and 3-polytopes are characterized in Section 7. Some equipartite d-polytopes are constructed in Section 8. They have $d+1$ and $d+2$ vertices for odd and even d, respectively. Other constructions yield d-polytopes with $2d$ and $2d+2$ vertices for all d. It will be interesting to construct equipartite polytopes with the number of vertices between $d+3$ and $2d-2$ (if they exist).

Some of our constructions attempt to generalize constructions of equipartite 3-polytopes. This process can be tricky. For instance, there are three distinct types of equipartite 3-simplices, but we have no clue as to the number of distinct symmetry types of equipartite $(2d+1)$-simplices for $d \geq 2$ (we identified three distinct types but we do not know whether there are any other types). In the other extreme, we identified three distinct types of equipartite 3-polytopes with eight vertices (the largest possible number of vertices in $\mathbb{R}^3$) but only one type of equipartite polytope with $2d+2$

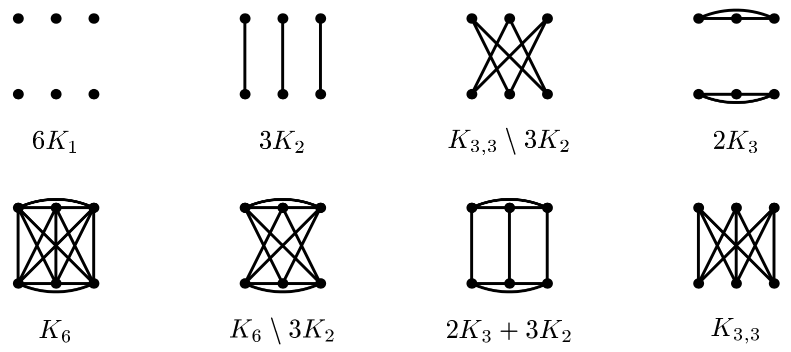

Figure 1: Equipartite graphs of order six.

vertices in $\mathbb{R}^d$.

2 Definitions and notation

We use a terminology standard in graph theory which can be found, e.g., in [4, 10]. If G is a graph, $V(G)$ and $E(G)$ denote its vertex and edge set, respectively. A graph G is *d-regular* if the degree of each vertex is d. A *closed neighborhood* $N(v)$ of a vertex v in G is the set consisting of the vertex v and all neighbors of v in G. If $A \subseteq V(G)$, then $G[A]$ stands for the subgraph induced by the vertices of A. The symbols K_n, $K_{a,b}$ and C_l denote the complete graph of order n, the complete bipartite graphs with parts of sizes a and b and the cycle of length l, respectively. A union of k vertex-disjoint copies of a graph G is denoted by kG. We write $G + H$ for an edge-disjoint union of two graphs G and H on the same vertex set; the graph $G + H$ will be always uniquely determined by the graphs G and H. Similarly, $G \setminus H$ stands for a graph G without a subgraph isomorphic to H. Again, the graph $G \setminus H$ will be always uniquely determined by the graphs G and H. This notation is used in Figures 1 and 2.

The d-dimensional Euclidean space is denoted by $\mathbb{R}^d$. A *d-polytope* is a convex hull of some $n \geq d+1$ points of $\mathbb{R}^d$ which are not contained in $(d-1)$-dimensional affine flat. A *symmetry* of a d-polytope $P \subseteq \mathbb{R}^d$ is an isometry τ of $\mathbb{R}^d$ that maps P onto P. In a suitable coordinate system, each symmetry can be represented by an orthogonal transformation. If P is an equipartite

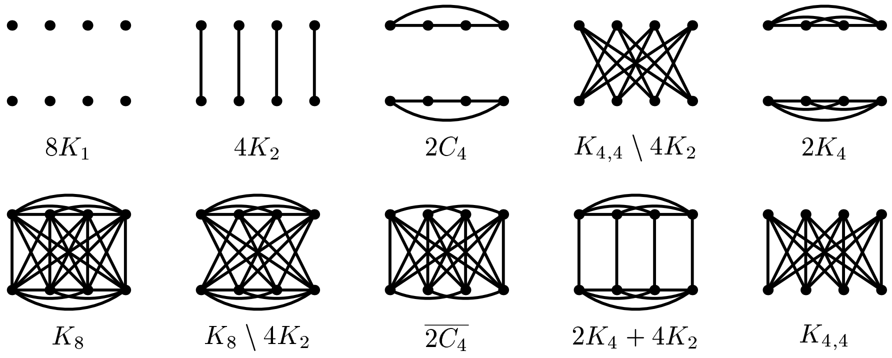

Figure 2: Equipartite graphs of order eight.

polytope, then the 1-skeleton of P is an equipartite graph. It is easy to construct examples of polytopes whose 1-skeleton is an equipartite graph, but no combinatorially equivalent polytope is equipartite. The simplest examples are neighborly d-polytopes with many vertices. They even fail to be vertex-transitive.

Two d-polytopes P and Q have the same *symmetry type* if they are combinatorially equivalent and their symmetry groups are isomorphic under an isomorphism compatible with the combinatorial equivalence [9].

A useful distinction for transitive actions of permutation groups on a set A_0 is that of primitivity and imprimitivity [7]. A partition $\pi(A_0)$ is *stable* under the action of a group Γ on A_0 if $gA \in \pi(A_0)$ for all $g \in \Gamma$ and $A \in \pi(A_0)$. The partitions $\pi_1(A_0) = \{A_0\}$ and $\pi_0(A_0) = \{\{a\} | a \in A_0\}$ are stable under any group Γ acting on A_0. These two partitions are called *trivial partitions* of A_0. The action of a permutation group Γ is *primitive* if the trivial partitions π_0 and π_1 are the only partitions which are stable under Γ.

If the action of a permutation group Γ on a set A_0 is not primitive and if $\pi(A_0)$ is a non-trivial partition of A_0 stable under Γ, then the elements of $\pi(A_0)$ are called *imprimitivity classes*. Note that if P is a centrally symmetric polytope, then the action of its symmetry group on the vertices is imprimitive, since the partition consisting of antipodal pairs of vertices is stabilized by the symmetry group. For odd d, the regular d-simplices are equipartite polytopes for which the symmetry group acts primitively. For all other examples of equipartite polytopes that we were able to find, the symmetry group acts imprimitively on the vertices. We believe that this is the case for all equipartite polytopes (except for simplices).

A permutation group Γ acting on a set A_0 of size $2n$ has the *interchange property* [1] if for every n-element subset $A \subseteq A_0$, there is a group element $g \in \Gamma$ which interchanges A with its complement. Note that a polytope P is equipartite if and only if its symmetry group, acting as a permutation group on the vertices of P, has the interchange property.

3 Preliminaries

We first note that equipartite polytopes have very high degree of symmetry. A polytope P is *isogonal* if for any two vertices of P, there is a symmetry of P that maps one of them onto the other. We now show that each equipartite polytope is isogonal:

Proposition 1 *If a polytope P with $2n$ vertices is equipartite, then P is also isogonal.*

Proof: Consider a graph G whose vertices are the vertices of P and two of them are adjacent if there is a symmetry of P that maps one of them onto the other. Clearly, the graph G is well-defined. If G contains a vertex v of degree at most $n-1$, choose a subset $A \subseteq V(G)$ with $|A| = n$ so that v is adjacent to no vertex of A. Since the polytope P is equipartite, there is a symmetry that maps the vertices of A onto the complementary set of vertices of P. But this means that v must have a neighbor in A contradicting the definition of A. Hence, the minimum degree of G is at least n. Therefore, the graph G is connected (its order is $2n$). Since a composition of two symmetries of P is a symmetry of P, it follows that G is the complete graph and the polytope P is isogonal.

$\blacksquare$

From the proof of Proposition 1, it is obvious that every equipartite graph is vertex-transitive. In the sequel, we show that weak equipartiteness implies vertex transitivity and also equipartiteness.

Let us now state two propositions on equipartite graphs. The proof of the first one follows directly from the definition.

Proposition 2 *The complement of a weakly equipartite graph is weakly equipartite.*

Proposition 3 *Every weakly equipartite graph G of order $2n$ is regular.*

Proof: Consider a graph G of order $2n$ which is not regular. Let $v_1, \ldots, v_{2n}$ be the vertices of G and let d_i be the degree of the vertex v_i. We can assume that $d_1 \geq d_2 \geq \ldots \geq d_{2n}$. Since G is not regular, it holds that $d_1 > d_{2n}$. Split the vertex set of G into two parts $A = \{v_1, \ldots, v_n\}$ and $B = \{v_{n+1}, \ldots, v_{2n}\}$. Let m_{AB} be the number of edges ab of G with $a \in A$ and $b \in B$. The numbers of edges of the subgraphs $G[A]$ and $G[B]$ are $m_A = (d_1 + \ldots + d_n - m_{AB})/2$ and $m_B = (d_{n+1} + \ldots + d_{2n} - m_{AB})/2$, respectively. Since $d_1 \geq d_2 \geq \ldots \geq d_{2n}$ and $d_1 > d_{2n}$, we have $m_A > m_B$. But then the graphs $G[A]$ and $G[B]$ are not isomorphic. Thus, the graph G is not weakly equipartite.

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We further restrict the vertex degrees which can appear in weakly equipartite graphs:

Lemma 4 *If G is a weakly equipartite graph of order $2n$, then G is a d-regular graph where*

$$d \in \{0, 1, n-3, n-2, n-1, n, n+1, n+2, 2n-2, 2n-1\}.$$

Proof: Fix a weakly equipartite graph G of order $2n$. By Proposition 3, the graph G is regular. Let d be the common degree of the vertices of G. Observe that if $n \leq 5$, then the statement of the lemma trivially holds because $n - 3 \leq 2$ and $2n - 2 \leq (n + 2) + 1$. Hence, we only consider the case that $n \geq 6$. Assume for the sake of contradiction that $2 \leq d \leq n - 4$. The case $n + 3 \leq d \leq 2n - 3$ is symmetric by Proposition 2.

We show that the vertex set $V(G)$ can be partitioned into $2\lceil n/(d+1)\rceil$ parts A_i, $1 \leq i \leq 2\lceil n/(d+1)\rceil$, with the following property: Each $G_i = G[A_i]$ contains a vertex γ_i adjacent to all the remaining vertices of G_i. In addition, the degree of γ_1 in G_1 is d (thus $|A_1| = d + 1$).

We first find disjoint subsets $N_1, \ldots, N_{2\lfloor n/(d+1)\rfloor}$ of $V(G)$ such that each N_i is a closed neighborhood of a vertex, say ν_i, of G. Set N_1 to be the closed neighborhood of any vertex of G. Split the vertex set of G into two parts A and B of size n such that $N_1 \subseteq A$. Since the graph G is weakly equipartite, B contains a closed neighborhood N_2 of a vertex of G. If $2(d + 1) > n$, the sets N_1 and N_2 are the sought subsets (note that $2\lfloor n/(d + 1)\rfloor = 2$ in such case). Otherwise, split the vertex set of G into two parts A' and B' such that $N_1 \cup N_2 \subseteq A'$. Again, since the graph G is weakly equipartite, B

contains closed neighborhoods N_3 and N_4 of some vertices of G. Continue in this way until the sets $N_1, \ldots, N_{2\lfloor n/(d+1)\rfloor}$ are found.

We now construct the sets $A_1, \ldots, A_{2\lceil n/(d+1)\rceil}$. Set $A_i = N_i$ for $1 \le i \le \lfloor n/(d+1)\rfloor$ (in this way, the first half of the sets N_i is used). In case that $d+1$ does not divide n, the set $A_{\lceil n/(d+1)\rceil}$ is a subset of $N_{\lfloor n/(d+1)\rfloor+1}$ of size $n - |A_1| - \ldots - |A_{\lfloor n/(d+1)\rfloor}| = n - \lfloor n/(d+1)\rfloor(d+1)$ which contains the vertex $\nu_{\lceil n/(d+1)\rceil}$. Let $A = A_1 \cup \ldots \cup A_{\lceil n/(d+1)\rceil}$ and $B = V(G) \setminus A$. Note that $|A| = n$ by the choice of $A_1, \ldots, A_{\lceil n/(d+1)\rceil}$. Hence, the subgraphs $G[A]$ and $G[B]$ are isomorphic. For $i > \lceil n/(d+1)\rceil$, set A_i to be an isomorphic copy of $A_{i-\lceil n/(d+1)\rceil}$ in $G[B]$. Note that each set A_i contains a vertex γ_i adjacent to all the other vertices of A_i. Moreover, the set A_1 is the closed neighborhood of $\gamma = \nu_1$.

Let $A_0 = A_1 \cup \{\gamma_2, \ldots, \gamma_{2\lceil n/(d+1)\rceil}\}$. Since $|A_0| = d + 2\lceil n/(d+1)\rceil \le n$ (the upper bound follows from $2 \le d \le n-4$ and $n \ge 6$), the vertex set $V(G)$ can be split into two n-element parts A and B such that $A_0 \subseteq A$. Note first that the maximum degree of $G[A]$ is d because it contains the subgraph $G[N_1]$. On the other hand, each vertex v of B is adjacent to at least one vertex of A (if v is in $A_i \cap B$, then this vertex is γ_i) and thus the maximum degree of $G[B]$ is at most $d-1$. Hence, the subgraphs $G[A]$ and $G[B]$ are not isomorphic — a contradiction.

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4 Weakly equipartite graphs with small degrees

The proof of the theorem which characterizes weakly equipartite graphs is split into several steps. We have already observed some general properties of weakly equipartite graphs, in particular, that they are regular graphs with very restricted degrees. Next, we focus on d-regular graphs of order $2n$ with $d \le n-1$. We distinguish two cases based on that whether the graph is disconnected or connected. In Subsection 4.1, we show that the only disconnected weakly equipartite graphs are $2nK_1$, nK_2, $2C_4$ and $2K_n$. In Subsection 4.2, we establish that in most cases the only connected weakly equipartite bipartite graph of order $2n$ with degrees smaller than n is the graph $K_{n,n} \setminus nK_2$. Our results are then glued together to provide a full characterization of equipartite and weakly equipartite graphs in the next section.

4.1 Disconnected weakly equipartite graphs

We first show that the orders of all the components of a disconnected weakly equipartite graph are the same:

Lemma 5 *If G is a disconnected weakly equipartite graph, then all its components have the same order.*

Proof: Consider a weakly equipartite graph G of order $2n$ with k components and let $\Gamma_1, \ldots, \Gamma_k$ be the components of G. Further, let n_i be the order of Γ_i. We can assume that $n_1 \geq \ldots \geq n_k$. Finally, let k_0 be the smallest integer such that $n_1 + \ldots + n_{k_0} \geq n$. Observe that $k_0 \leq \lceil k/2 \rceil$. Set $w = n - n_1 - \ldots - n_{k_0 - 1}$ and let W be a subset of vertices of Γ_{k_0} of size w such that the subgraph $\Gamma_{k_0}[W]$ is connected. Split the vertex set of G into two parts A and B as follows:

$$\begin{aligned} A &= V(\Gamma_1) \cup \ldots \cup V(\Gamma_{k_0-1}) \cup W \\ B &= V(G) \setminus A \end{aligned}$$

The number of components of $G[A]$ is k_0 by the choice of the set A. It remains to estimate the number of components of $G[B]$. Note that this number must be also k_0 because the graph G is weakly equipartite.

If $n_1 + \ldots + n_{k_0} = n$, the number of components of $G[B]$ is $k - k_0$ (the subgraph $G[B]$ is comprised precisely by the components $\Gamma_{k_0+1}, \ldots, \Gamma_k$). As noted above, it must be $k - k_0 = k_0$ and thus $k_0 = n$. This together with $n_1 \geq \ldots \geq n_k$ and $n_1 + \ldots + n_k = 2n$ immediately yields that $n_1 = \ldots = n_k$. Hence, all the components of G have the same order.

In the rest, we consider the case that $n_1 + \ldots + n_{k_0} > n$. Since the number of components of $G[B]$ is at least $k - k_0 + 1$, it must hold that $k_0 \geq k - k_0 + 1$. This immediately yields that k is odd and $k_0 = \lceil (k+1)/2 \rceil$. Hence, the number of components of $G[B]$ is exactly $k - k_0 + 1$ and the graph $\Gamma_{k_0} \setminus W$ is connected. The orders of components of $G[A]$ are $n_1, \ldots, n_{k_0-1}, w$ and the orders of components of $G[B]$ are $n_{k_0} - w, n_{k_0+1}, \ldots, n_k$. Since $G[A]$ and $G[B]$ are isomorphic, we can deduce from $n_1 \geq \ldots \geq n_k$ and $w > 0$ that $n_1 = n_{k_0+1}$, $n_2 = n_{k_0+2}$, etc. This is possible only if $n_1 = n_2 = \ldots = n_k$. Hence, all the components of G have the same order.

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In order to prove Lemma 7 below, we need the following proposition [8, Lemma 1.15]:

Proposition 6 *Let G be a 2-connected graph. If G is neither a cycle nor a complete graph, then G contains two non-adjacent vertices u and v such that the graph $G \setminus \{u, v\}$ is connected.*

Lemma 7 *Each component of a disconnected weakly equipartite graph G is a cycle or a complete graph.*

Proof: Consider a weakly equipartite graph G of order $2n$ with $k \geq 2$ components. Let $\Gamma_1, \ldots, \Gamma_k$ be the components of G. By Lemma 5, the components Γ_i have the same order $\gamma = 2n/k$. We now construct two disjoint subsets A_0 and B_0 of vertices of G with $|A_0| = |B_0| = n - \gamma = n - 2n/k$. If k is odd, let W be a subset of $V(\Gamma_3)$ such that $\Gamma_3[W]$ is connected and $|W| = \gamma/2 = n/k$ (the number n/k is an integer because k is odd). The sets A_0 and B_0 are chosen as follows:

$$\begin{array}{rclcl} A_0 & = & W & \cup & V(\Gamma_4) \cup V(\Gamma_5) \cup \ldots \cup V(\Gamma_{(k+1)/2+1}) \\ B_0 & = & (V(\Gamma_3) \setminus W) & \cup & V(\Gamma_{(k+1)/2+2}) \cup \ldots \cup V(\Gamma_{k-1}) \cup V(\Gamma_k) \end{array}$$

If k is even, the sets A_0 and B_0 are chosen as follows:

$$\begin{array}{rcl} A_0 & = & V(\Gamma_3) \cup \ldots \cup V(\Gamma_{k/2+1}) \\ B_0 & = & V(\Gamma_{k/2+2}) \cup \ldots \cup V(\Gamma_k) \end{array}$$

Observe that regardless of the parity of k, the subgraph $G[A_0]$ consists of *precisely* $\lceil (k-2)/2 \rceil$ components and the subgraph $G[B_0]$ of *at least* $\lceil (k-2)/2 \rceil$ components (the graph $\Gamma_3 \setminus W$ might be disconnected).

We now show that the component Γ_1 must be a cycle or a complete graph. First, we consider the case that Γ_1 contains a cut-vertex v_1. Let v_2 be a vertex of Γ_2 such that $\Gamma_2 \setminus v_2$ is connected. Split the set $V(G)$ into two parts A and B:

$$A = A_0 \cup \{v_1\} \cup (V(\Gamma_2) \setminus \{v_2\})$$
$$B = B_0 \cup \{v_2\} \cup (V(\Gamma_1) \setminus \{v_1\})$$

The number of components of $G[A]$ is exactly $\lceil (k-2)/2 \rceil + 2$, but the number of components of $G[B]$ is at least $\lceil (k-2)/2 \rceil + 3$ by the choice of the vertices v_1 and v_2. However, since G is a weakly equipartite graph, $G[A]$ and $G[B]$ must be isomorphic — a contradiction. We can conclude that Γ_1 contains no cut-vertex, i.e., Γ_1 is 2-connected.

Assume now that G_1 is neither a cycle nor a complete graph. By Proposition 6, Γ_1 contains two non-adjacent vertices u and v such that $\Gamma_1 \setminus \{u, v\}$

is connected. Let $u'v'$ be any edge of Γ_2. We now consider the following partition of the vertex set of G:

$$\begin{array}{rcl} A & = & A_0 \cup \{u',v'\} \cup (V(\Gamma_1) \setminus \{u,v\}) \\ B & = & B_0 \cup \{u,v\} \cup (V(\Gamma_2) \setminus \{u',v'\}) \end{array}$$

Observe that the number of components of $G[A]$ is exactly $\lceil (k-2)/2 \rceil + 2$ and the number of components of $G[B]$ is at least $\lceil (k-2)/2 \rceil + 3$. Again, this contradicts the fact that G is weakly equipartite.

Since the choice of Γ_1 among the components of G was arbitrary, all the components of G are cycles or complete graphs.

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We are now ready to characterize all disconnected weakly equipartite graphs:

Theorem 8 *Any disconnected weakly equipartite graph G is one of the following graphs:*

$$2nK_1,\ nK_2,\ 2C_4 \text{ and } 2K_n.$$

Proof: It is straightforward to verify that the graphs $2nK_1$, nK_2, $2C_4$ and $2K_n$ are weakly equipartite. Consider a weakly equipartite disconnected graph G. The graph G is regular by Proposition 3. Let d be the common degree of all its vertices. All the components of G have the same order by Lemma 5.

If $d = 0$, then G is $2nK_1$. If $d = 1$, then G is nK_2. Hence, we can assume $d \geq 2$. Under this assumption, we show that G consists of exactly two components. Let $\Gamma_1, \ldots, \Gamma_k$ be the components of G and assume for the sake of contradiction that $k \geq 3$. Observe that if $d = 2$, then all the components $\Gamma_1, \ldots, \Gamma_k$ are cycles of the same length and if $d \neq 2$, then they all are complete graphs of the same order by Lemma 7.

If k is even, we proceed as follows: Let A be a set consisting of the vertices of the components $\Gamma_1, \ldots, \Gamma_{k/2}$ and B a set consisting of the vertices of the components $\Gamma_{k/2+1}, \ldots, \Gamma_k$. Let a_1 be a vertex of Γ_1 and a_2 a vertex of Γ_2. Note that both $\Gamma_1 \setminus a_1$ and $\Gamma_2 \setminus a_2$ are connected. Finally, let $b_1 b_2$ be an edge of $\Gamma_{k/2+1}$. Note that $\Gamma_{k/2+1} \setminus \{b_1, b_2\}$ is connected since $\Gamma_{k/2+1}$ is a cycle or complete graph. Consider now the sets A' and B' obtained from A and B by interchanging the pair of the vertices a_1 and a_2 and the pair of the vertices b_1 and b_2. Note that the number of components of $G[A']$ is

$k+1$ and the number of components of $G[B']$ is $k+2$. This contradicts the assumption that the graph G is weakly equipartite.

We now consider the case that $k \geq 3$ is odd. Let ν be the common order of the components of G. Since k is odd, the number ν must be even. Split Γ_1 into two connected parts A_0 and B_0 of orders $\nu+1$ and $\nu-1$ (recall that Γ_1 is a cycle or a complete graph). Let uv be an edge of Γ_2 such that $\Gamma_2 \setminus \{u,v\}$ is connected. Consider now the following partition of the vertex set $V(G)$:

$$\begin{aligned} A &= A_0 \cup (V(\Gamma_2) \setminus \{u,v\}) \cup V(\Gamma_3) \cup \ldots \cup V(\Gamma_{(k+1)/2}) \\ B &= B_0 \cup \{u,v\} \cup V(\Gamma_{(k+3)/2}) \cup \ldots \cup V(\Gamma_k) \end{aligned}$$

By the choice of A_0, B_0, u and v, the subgraph $G[A]$ consists of exactly $(k+1)/2$ components and the number of components of $G[B]$ is at least $(k+1)/2+1$. This again contradicts that the graph G is weakly equipartite.

We have now proven that G consists of two components. If both the components are complete graphs, then G is $2K_n$. Otherwise, they are both cycles, i.e., G is $2C_l$. We show that it must hold that $l \leq 4$ and thus G is either $2K_3$ or $2C_4$. Assume for the sake of contradiction that $l \geq 5$. Let A and B be the vertex sets of the two cycles of G. Let a_1 and a_2 be two non-adjacent vertices of the first cycle and b_1 and b_2 be two adjacent vertices of the second cycle. Let us consider sets A' and B' obtained from the sets A and B by interchanging the pair of the vertices a_1 and a_2 and the pair of the vertices b_1 and b_2. Observe that the number of components of $G[A']$ is 3 and the number of components of $G[B']$ is 4. This is impossible because G is a weakly equipartite graph — a contradiction.

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It is noteworthy that the only disconnected (weakly) equipartite graph with two components that are not complete graphs is the graph $2C_4$. This corresponds to the existence of three symmetry types of equipartite boxes in $\mathbb{R}^3$.

4.2 Connected weakly equipartite graphs

First, we state a lemma that each d-regular weakly equipartite graph of order $2n$ with $d \leq n-1$ contains a subgraph on n vertices with a very special structure:

Lemma 9 *Let G be a weakly equipartite d-regular graph of order $2n$ with $d \leq n-1$ and let v_0 be an arbitrary vertex of G. Then, there is a subset $A \subseteq V(G)$ with $|A| = n$ such that $G[A]$ consists of a single component of order $d+1$ and the remaining components of $G[A]$ are isolated vertices. In addition, the component of order $d+1$ of $G[A]$ is precisely the closed neighborhood of the vertex v_0 in G.*

Proof: Let A_0 be a subset of n vertices of G which contains the closed neighborhood of the vertex v_0 such that the number of edges of $G[A_0]$ is the least possible. If A_0 is of the form described in the statement of the lemma, we are done. Otherwise, there exists a vertex v which is neither v_0 nor a neighbor of v_0 joined by an edge to another vertex of A_0. Let $B_0 = V(G) \setminus A_0$ and v_0' the counterpart of the vertex v_0 in $G[B_0]$. Note that all the d neighbors of v_0' are contained in B_0.

Let us consider the set $A_0' = (A_0 \cup \{v_0'\}) \setminus \{v\}$. Clearly, the set A_0' contains the closed neighborhood of the vertex v_0. In addition, the vertex v_0' is an isolated vertex of $G[A_0']$ since all the neighbors of v_0' are contained in the set B_0. Therefore, the number of edges of $G[A_0']$ is smaller than the number of edges of $G[A_0]$ which contradicts the choice of the set A_0.

■

We now show that all regular connected weakly equipartite graphs of order $2n$ with maximum degree at most $n-1$ are bipartite:

Lemma 10 *If G is a weakly equipartite d-regular connected graph of order $2n$ with $\max\{2, n-d\}+1 \leq d \leq n-1$.*

Proof: Let $k = n-d-1$. Note that k is 0, 1 or 2 because d is be at least $n-3$ by Lemma 4. Fix a set $A \subseteq V(G)$ of size n such that the subgraph $G[A]$ consists exactly of k isolated vertices and a component of order $d+1$ which contains a vertex of degree d. Such a set A exists by Lemma 9. Let Γ_A be the component of order $d+1$ of $G[A]$, γ_A a vertex of degree d contained in Γ_A and X_A the set consisting of the k isolated vertices of $G[A]$. Since the graph G is weakly equipartite, the subgraph $G[B]$ with $B = V(G) \setminus A$ is isomorphic to $G[A]$. Let Γ_B, γ_B and X_B be isomorphic images of Γ_A, γ_A and X_A in $G[B]$, respectively. In addition, let $\Gamma_A' = \Gamma_A \setminus \gamma_A$ and $\Gamma_B' = \Gamma_B \setminus \gamma_B$ (see Figure 3). By the choice of Γ_B and γ_B, the graphs Γ_A' and Γ_B' are isomorphic.

We show that both the graphs Γ_A' and Γ_B' consist of isolated vertices by considering three distinct cases. Before this, note that the set Γ_A' contains a vertex adjacent to a vertex from the set B since the graph G is connected.

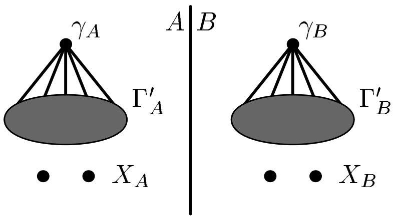

Figure 3: Notation used in the proof of Lemma 10.

First, assume for the sake of contradiction that the graphs Γ'_A and Γ'_B are connected. Let x be a vertex of Γ'_A adjacent to a vertex from the set B. Let us partition the vertex set of G into two sets A' and B' as follows:

$$\begin{aligned} A' &= (A \setminus \{x\}) \cup \{\gamma_B\} \\ B' &= (B \setminus \{\gamma_B\}) \cup \{x\} \end{aligned}$$

The subgraph $G[A']$ consists precisely of $k+2$ components: One of them is formed by the vertex γ_A and its $d-1$ neighbors in A' and the remaining components are isolated vertices, namely, the vertex γ_B and the k vertices of X_A. On the other hand, the subgraph $G[B']$ consists of at most $k+1$ components. In order to see this, note that $G[B \setminus \{\gamma_B\}]$ consists of $k+1$ components (Γ'_B and the isolated vertices of X_B). Since a vertex x is joined by an edge to a vertex of B and this vertex cannot be γ_B because all the neighbors of γ_B are in the set B, the vertex x is not isolated in $G[B']$. Hence, $G[B']$ consists of at most $k+1$ components. Since the numbers of components of $G[A']$ and $G[B']$ are different, we obtain a contradiction with our assumption that G is weakly equipartite.

Let us now consider the case that the graphs Γ'_A and Γ'_B are formed by two components each. Let k_1 and k_2 be the orders of the two components of Γ'_A. We can assume that $k_1 \geq k_2$. Since the graph G is d-regular, each vertex of Γ'_A has at least one neighbor among the vertices of B. Recall that no vertex of A is adjacent to the vertex γ_B. Choose x to be any vertex of the first (larger) component of Γ'_A. We consider the following partition of the set $V(G)$:

$$\begin{aligned} A' &= (A \setminus \{x\}) \cup \{\gamma_B\} \\ B' &= (B \setminus \{\gamma_B\}) \cup \{x\} \end{aligned}$$

The graph $G[A']$ consists of exactly $k+2$ components: One of the components is formed by the vertex γ_A and its $d-1$ neighbors and the remaining components are formed by isolated vertices, namely the vertex γ_B and the vertices of X_A. On the other hand, $G[B \setminus \{\gamma_B\}]$ consists of exactly $k+2$ components, namely two components of $\Gamma_B \setminus \gamma_B$ and k isolated vertices of X_B. A vertex x is joined to at least one of these $k+2$ components. Since the graph G is weakly equipartite, the graphs $G[A']$ and $G[B']$ are isomorphic. In particular, $G[B']$ must contain $k+1$ isolated vertices. But this is possible only if $k_2 = 1$ and the vertex x is adjacent only to the vertices of the component of Γ'_B of order k_1. Let y be the only vertex of the component of order $k_2 = 1$ of Γ'_B. Since we could choose x to be any vertex of the larger component of Γ'_A, we can conclude that the vertex y is adjacent to no vertex of the component of order k_1 of Γ'_A. In the following two paragraphs, we show that $d \leq \max\{2, n-d\} = \max\{2, k+1\}$ which contradicts our assumption on the degree d.

Let us first consider the case that $k = 0$. By the assumption on the structure of $G[B]$, we know that γ_B is the only neighbor of y in B. Hence, the only two vertices to which y could be adjacent are γ_A and the counterpart of y in Γ'_A. But we know that y is not adjacent to the vertex γ_A (all the neighbors of γ_A are in A). Thus the degree of y is at most 2, i.e., $d \leq 2$.

Next, we consider the case that $k > 0$. Let y' be one of the isolated vertices in $G[B]$. Note that all the neighbors of y' are contained in A. Therefore, y' can be adjacent only to the single vertex of the component of order $k_2 = 1$ of Γ'_A and the k vertices from X_A (recall that all the neighbors of γ_A are in A). Thus, the degree of y' is at most $k+1$, i.e., $d \leq k+1$.

We have now excluded the case that the graphs Γ'_A and Γ'_B are formed by two components each.

The final case is that the graph Γ'_A consists of least three components. Note that the degree of any vertex from $V(\Gamma'_A)$ in $G[A]$ as well as of any vertex from $V(\Gamma'_B)$ in $G[B]$ is at most $d-2$. Let x be any vertex of Γ'_A. Split the vertex set of G to two parts A' and B' as follows:

$$\begin{aligned} A' &= (A \setminus \{x\}) \cup \{\gamma_B\} \\ B' &= (B \setminus \{\gamma_B\}) \cup \{x\} \end{aligned}$$

The degree of the vertex γ_A in $G[A']$ is $d-1$. We now estimate the degrees of the vertices in $G[B']$. Each of the vertices of X_B can now have degree at most 1 which is strictly smaller than $d-1$ (recall $d \geq 3$). The degrees of the vertices of $V(\Gamma'_B)$ in $G[B]$ are at most $d-2$. Now, the vertex γ_B, which is adjacent to each vertex of $V(\Gamma'_B)$, is moved to the set A and the vertex x is added. Hence, in $G[B']$, each of the vertices of $V(\Gamma'_B)$ has degree at most $d-2$. Since the graph G is weakly equipartite, the subgraphs $G[A']$ and $G[B']$ are isomorphic. All the vertices of $G[B']$ with a possible exception of x has degree at most $d-2$. Thus, the isomorphism between $G[A']$ and $G[B']$ must map the vertex γ_A to the vertex x. In particular, the vertex x has $d-1$ neighbors in the set $B \setminus \{\gamma_B\}$. The degree of x in $G[A]$ is one because the graph G is d-regular. Since the choice of x was arbitrary, we infer that both the graphs Γ'_A and Γ'_B are comprised solely by isolated vertices.

Split now the vertex set of G as follows:

$$\begin{aligned} V_1 &= V(\Gamma'_A) \cup X_A \cup \{\gamma_B\} \\ V_2 &= V(\Gamma'_B) \cup X_B \cup \{\gamma_A\} \end{aligned}$$

Since all the neighbors of γ_B are in the set B, the set V_1 is independent. The set V_2 is also independent because the graph G is weakly equipartite. Then, the graph G is obviously bipartite (with parts V_1 and V_2) as desired.

■

Finally, we characterize weakly equipartite connected bipartite regular graphs:

Lemma 11 *Let G be a connected bipartite d-regular weakly equipartite graph of order $2n$ with $3 \leq d \leq n-1$. Then, $G = K_{n,n} \setminus nK_2$.*

Proof: Let V_1 and V_2 be the two independent sets to which G can be partitioned. Since G is regular, we have $|V_1| = |V_2|$. In the rest, we show $d = n-1$. This immediately yields the statement of the lemma. By Lemma 4, it is enough to exclude the cases that $d = n-2$ and $d = n-3$.

If $d = n-2$, let w be a vertex of V_1 and let x and y be the two vertices of V_2 which are not adjacent to w. Note that $n \geq 5$ because $d \geq 3$. Since $2d = 2n-4 > n$, there is a common neighbor w' of x and y.

If $d = n-3$, let w be again a vertex of V_1 and let x, y and z be the three vertices of V_2 which are not adjacent to w. Note that $n \geq 6$ because $d \geq 3$. Since $3d = 3n-9 > n$, at least two of the vertices x, y and z have a common neighbor. Assume that x and y are such two vertices and w' is their common neighbor.

We now proceed jointly for both the cases $n = d-3$ and $n = d-2$. Let us split the vertex set of G into sets A and B as follows:

$$A = (V_1 \setminus \{w, w'\}) \cup \{x, y\}$$

$$B = (V_2 \setminus \{x,y\}) \cup \{w,w'\}$$

The degrees of x and y in $G[A]$ are exactly $d-1$ by the choice of w and w'. Each vertex of V_1 is adjacent to at least one vertex of $V_2 \setminus \{x,y\}$ (recall that d is at least three) and thus each vertex of V_1 has degree at most $d-1$ in $G[A]$. We can conclude that the maximum degree of $G[A]$ is $d-1$. On the other hand, $G[B]$ contains a vertex of degree d (the vertex w). Hence, the subgraphs $G[A]$ and $G[B]$ are not isomorphic. Therefore, d is neither $n-3$ nor $n-2$.

We conclude that the graph G is an $(n-1)$-regular bipartite graph of order $2n$.

■

5 Characterization of equipartite and weakly equipartite graphs

Before we prove Theorem 14, we state two lemmas on 2- and 3-regular weakly equipartite graphs whose cases were not covered in the previous two sections:

Lemma 12 *If C_l is a weakly equipartite cycle, then its length l is either four or six.*

Proof: Clearly, the graphs C_4 and C_6 are weakly equipartite. Let us now consider a cycle $C_l = v_1 \dots v_l$ of an even length $l > 6$. Split the set $V(G)$ into two parts A and B:

$$\begin{aligned} A &= \{v_1, v_4, \dots, v_{l/2+2}\} \\ B &= \{v_2, v_3, v_{l/2+3}, \dots, v_l\} \end{aligned}$$

Clearly, the subgraphs $G[A]$ and $G[B]$ are not isomorphic. Hence, no cycle C_l with $l > 6$ is a weakly equipartite graph.

■

Before proving Theorem 14, we need to exclude another case not covered by the previous lemmas:

Lemma 13 *There is no weakly equipartite cubic graph of order* 12.

Proof: Let G be a cubic graph of order $2n = 12$. Assume for the sake of contradiction that G is weakly equipartite.

Let us first consider the case that the graph G is triangle-free. Let v_0 be any vertex of G and let A be a set of vertices of G as in Lemma 9, i.e., the subgraph $G[A]$ is isomorphic to $K_{1,3} + K_1 + K_1$. Let $B = V(G) \setminus A$. Since the graph G is weakly equipartite, the subgraph $G[B]$ is isomorphic to $G[A]$. Let v_0' be the counterpart of v_0 in $G[B]$. Since G is cubic, all the neighbors of v_0 are in A and all the neighbors of v_0' in B. In particular, the sets $A' = (A \setminus \{v_0\}) \cup \{v_0'\}$ and $B' = (B \setminus \{v_0\}) \cup \{v_0'\}$ are independent. Since $A' \cup B' = V(G)$, the graph G is bipartite. But there is no weakly equipartite cubic bipartite graph of order 12 by Lemma 11.

The remaining case is that G contains a triangle, say a triangle $v_1v_2v_3$. Let A_1 be a subset of six vertices of $V(G)$ which contains the vertices v_1, v_2 and v_3 and let $B_1 = V(G) \setminus A_1$. Since the graph G is weakly equipartite, the subgraph $G[B_1]$ is isomorphic to the subgraph $G[A_1]$. In particular, $G[B_1]$ contains a triangle, say a triangle $v_4v_5v_6$. Let $A_2 = \{v_1, v_2, v_3, v_4, v_5, v_6\}$ and $B_2 = V(G) \setminus A_2$. The subgraphs $G[A_2]$ and $G[B_2]$ are isomorphic because G is weakly equipartite. In particular, the subgraph $G[B_2]$ contains two vertex-disjoint triangles, say triangles $v_7v_8v_9$ and $v_{10}v_{11}v_{12}$. Consider now the following two sets A and B:

$$\begin{aligned} A &= \{v_1, v_2, v_3, v_4, v_7, v_{10}\} \\ B &= \{v_5, v_6, v_8, v_9, v_{11}, v_{12}\} \end{aligned}$$

Since the graph G is weakly equipartite, the graphs $G[A]$ and $G[B]$ are isomorphic. Since $G[A]$ contains the triangle $v_1v_2v_3$, $G[B]$ contains a triangle, too. Observe now that the maximum degree of $G[B]$ is two because each vertex of B has at least one neighbor among the vertices v_4, v_7 and v_{10} (which are contained in A). Thus, a triangle contained in $G[B]$ actually forms a component of $G[B]$. In addition, the graph $G[B]$ contains a perfect matching (consider the edges v_5v_6, v_8v_9 and $v_{11}v_{12}$). But this is impossible because one of the components of $G[B]$ is a triangle — a contradiction.

■

We are now ready to characterize weakly equipartite graphs. Let us recall that all weakly equipartite graphs of order six and eight are depicted in Figures 1 and 2.

Theorem 14 *A graph G is weakly equipartite if and only if it is one of the following graphs:*

$$2nK_1,\ nK_2,\ 2C_4,\ K_{n,\,n} \setminus nK_2 \text{ and } 2K_n,$$

or one of their complements:

$$K_{2n},\ K_{2n} \setminus 2K_n,\ K_8 \setminus 2C_4,\ 2K_n + nK_2 \text{ and } K_{n,\,n}.$$

Proof: It is straightforward to verify that all the graphs listed in the statement of the theorem are weakly equipartite. We prove that no other graph is weakly equipartite. Fix a weakly equipartite graph G of order $2n$. By Lemma 4, the graph G is d-regular with $d \in \{0, 1, n-3, n-2, n-1\}$. By Proposition 2, we can assume that $d \leq n-1$ (otherwise, we consider the complement of G).

If the graph G is disconnected, then G is one of the graphs $2nK_1$, nK_2, $2C_4$ and $2K_n$ by Theorem 8. Let us assume in the rest that G is connected. In particular, $d \geq 2$ unless $G = K_2$. If $d = 2$, then G is a cycle and its length is either four or six by Lemma 12. Note that C_4 is $K_{2,2}$ and C_6 is $K_{3,3} \setminus 3K_2$. In the rest, we assume that $d \geq 3$.

If $n - d \leq 2$, then the graph G is bipartite by Lemma 10. If $n - d > 2$, we infer that $n-d = 3$ because $d \in \{n-3, n-2, n-1\}$. If $n \geq 7$ in addition to $n - d > 2$, the assumption of Lemma 10 is also satisfied:

$$d = n - 3 \geq \max\{2, n-d\} + 1 = \max\{2, 3\} + 1 = 4.$$

We conclude by Lemma 10 that if $d \neq n - 3$ or $n \geq 7$, the graph G is bipartite and by Lemma 11 that G is $K_{n,\,n} \setminus nK_2$.

The case which remains to consider is that $d = n - 3$ and $n \leq 6$. Recall that $d \geq 3$. Therefore, $d = 3$ and $n = 6$. However, there is no weakly equipartite cubic graph of order 12 by Lemma 13.

■

We finish this section by an immediate corollary of Theorem 14:

Corollary 15 *A graph G of order $2n$ is equipartite if and only if it is weakly equipartite.*

6 Equipartite polytopes

In this section, we show that every equipartite d-polytope has at most $2(d+1)$ vertices. Since the only equipartite 2-polytopes can be cycles (there are no other convex 2-polytopes), we can conclude:

Proposition 16 *An equipartite* 2*-polytope* P *has at most six vertices.*

We can now show that every equipartite d-polytope has at most $2(d+1)$ points:

Theorem 17 *If* P *is an equipartite* d*-polytope with* $2n$ *vertices, then* $n \leq d+1$.

Proof: If $d = 2$, the claim follows from Proposition 16. Henceforth, we assume that $d \geq 3$. Let $D = \{d_1, \ldots, d_k\}$ be the set of the lengths of all segments whose end-points are pairs of vertices of P. For each $i = 1, \ldots, k$, we define a graph G_i with $V(G_i) = X$ and edges corresponding to the pairs of vertices at distance d_i. The graphs G_i partition K_{2n} to edge-disjoint subgraphs. Since the polytope P is equipartite, the graph $\bigcup_{i \in I} G_i$ is also equipartite for each set $I \subseteq \{1, \ldots, k\}$. In particular, each G_i is a non-empty equipartite graph of order $2n$.

We show that $n \leq d+1$. Assume for contradiction the opposite, i.e., $n \geq d+2$. Since $\mathbb{R}^d$ does not contain $d+2$ distinct points with all distances equal, no graph G_i contains a clique of order n. Note that the only equipartite graphs of order $2n$ without a clique of order n are nK_2, $K_{n,n}$ and $K_{n,n} \setminus nK_2$. Hence, each G_i is isomorphic to nK_2, $K_{n,n}$ or $K_{n,n} \setminus nK_2$. Therefore, $k \geq 3$.

At most one of the graphs G_i can be isomorphic to nK_2. Indeed, if two of the graphs G_j and $G_{j'}$ were isomorphic to nK_2, then the graph $G_j \cup G_{j'}$ would be an equipartite 2-regular graph of order $2n \geq 2(d+2) \geq 10$. But there is no such equipartite graph by Theorem 14.

Since at most one of the graphs G_i is isomorphic to nK_2 and $k \geq 3$, two of the graphs G_i, say G_1 and G_2, are isomorphic to $K_{n,n}$ or $K_{n,n} \setminus nK_2$. Both G_1 and G_2 contain $K_{n,n} \setminus nK_2$ as a subgraph. Since the graphs G_i partition the complete graph K_{2n}, the graph G_2 is a subgraph of the complement of the graph G_1. This immediately yields that $K_{n,n} \setminus nK_2$ is a subgraph of its complement, i.e., $K_{n,n} \setminus nK_2 \subseteq 2K_n + nK_2$. However, this is not true for $n \geq 4$ — a contradiction.

■

7 Equipartite 2- and 3-polytopes

We first characterize equipartite 2-polytopes:

Theorem 18 *Equipartite polygons are precisely isogonal quadrangles and hexagons, i.e., they are precisely rectangles, regular hexagons and hexagons whose all interior angles are all equal to* 120° *and their sides alternate between two lengths.*

Proof: It is easy to see that isogonal quadrangles and hexagons, i.e., squares, non-square rectangles, regular hexagons and non-regular isogonal hexagons, are equipartite. These are all isogonal polygons with at most six vertices. Since there is no equipartite polygon with eight or more vertices by Theorem 17, the statement of the theorem now follows.

∎

Equipartite 3-polytopes are more interesting. In order to describe all of them, we start by recalling that the symmetry types of all isogonal 3-polytopes have been determined [6]. It is well-known that each isogonal 3-polytope is combinatorially equivalent to one of the Platonic or Archimedean solids (we include prisms and antiprisms among Archimedean solids). We can now show the following:

Theorem 19 *Equipartite* 3*-polytopes are precisely tetrahedra,* 3*-sided prisms,* 4*-sided prisms and* 3*-sided antiprisms.*

Proof: Each equipartite 3-polytope has at most 8 vertices by Theorem 17. It is easy to check that the only isogonal polytope on at most 8 vertices not listed in the statement of the theorem is a 4-sided antiprism. However, no 4-sided antiprism is equipartite (consider two successive vertices of one base and two non-adjacent vertices of the other). In order to assist the reader to verify the proof, we list all isogonal 3-polytopes with 10 and more vertices identified by their usual names and symbols [3]: a regular dodecahedron (5.5.5), a truncated cube (3.8.8), a truncated dodecahedron (3.10.10), a cuboctahedron (3.4.3.4), an icosidodecahedron (3.5.3.5), a truncated octahedron (4.6.6), a regular icosahedron (3.3.3.3.3), a great rhombicuboctahedron (4.6.8), a truncated icosahedron (5.6.6), a great rhombicosidodecahedron (4.6.10), a small rhombicuboctahedron (3.4.4.4.4), a small rhombicosidodecahedron (3.4.5.4), a snub cube (3.3.3.3.4), a snub dodecahedron

(3.3.3.3.5), a truncated tetrahedron (3.6.6), an n-prism and an n-antiprism with $n \geq 5$.

■

An inspection of symmetry types of isogonal tetrahedra, 3-sided prisms, 4-sided prisms and 3-sided antiprisms leads to the following exhaustive list of possible symmetry types of equipartite 3-polytopes (representatives of the symmetry types are depicted in Figure 4). The symmetry groups are denoted as in [2]:

- **Tetrahedra**
  - The 2-parameter type SIG1 with the symmetry group $[2,2]^+$, i.e., a convex hull of congruent non-parallel segments, perpendicular to the line connecting their midpoints but not to each other.
  - The 1-parameter type SIG1 with the symmetry group $[2^+,4]$, i.e., the limit case of the above in which the segments are perpendicular to each other, but the faces are not equilateral triangles, i.e., they are non-equilateral isosceles triangles.
  - The regular tetrahedron SIG5 with the symmetry group $[3,3]$.
- **3-sided prisms**
  - The 1-parameter type SIG60(3) with the symmetry group $[2,3]$, i.e., a straight prism with an equilateral triangle as a base.
- **4-sided prisms**
  - The 2-parameter type SIG14 with the symmetry group $[2,2]$, i.e., a rectangular box with three distinct dimensions.
  - The 1-parameter type SIG19 with the symmetry group $[2,4]$, i.e., a straight prism with a square as a base and with non-square mantle faces.
  - The cube SIG19 with the symmetry group $[3,4]$.
- **3-sided antiprisms**
  - The 2-parameter type SIG30 with the symmetry group $[2,3]^+$, i.e.. a convex hull of two congruent equilateral triangles in horizontal planes, perpendicular to the line connecting their centers, with sides not parallel. The side faces are either congruent

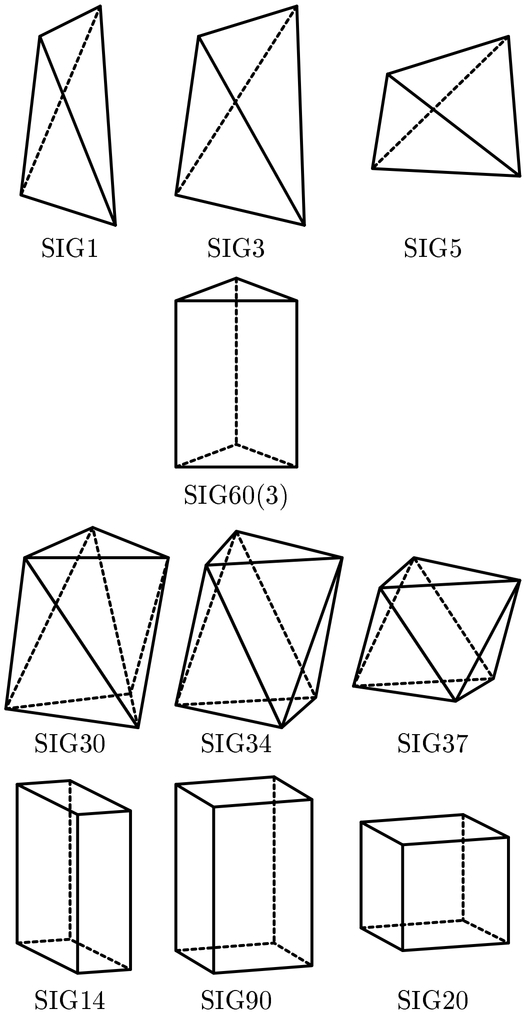

Figure 4: Examples of equipartite 3-polytopes for each possible symmetry type.

scalene triangles, or congruent non-horizontal isosceles triangles with non-horizontal bases.

– The 1-parameter type SIG34 with the symmetry group $[2^+, 6]$, i.e., the limit case of the above in which the side faces are non-equilateral isosceles triangles with horizontal bases.

– The regular octahedron SIG37 with the symmetry group $[3, 4]$.

We conclude this section with a construction of an equipartite 4-polytope with 8 vertices. It is unique in the sense that this construction does not generalize to higher dimensions.

Take two congruent rectangles lying in a pair of orthogonal 2-dimensional subspaces of R^4. For instance, consider the two rectangles with vertices $\{(a, b, 0, 0), (a, c, 0, 0), (d, b, 0, 0), (d, c, 0, 0)\}$ and $\{(0, 0, a, b), (0, 0, a, c), (0, 0, d, b), (0, 0, d, c)\}$. Note that for each two triples of vertices of a rectangle in $\mathbb{R}^2$ there is an isometry of the rectangle that maps one triple onto the other triple. Also any two vertices of a rectangle can be isometrically mapped onto the other two vertices. It is now easy to verify that the $4-$polytope obtained by the convex hull of these 8 points in $\mathbb{R}^4$ is equipartite.

8 Constructions of equipartite polytopes

In this section, we develop tools for constructing equipartite polytopes. We start with explicit constructions of d-polytopes with $2d$ vertices. We also identify three distinct types of equipartite $(2d+1)$-simplices, construct equipartite $2d$-polytopes with $2d + 2$ vertices and equipartite d-polytopes with $2d$ and $2d + 2$ vertices.

Lemma 20 *The cross polytope $P = conv(\pm e_1, \dots, \pm e_d)$ where e_i are the unit vectors in $\mathbb{R}^d$ is an equipartite polytope.*

Proof: Let $F = \{f_1, \dots, f_d\}$ be any subset of d points from $\{\pm e_1, \dots, \pm e_d\}$ and let G be its complement. We shall construct an orthogonal matrix P that maps F onto G. Let:

- $F_1 = \{i | e_i \in F, -e_i \notin F\}$
- $F_2 = \{i | -e_i \in F, e_i \notin F\}$
- $F_3 = \{i | e_i \in F, -e_i \in F\}$

Let G_1, G_2, G_3 be defined similarly. Clearly, $F_1 = G_2$, $F_2 = G_1$, $F_3 \cap G_3 = \emptyset$ and $|F_3| = |G_3|$. Let $\psi : F_3 \to G_3$ be a bijection from F_3 onto G_3 such that $\psi(e_i) = -\psi(-e_i)$. Define the matrix P by:

$$P_{i,j} = \begin{cases} -1 & \text{if } i = j, i \in F_1 \cup F_2, \\ 1 & \text{if } i \in F_3, j = \psi(i), \\ 1 & \text{if } j \in G_3, j = \psi(i), \\ 0 & \text{otherwise.} \end{cases}$$

Note that P is a permutation matrix in which some of the 1-entries are replaced by -1. So, P is an orthogonal matrix and it is simple to verify that $P : F \to G$ is a bijection.

■

Lemma 22 generalizes the proof above. It is based on the following well-known basic property of orthogonal real matrices:

Lemma 21 *If $\{u_1, \ldots, u_k\}$ and $\{v_1, \ldots, v_k\}$ are two sets of unit vectors in R^d such that $\langle u_i, u_j \rangle = \langle v_i, v_j \rangle$ $\forall \{i,j\}$, then there is an orthogonal matrix τ such that $\tau(u_i) = v_i$.*

Lemma 22 *Let $U = \{u_1, \ldots, u_k\}$ and $V = \{v_1, \ldots, v_k\}$ be two disjoint sets of unit vectors in $\mathbb{R}^d$. Suppose that $\langle u_i, u_j \rangle = \langle v_i, v_j \rangle = \alpha$ for all $1 \le i < j \le k$, $\langle u_i, v_j \rangle = \langle u_j, v_i \rangle = \beta$ for all $1 \le i < j \le k$ and $\langle u_i, v_i \rangle = \gamma$ for all $1 \le i \le k$. Let P be the convex hull of the end-points of the vectors of $U \cup V$. Then, the polytope P is equipartite.*

Proof: We first note that $k \le d+1$ and that there is essentially one configuration of $d+1$ equiangular unit vectors in R^d; the unit vectors connecting the center of the regular d-simplex to its vertices. The angle between any two of these vectors is $\arccos -\frac{1}{d}$.

Let A be an arbitrary k-element subset of the vectors $\{u_1, \ldots, u_k, v_1, \ldots, v_k\}$ and let B be the remaining k vectors. By Lemma 21 it is enough to show that the vectors in A and B can be sequenced so that $\langle a_i, a_j \rangle = \langle b_i, b_j \rangle$ $\forall \{i,j\}$. The proof follows the same approach used in the proof of Lemma 20. In A we first list the vectors u_i for which $u_i \in A$ but $u_i \notin B$. We then list the vectors v_i for which $v_i \in A$ but $v_i \notin B$. We finally list the pairs (in order) of vectors u_i, v_i for which both vectors are in A. We sequence the vectors in B similarly (note that each of the 3 sets of vectors both in A and B contain the same number of vectors).

It is now simple to check that this sequencing guarantees the existence of an orthogonal transformation that maps A onto B.

■

We now present three constructions of equipartite polytopes:

Theorem 23 *For every $d \geq 2$, there is an equipartite d-polytope with $2d+2$ vertices.*

Proof: Let $u_1, \ldots, u_{d+1}$ be a set of $d+1$ equiangular unit vectors in $\mathbb{R}^d$. Note that the convex hull of the end-points of the vectors $u_1, \ldots, u_{d+1}$ is a regular d-simplex. Set $v_i = -u_i$ for each $i = 1, \ldots, d+1$ and let P be the convex hull of the end-points of the $2d+2$ vectors $u_1, \ldots, u_{d+1}, v_1, \ldots, v_{d+1}$. P is a centrally symmetric convex d-polytope with $2d+2$ vertices. By Lemma 22, the polytope P is equipartite.

■

Theorem 24 *The prism over a regular $(d-1)$-simplex is an equipartite d-polytope with $2d$ vertices for every $d \geq 2$.*

Proof: Let $w_1, \ldots, w_d$ be a set of d equiangular unit vectors contained in a $(d-1)$-dimensional subspace of $\mathbb{R}^d$. For $0 < \alpha < 1$, We define a d-polytope P_α to be the convex hull of the end-points of the $2d$ vectors

$$\alpha w_1 + \sqrt{1-\alpha^2}\, e, \ldots, \alpha w_d + \sqrt{1-\alpha^2}\, e$$

$$\alpha w_1 - \sqrt{1-\alpha^2}\, e, \ldots, \alpha w_d - \sqrt{1-\alpha^2}\, e$$

where e is the unit vector orthogonal to the subspace of $\mathbb{R}^d$ spanned by the vectors $w_1, \ldots, w_d$. Clearly, each prism over a regular $(d-1)$-simplex is conformable to the polytope P_α for a suitable $\alpha, 0 < \alpha < 1$. Since P_α is equipartite by Lemma 22, the claim of the theorem now readily follows.

■

Theorem 25 *The convex hull of two regular isomorphic d-simplices centered at the origin which lie in orthogonal d-dimensional subspaces of R^{2d} is an equipartite $(2d)$-polytope with $2d+2$ vertices.*

Proof: Let $u_1, \ldots, u_d$ and $v_1, \ldots, v_d$ be vectors from the origin to the vertices of each of the two regular d-simplices. We can assume without loss of generality that all the vectors $u_1, \ldots, u_d$ and $v_1, \ldots, v_d$ are unit. Lemma 22 now implies that the polytope described in the statement of the theorem is equipartite.

■

Note that the construction described in Theorem 25 provides a cyclic neighborly $(2d)$-polytope with $2d+2$ vertices. Let us remark that cyclic d-polytopes for $d \geq 4$ have the complete graph as their 1-skeleton and yet they fail to be equipartite when they have more than $2d+2$ vertices. Furthermore, by checking the distances between points on various moment curves used to construct cyclic polytopes with more than $d+2$ vertices [5, Section 4.7], one can show that they fail to be vertex transitive.

We now focus on the number of distinct symmetry types of equipartite $(2d+1)$-simplices:

Theorem 26 *For each $d \geq 2$, there are at least three distinct symmetry types of equipartite $(2d+1)$-simplices.*

Proof: The regular $(2d+1)$-simplex is clearly equipartite. Another $(2d+1)$-simplex, of a different symmetry type, can be obtained as follows:

Let $\{X_1, \ldots, X_{d+1}\}$ and $\{Y_1, \ldots, Y_{d+1}\}$ be the vertices of two regular d-simplices centered at the origin. Let $A_i = (X_{i,1}, \ldots, X_{i,d}, -1, 0, \ldots, 0) \in \mathbb{R}^{2d+1}$ and $B_i = (0, \ldots, 0, +1, Y_{i,1}, \ldots, Y_{i,d}) \in \mathbb{R}^{2d+1}$ for $1 \leq i \leq d+1$. Let a_i and b_i be a vector from the origin to the point A_i and B_i, respectively. In order to show that the convex hull C of the points $A_1, \ldots, A_d, B_1, \ldots, B_d$ is a $(2d+1)$-simplex, we verify that the vectors $a_1, \ldots, a_d, b_1, \ldots, b_d$ are affinely independent. Indeed:

$$\det \begin{pmatrix} 1 & x_{1,1} & \cdots & x_{1,d} & -1 & 0 & \cdots & 0 \\ 1 & x_{2,1} & \cdots & x_{2,d} & -1 & 0 & \cdots & 0 \\ \vdots & \vdots & & \vdots & \vdots & \vdots & & \vdots \\ 1 & x_{d+1,1} & \cdots & x_{d+1,d} & -1 & 0 & \cdots & 0 \\ 1 & 0 & \cdots & 0 & 1 & y_{1,1} & \cdots & y_{1,d} \\ 1 & 0 & \cdots & 0 & 1 & y_{2,1} & \cdots & y_{2,d} \\ \vdots & \vdots & & \vdots & \vdots & \vdots & & \vdots \\ 1 & 0 & \cdots & 0 & 1 & y_{d+1,1} & \cdots & y_{d+1,d} \end{pmatrix} =$$

$$\det \begin{pmatrix} 1 & x_{1,1} & \cdots & x_{1,d} & 0 & 0 & \cdots & 0 \\ 1 & x_{2,1} & \cdots & x_{2,d} & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & & \vdots & \vdots & \vdots & & \vdots \\ 1 & x_{d+1,1} & \cdots & x_{d+1,d} & 0 & 0 & \cdots & 0 \\ 1 & 0 & \cdots & 0 & 2 & y_{1,1} & \cdots & y_{1,d} \\ 1 & 0 & \cdots & 0 & 2 & y_{2,1} & \cdots & y_{2,d} \\ \vdots & \vdots & & \vdots & \vdots & \vdots & & \vdots \\ 1 & 0 & \cdots & 0 & 2 & y_{d+1,1} & \cdots & y_{d+1,d} \end{pmatrix} =$$

$$\det \begin{pmatrix} 1 & x_{1,1} & \cdots & x_{1,d} \\ 1 & x_{2,1} & \cdots & x_{2,d} \\ \vdots & \vdots & & \vdots \\ 1 & x_{d+1,1} & \cdots & x_{d+1,d} \end{pmatrix} \cdot \det \begin{pmatrix} 2 & y_{1,1} & \cdots & y_{1,d} \\ 2 & y_{2,1} & \cdots & y_{2,d} \\ \vdots & \vdots & & \vdots \\ 2 & y_{d+1,1} & \cdots & y_{d+1,d} \end{pmatrix} \neq 0$$

Observe now that the vectors $a_1, \ldots, a_d$ and $b_1, \ldots, b_d$ satisfy the assumption of Lemma 22. Hence, C is an equipartite $(2d+1)$-simplex. Note that the simplex C is not regular. The symmetry group of this simplex acts imprimitively with two classes of imprimitivity, namely A and B, and no other types of imprimitivity. In particular, for $d \geq 2$, there are no imprimitivity classes of size 2. The symmetry group of this simplex is the wreath product of the groups S_{d+1} and S_2.

Finally, there is a third symmetry type of equipartite $(2d+1)$-simplices. Let $a_1, \ldots, a_{d+1}$ be $d+1$ equiangular unit vectors in $\mathbb{R}^d$ and let $e_1, \ldots, e_{d+1}$ be $d+1$ vectors of an orthonormal basis of $\mathbb{R}^{d+1}$. Note that the end-points of the vectors $a_1, \ldots, a_{d+1}$ form a regular d-simplex. We now define vectors $b_1, \ldots, b_{2d+2}$: The vector b_{2i-1} is equal to $(a_i|e_i)$ and b_{2i} to $(a_i|-e_i)$ where $(a|e)$ is a $(2d+1)$-dimensional vector obtained by concatenation of the vectors a and e. It is easy to see that the vectors $b_1, \ldots, b_{2d+2}$ are affinely independent vectors and hence the convex hull P of their end-points is a $(2d+1)$-simplex.

In order to see that the $(2d+1)$-simplex P is equipartite, note first that $\langle b_{2i-1}, b_{2i} \rangle = 0$ for each $i = 1, \ldots, d+1$ and $\langle b_{2i-1}, b_{2k} \rangle = \langle b_{2i}, b_{2k} \rangle = \arccos \frac{-1}{d}$ when $i \neq k$. Consider now an arbitrary permutation $\pi : \{1, \ldots, d+1\} \to \{1, \ldots, d+1\}$. Let π' be a mapping which maps, for $i = 1, \ldots, d+1$, the numbers $2i-1$ and $2i$ to the numbers $2\pi(i)-1$ and $2\pi(i)$, respectively, and let τ be a mapping which maps the end-point of the vector b_i, $i = 1, \ldots, 2d+2$, to the end-point of the vectors $b_{\pi'(i)}$. Since $\langle b_i, b_{i'} \rangle = \langle b_{\pi'(i)}, b_{\pi'(i')} \rangle$ for all $1 \leq i, i' \leq 2d+2$, the mapping τ is an isometry of the simplex P. The mapping τ is also an isometry of P if we redefine $\pi'(2i-1)$ to $2\pi(i)$ and $\pi'(2i)$ to $2\pi(i)-1$ for an arbitrary index i.

It is now a simple matter to verify that the isometries just described imply that the simplex P is an equipartite polytope. Note that the symmetry group of P is imprimitive with $d+1$ classes of imprimitivity, corresponding to the pairs of vertices b_{2i-1} and b_{2i} for $i = 1, \ldots, d+1$. Therefore, the symmetry group of P is the wreath product of the groups S_2 and S_{d+1}.

■

We now describe a general construction of equipartite polytopes from equipartite graphs. If G is a graph with vertices $v_1, \ldots, v_n$, the matrix $A_G(\alpha, \beta)$ for (not necessarily positive) real numbers α and β is defined as follows:

$$A_G(\alpha,\beta)_{ij} = \begin{cases} 0 & \text{if } i = j, \\ \alpha & \text{if } v_i v_j \in E(G) \text{ and} \\ \beta & \text{otherwise.} \end{cases}$$

Note that if $\overline{G}$ is the complement of a graph G, then $A_{\overline{G}}(\alpha,\beta) = A_G(\beta,\alpha)$.

Theorem 27 *If G is an equipartite graph of order $n = 2m$, the smallest eigenvalue λ of the matrix $A_G(\alpha,\beta)$ has multiplicity k and $\lambda \neq 0, -\alpha, -\beta$, then there is an equipartite d-polytope with n vertices where $d = n-k-1$ or $d = n-k$.*

Proof: For the sake of brevity, we assume that the vertex set of G consists of the numbers $1, \ldots, n$. The matrix $B = I - \lambda^{-1} A_G(\alpha,\beta)$ is a positive semidefinite matrix of rank $n-k$. Since the graph G is equipartite, it is r-regular for some $r \geq 0$ by Proposition 3. Hence, each row of the matrix $A_G(\alpha,\beta)$ contains the same number of α's. Therefore, the unit vector $(n^{-1/2}, \ldots, n^{-1/2}) \in \mathbb{R}^n$ is an eigenvector of $A_G(\alpha,\beta)$ corresponding to the eigenvalue $r\alpha + (n-r-1)\beta$. Let $v_1, \ldots, v_n$ be a complete set of mutually orthogonal unit eigenvectors of the matrix B and let V be the matrix whose rows are the vectors $v_1, \ldots, v_n$. Note that $VBV^{\mathrm{tr}} = D$ for a diagonal matrix D of order n with k diagonal entries equal to zero. By properly arranging the eigenvectors $v_1, \ldots, v_n$, we may assume that $D_{ii} = 0$ for $i = n-k+1, \ldots, n$.

Observe now that $B = (\sqrt{D}V)^{\mathrm{tr}}(\sqrt{D}V)$. Let $u_1, \ldots, u_n$ be the column vectors of the matrix $\sqrt{D}V$. Note that $\langle u_i, u_j \rangle = B_{ij}$ for all $1 \leq i, j \leq n$. Since the diagonal entries of the matrix B are equal to one, each of the vectors $u_1, \ldots, u_n$ is unit. In addition, all the vectors $u_1, \ldots, u_n$ are distinct because no entry of the matrix $A_G(\alpha,\beta)$ is equal to $-\lambda$. Since the last k rows of the matrix $\sqrt{D}V$ are zero vectors, the vectors $u_1,\ldots,u_n$ determine n distinct unit vectors of an $(n-k)$-dimensional subspace of $\mathbb{R}^n$. Let P be the convex hull of their end-points. If the vectors $u_1,\ldots,u_n$ are affinely independent, P is an $(n-k)$-polytope. If the smallest eigenvalue λ is strictly smaller than $r\alpha+(n-r-1)\beta$, then the vectors $u_1,\ldots,u_n$ have the same first coordinate and P is an $(n-k-1)$-polytope. In both cases, $\langle u_i,u_j\rangle=-\frac{\alpha}{\lambda}$ if $ij\in E(G)$ and $\langle u_i,u_j\rangle=-\frac{\beta}{\lambda}$ otherwise.

Let us show that the polytope P is equipartite. Consider an arbitrary subset $X\subseteq\{1,\ldots,n\}$ of size m and set $Y=\{1,\ldots,n\}\setminus X$. We aim to show that there is a symmetry of P which interchanges the end-points of the vectors u_i with $i\in X$ and the end-points of the vectors u_i with $i\in Y$. Since G is equipartite, there exists an automorphism τ_G of G which maps X onto Y. Define a mapping $\tau:\{1,\ldots,2n\}\to\{1,\ldots,2n\}$ which maps the end-points of a vector u_i to end-points of the vector $u_{\tau_G(i)}$. Observe that $\langle u_i,u_j\rangle=\langle u_{\tau(i)},v_{\tau(j)}\rangle$ for every $1\le i,j\le 2n$. Hence, τ can be viewed as a symmetry of P. Since the choice of X was arbitrary, the polytope P is equipartite. This completes the proof of Theorem 27.

■

We now briefly describe representatives of equipartite polytopes which we obtain for various equipartite graphs G and choices of α and β in Theorem 27. The actual polytopes obtained for a particular choice of α and β may differ from the described representative but their symmetry types are the same. The smallest eigenvalue of the matrix is denoted by λ as in Theorem 27.

- $A_{2nK_1}(\alpha,\beta)=A_{K_{2n}}(\beta,\alpha)$
  The eigenvalues of the matrix $A_{2nK_1}(\beta,\beta)$ are $(n-1)\beta$ with multiplicity one and $-\beta$ with multiplicity $2n-1$.

  If $\lambda=(n-1)\beta$, we obtain a regular $(2n-1)$-simplex.

  If $\lambda=-\beta$, the assumptions of Theorem 27 are not satisfied.

- $A_{nK_2}(\alpha,\beta)=A_{K_{2n}\setminus nK_2}(\beta,\alpha)$
  The eigenvalues of the matrix $A_{nK_2}(\alpha,\beta)$ are $\alpha+(2n-2)\beta$, $\alpha-2\beta$ and $-\alpha$ with multiplicities one, $n-1$ and n.

  We first consider the case that $\lambda=\alpha+(2n-2)\beta$. Let $u_1,\ldots,u_n$ be n equiangular unit vectors lying in an $(n-1)$-dimensional subspace of $\mathbb{R}^{2n-1}$ and $e_1,\ldots,e_n$ be n mutually orthogonal unit vectors which are also orthogonal to the vectors $u_1, \dots, u_n$. The $2n$ vertices of the constructed $(2n-1)$-polytope lie at the end-points of the vectors $u_1 + e_1, \dots, u_n + e_n, u_1 - e_1, \dots, u_n - e_n$.

If $\lambda = \alpha - 2\beta$, we obtain an equipartite n-polytope with $2n$ vertices lying at the end-points of the vectors e_i and $-e_i$ where $e_1, \dots, e_n$ are n mutually orthogonal unit vectors in $\mathbb{R}^n$.

If $\lambda = -\alpha$, the assumptions of Theorem 27 are not satisfied.

- $A_{K_{n,n}\setminus nK_2}(\alpha, \beta) = A_{2K_n+nK_2}(\beta, \alpha)$
The eigenvalues of the matrix $A_{K_{n,n}\setminus nK_2}(\alpha, \beta)$ are $(n-1)\alpha + n\beta$ and $-(n-1)\alpha + (n-2)\beta$ both with multiplicity one, $-\alpha$ and $\alpha - 2\beta$ both with multiplicity $n-1$.

  We first consider the case that $\lambda = -(n-1)\alpha + (n-2)\beta$. Let $u_1, \dots, u_n$ be n distinct equiangular unit vectors lying in an $(n-1)$-dimensional subspace of $\mathbb{R}^{2n-2}$. Let $\tilde{u}_1, \dots, \tilde{u}_n$ be further n distinct equiangular unit vectors lying in the $(n-1)$-dimensional subspace of $\mathbb{R}^{2n-2}$ orthogonal to the subspace spanned by the vectors. The vertices of the obtained $(2n-2)$-polytope with $2n$ vertices lie at the end-points of the vectors $u_1 + \tilde{u}_1, \dots, u_n + \tilde{u}_n, u_1 - \tilde{u}_1, \dots, u_n - \tilde{u}_n$.

  If $\lambda = (n-1)\alpha + n\beta$, the obtained $(2n-1)$-polytope P with $2n$ vertices is similar to that for $\lambda = -(n-1)\alpha + (n-2)\beta$ but the vertices of P corresponding to the vectors $u_1 + \tilde{u}_1, \dots, u_n + \tilde{u}_n$ and to the vectors $u_1 - \tilde{u}_1, \dots, u_n - \tilde{u}_n$ are shifted in the direction orthogonal to the subspace spanned by the vectors $u_1, \dots, u_n, \tilde{u}_1, \dots, \tilde{u}_n$.

  If $\lambda = -\alpha$, the assumptions of Theorem 27 are not satisfied.

  If $\lambda = \alpha - 2\beta$, we obtain an n-polytope which is a prism over a regular $(n-1)$-simplex (this type of equipartite polytopes was described in Theorem 24).

- $A_{2K_n}(\alpha, \beta) = A_{K_{n,n}}(\beta, \alpha)$
The eigenvalues of the matrix $A_{2K_n}(\alpha, \beta)$ are $(n-1)\alpha + n\beta$ and $(n-1)\alpha - n\beta$ both with multiplicity one and $-\alpha$ with multiplicity $2n-2$.

  We first consider the case that $\lambda = (n-1)\alpha - n\beta$. In this case, the obtained $(2n-2)$-polytope with $2n$ vertices is the convex hull of two regular $(n-1)$-simplices in $\mathbb{R}^{2n-2}$ which lie in orthogonal $(n-1)$-dimensional subspaces of $\mathbb{R}^{2n-2}$. Let us remark that this type of equipartite polytopes was described in Theorem 23.

If $\lambda = (n-1)\alpha + n\beta$, the obtained $(2n-1)$-polytope with $2n$ vertices is similar to that for $\lambda = (n-2)\alpha - (n-1)\beta$ but the two $(n-1)$-simplices are mutually shifted in the direction orthogonal to the subspaces of the two $(n-1)$-simplices.

If $\lambda = -\alpha$, the assumptions of Theorem 27 are not satisfied.

- $A_{2C_4}(\alpha, \beta)$
  The eigenvalues of the matrix $A_{2C_4}(\alpha, \beta)$ are $2\alpha + 5\beta$ and $2\alpha - 3\beta$ with multiplicities one, $\beta - 2\alpha$ with multiplicity two and $-\beta$ with multiplicity four.

  If $\lambda = 2\alpha + 5\beta$, the obtained 7-polytope is the convex hull of two orthogonal regular 3-simplices which are mutually shifted in the direction orthogonal to the subspace spanned by the two 3-simplices.

  If $\lambda = 2\alpha - 3\beta$, the obtained 6-polytope is the convex hull of two orthogonal regular 3-simplices (as described in Theorem 23).

  If $\lambda = \beta - 2\alpha$, the obtained 5-polytope is the convex hull of two orthogonal squares which are mutually shifted in the direction orthogonal to the planes of the two squares.

  If $\lambda = -\beta$, the assumptions of Theorem 27 are not satisfied.

Let us remark that if two (or more) of eigenvalues of the above considered matrices coincide, no polytope of a new symmetry type arises, e.g., $A_{2K_n}(\alpha, 0)$ has an eigenvalue $(n-1)\alpha$ with multiplicity two and the obtained $(2n-2)$-polytope is still the convex hull of two orthogonal $(n-1)$-simplices.

9 Conclusion

Throughout the paper, we have already stated two questions related to the notion of equipartite polytopes which we were not able to answer. We list them here:

Problem 1 *For which k does there exist an equipartite d-polytope with $2k$ vertices? In particular, does there exist an equipartite d-polytope with $2k$ vertices for $d \geq 7$ and $d/2 + 1 < k < d - 1$?*

Problem 2 *Classify all possible symmetry groups of equipartite polytopes. In particular, is it true that the symmetry group of each equipartite polytope P which is not a regular simplex acts imprimitively on the vertices of P?*

Several possible generalizations of the notions of equipartite graphs and polytopes come easily to one's mind. Some of them lead to new interesting questions, some do not provide any new results.

The first possible generalization is to consider, in addition to equipartite polytopes, equipartite sets of points in $\mathbb{R}^d$. A set of $2n$ points is (weakly) equipartite if for any partitioning to two parts of n points, there is an isometry carrying one of the parts onto the other (there is an isometry between the parts, respectively). However, it can be shown that if a set of points is equipartite (or weakly equipartite), then the set is convex and thus the points form vertices of a d-polytope.

Another generalization may be to consider partitions into more than two parts. It is not hard to show that the only graphs which are equipartite when splitting to $k \geq 3$ parts are the complete graphs K_{kn} and the trivial graphs nkK_1. If the weak equipartiteness is considered, the list is enhanced by all graphs of order k (which are weakly equipartite when splitting to k parts from trivial reasons). This yields that the only equipartite polytopes with respect to splitting to $k \geq 3$ parts are the regular simplices.

Finally, some relaxations of the notion of equipartiteness also seem to be of some interest:

Problem 3 *A graph G of order $2n$ is degree-equipartite if for every n-element set $A \subseteq V(G)$, the degree sequences of the graphs $G[A]$ and $G[V(G)\setminus A]$ are the same. Which graphs G are degree-equipartite? In particular, is there a degree-equipartite graph which is not equipartite?*

Problem 4 *A set X of $2n$ points in $\mathbb{R}^d$ is ε-almost weakly equipartite if for every partition of X into two n-element subsets X_1 and X_2 there is a bijection τ between X_1 and X_2 such that $\frac{1}{1+\varepsilon}\mathrm{dist}(a,b) \leq \mathrm{dist}(\tau(a),\tau(b)) \leq (1+\varepsilon)\mathrm{dist}(a,b)$ for every $a,b \in X_1$. What is the largest cardinality of an ε-almost equipartite set of points in $\mathbb{R}^d$?*

Acknowledgement

The authors would like to thank László Babai for fruitful discussions on groups, Zdeněk Dvořák and Riste Škrekovski for helpful discussions on equipartite graphs and Peter Horák for his comments on the presentation of the results contained in this paper.

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