Planar point sets with a small number of empty convex polygons

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Abstract

A subset A of a finite set P of points in the plane is called an *empty polygon*, if each point of A is a vertex of the convex hull of A and the convex hull of A contains no other points of P. We construct a set of n points in general position in the plane with only $\approx 1.62n^2$ empty triangles, $\approx 1.94n^2$ empty quadrilaterals, $\approx 1.02n^2$ empty pentagons, and $\approx 0.2n^2$ empty hexagons.

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1 Introduction

Results. We say that a set P of points in the plane is in *general position* if it contains no three points on a line.

Let P be a finite set of points in general position in the plane. We call a subset A of k points in P an *empty k-gon* if the convex hull of A is a k-gon containing no point of $P \setminus A$.

Let $g_k(n)$ be the minimum number of empty k-gons in a set of n points in general position in the plane. Horton [H83] proved that $g_k(n) = 0$ for any $k \geq 7$ and any $n \in \mathbb{N}$. The following bounds on $g_k(n), k = 3, 4, 5, 6$, have been known:

$$n^2 - O(n \log n) \leq g_3(n) \leq \frac{3771}{2240} n^2 = 1.683...n^2,$$

$$\frac{1}{2} n^2 - O(n) \leq g_4(n) \leq \frac{976}{448} n^2 = 2.131...n^2,$$

$$\left\lfloor \frac{n-4}{6} \right\rfloor \leq g_5(n) \leq \frac{393}{320} n^2 = 1.228...n^2,$$

$$g_6(n) \leq \frac{666}{2240} n^2 = 0.297...n^2.$$

The upper bounds have been shown in [D00], improving previous bounds of [KM88, BF87, V95]. The lower bound on $g_3(n)$ has been shown in [BF87], the lower bound on $g_4(n)$ by Bárány (see [V95]) and by Dumitrescu [D00], and the lower bound on $g_5(n)$ in [BK01]. In this paper we give the following improved upper bounds:

Theorem 1

$$g_3(n) \leq \left(4 + \frac{35}{72} + \frac{16}{3}\alpha - \frac{16}{3}\beta\right) p \cdot n^2 + o(n^2) = 1.6195...n^2 + o(n^2),$$

$$g_4(n) \leq \left(5 + \frac{31}{56} + 8\alpha - 16\beta + \frac{16}{3}\gamma\right) p \cdot n^2 + o(n^2) = 1.9396...n^2 + o(n^2),$$

$$g_5(n) \leq \left(3 - \frac{1}{56} + \frac{16}{3}\alpha - 16\beta + \frac{32}{3}\gamma\right) p \cdot n^2 + o(n^2) = 1.0206...n^2 + o(n^2),$$

$$g_6(n) \leq \left(\frac{293}{504} + \frac{4}{3}\alpha - \frac{16}{3}\beta + \frac{16}{3}\gamma\right) p \cdot n^2 + o(n^2) = 0.2005...n^2 + o(n^2),$$

where

$$\begin{aligned}
p &= \frac{3}{\pi^2} = 0.3039635...,\\
\alpha &= \sum_{z \geq 3 \text{ odd}} \frac{1}{z^2} = \frac{\pi^2}{8} - 1 = 0.2337005...,\\
\beta &= \sum_{z \geq 3 \text{ odd}} \frac{1}{z^2 2^{\lfloor \log_2 z \rfloor}} = 0.07582879...,\\
\gamma &= \sum_{z \geq 3 \text{ odd}} \frac{1}{z^2 4^{\lfloor \log_2 z \rfloor}} = 0.03210483....
\end{aligned}$$

Our construction seems to be the final one of the type developed in [V95, D00], and is, perhaps, the best possible up to the additive $o(n^2)$-factor. Several exciting questions remain open. The most interesting is whether $g_6(n) > 0$ for sufficiently large n (e.g. see [E75]). In other words, is it true that if P is a finite set of points in general position in the plane with $|P|$ large enough, then P contains an empty hexagon. Another question is whether $g_3(n) \geq (1+\varepsilon)n^2$ holds for large enough n for some fixed $\varepsilon > 0$. This would be the case if one could show that $g_5(n) > \varepsilon n^2$ for some fixed $\varepsilon > 0$. These questions have turned out to be more difficult than expected: for instance the innocent looking $g_6(n) > 0$ has been a challenge for more than 30 years now.

2 The construction

Our construction giving the upper bounds in Theorem 1 is a set obtained from the grid $\sqrt{n} \times \sqrt{n}$ by a little perturbation (due to monotonicity of $g_k(n)$ it suffices to prove Theorem 1 when n is a square of integer). Throughout the rest of the paper, n is a square of integer, and Λ is the grid $\{1, 2, \ldots, \sqrt{n}\} \times \{1, 2, \ldots, \sqrt{n}\}$. The perturbed set will be denoted by Λ^*. The construction of Λ^* uses so-called Horton sets [V92] which generalize a construction of Horton [H83] giving $g_7(n) = 0$ for any n.

Horton sets. Let H be a finite set of points in general position in the plane such that no two points have the same x-coordinate, and let $h_0, h_1, \ldots, h_m$

be the points of H listed by increasing x-coordinate. We say that a subset $H' \subseteq H$ lies *far below* a subset $H'' \subseteq H$ (and H'' lies *far above* H'), if the entire set H'' lies above every line trough a pair of points of H' and the entire set H' lies below every line trough a pair of points of H''. For $0 \le i < j$, we define a subset $H_{i,j}$ of H as the set of points h_k with $k \equiv i (\mathrm{mod}\ j)$. The set H is called a *Horton set* if, for every $j = 2, 4, 8, 16, \ldots$ and every integer i with $0 \le i < j/2$, the set $H_{i,j}$ lies far below or far above $H_{i+j/2,j}$. It was shown in [V92] that if H is Horton, then also each $H_{i,j}, 0 \le i < j$, is Horton. Obviously, if H is Horton then also each contiguous segment of H (i.e., a set of points h_k of H with $k_0 \le k \le k_1$) is Horton.

Construction of Λ^*. Set $m := \sqrt{n} - 1 \ge 1$ and $\varepsilon := 1/(10m)$. We construct an auxiliary *random Horton set* $H = H(\varepsilon)$ of size $m+1$ as follows. We choose randomly and independently for each $i, j, 0 \le i < j/2, 2 \le j = 2^l \le m$, the mutual position of the sets $H_{i,j}$, $H_{i+j/2,j}$ (whose union is the set $H_{i,j/2}$): the set $H_{i,j}$ will lie with probability $1/2$ far above $H_{i+j/2,j}$ and with probability $1/2$ far below $H_{i+j/2,j}$. For a given choice of mutual positions, we define H as the set of points $h_k = (k, \varepsilon \sum_{l=1}^{\lfloor \log_2 m \rfloor} \pm (m+1)^{-l}), k = 0, \ldots, m$, where the choice of $+$ or $-$ at $(m+1)^{-l}$ corresponds to the choice of mutual position of those sets $H_{i,2^l}$, $H_{i+2^{l-1},2^l}$ whose union $H_{i,2^{l-1}}$ contains h_k (we take $+(m+1)^{-l}$ in the sum if h_k lies in that of the sets $H_{i,2^l}$, $H_{i+2^{l-1},2^l}$ which lies far above the other of these sets; otherwise we take $-(m+1)^{-l}$). The x-coordinates of the points of $H = H(\varepsilon)$ are $0, 1, \ldots, m$ and the y-coordinates lie in the interval $(-\varepsilon, \varepsilon)$. For $\varepsilon' > 0$, we consider another, analogously defined random Horton set $H' = H'(\varepsilon')$ of size $m+1$. Further, we consider the set $H'' = H''(\varepsilon')$ obtained from $H' = H'(\varepsilon')$ by the interchange of the axes, i.e. $H'' = T(H')$, where $T : (x, y) \mapsto (y, x)$. We define Λ^* as the Minkowski sum of the sets $H = H(\varepsilon)$ and $H'' = H''(\varepsilon')$, where $\varepsilon' = \varepsilon'(m) > 0$ is sufficiently small compared to $\varepsilon = 1/(10m)$ (e.g., $\varepsilon' = 1/(20m(m+1)^{1+\log_2 m})$ will do). The set Λ^* approximates Λ. For a point X in Λ, we denote by X^* the corresponding point of Λ^*. We usually use letters I, J, K, L, R, S, T to denote points in Λ. We denote their coordinates by $I = (i, y(I)), J = (j, y(J))$, etc. (We use such a notation since we mostly work with the first coordinate). It follows from the choice of $\varepsilon, \varepsilon'$ that for any three points $I, J, K \in \Lambda$ the following holds:

Observation 2 *(i) If $I, J, K \in \Lambda$ are not collinear, then the triples I, J, K and I^*, J^*, K^* have the same orientation.*

(ii) If $I, J, K \in \Lambda$ lie on a non-vertical common line, then the orientation of the triple I^*, J^*, K^* is equal to the orientation of the triple h_i, h_j, h_k of points of H.

(iii) If $I, J, K \in \Lambda$ lie on a vertical common line, then the orientation of the triple I^*, J^*, K^* is determined by the orientation of the corresponding triple of points of H'.

It follows from Observation 2 that the points of Λ^* corresponding to the intersection of a non-vertical line with Λ form a set having the same order type as a contiguous part of some set $H_{i,j}, 0 \leq i < j$. Consequently, such points form a Horton set (see Claim 3.10 in [V92]).

Observation 3 *The points of Λ^* corresponding to the intersection of a non-vertical line with Λ form a random Horton set G. That is, randomly and independently for each $i, j, 0 \leq i < j/2, 2 \leq j = 2^l \leq m$, the set $G_{i,j}$ lies with probability $1/2$ far above $G_{i+j/2,j}$ and with probability $1/2$ far below $G_{i+j/2,j}$.*

Notation. The *lattice* is the usual lattice of points in the plane with integer coordinates. A *lattice point* is a point of the lattice. We say that a line is a *lattice line*, if it contains infinitely many lattice points. For a non-vertical lattice line l, we denote by l^+ (resp. l^-) the closest lattice line above (below) l and parallel to l. A *lattice segment* is a segment connecting two lattice points. We say that a lattice segment is *s-prime*, if it contains $s+1$ lattice points (including its endpoints). If a lattice segment is 1-prime (i.e., its relative interior contains no lattice points), then we call it a *prime segment*. Otherwise we call it a *non-prime segment*.

If $I_1^* I_2^* \dots I_k^*$ is an empty k-gon in Λ^*, then we also say that $I_1 I_2 \dots I_k$ is an *empty k-gon* (it may be degenerate).

For an empty k-gon $P = L_1 L_2 \dots L_k$ with all vertices in Λ, we define the *base of* P as the segment $L_v L_w$ connecting the vertex L_v having the smallest x-coordinate with the vertex L_w having the largest x-coordinate. If P has more vertices with the smallest x-coordinate, then we choose for L_v that one with the smallest y-coordinate. Similarly, if P has more vertices with the largest x-coordinate, then we choose for L_w that one with the largest y-coordinate.

If the base of an empty polygon P is prime, non-prime, or s-prime, then we say that P is *prime*, *non-prime*, or *s-prime*, respectively.

We say that a (possibly degenerate) polygon P with all vertices in Λ is a *t-line polygon*, if t is the least number such that the vertices of P lie on t neighboring parallel lattice lines.

3 Structure of the proof

We note first that Λ^* contains no empty 7-gon. This was proved in [V95]: the reason is that Λ^* is built from Horton sets.

For each $k = 3, 4, 5, 6$, we distinguish five types of empty k-gons and estimate the expected number of empty k-gons for each of them separately. Here are the five types of empty k-gons:

- 3-line k-gons,
- 2-line prime k-gons,
- 2-line non-prime k-gons,
- 1-line 2^s-prime k-gons ($s \in \mathbb{N}$),
- 1-line r-prime k-gons ($r \neq 2^s$),

It follows from Observation 4 below that every empty polygon in Λ^* is 1-, 2-, or 3-line. Thus, the above five types embrace all empty polygons in Λ^*.

Observation 4 ([V92]) *If the convex hull of a subset S of Λ has no lattice point in the interior, then S lies either on one line, or on two parallel lines with no lattice point strictly between them, or on the perimeter of a lattice triangle with exactly one lattice point in the relative interior of each side.*

Next, let P be a finite point set, of n points, say, in the plane in general position. Consider the complex, $\mathcal{C}$, of empty convex polygons in P. $\mathcal{C}$ is clearly a simplicial complex. Let $f_k(P)$ be its f-vector ($k = 1, 2, \ldots$), that is, $f_k(P)$ is the number of empty convex k-gons in P. Clearly $f_1(P) = n$, and $f_2(P) = \binom{n}{2}$. It is proved by Edelman and Rainer [ER00] that $\mathcal{C}$ is contractible. Then it satisfies the Euler equation:

$$f_1(P) - f_2(P) + f_3(P) - f_4(P) \ldots = 1.$$

There is another linear relation satisfied by the f-vector: it is shown by Ahrens et al. [AGM99] that

$$f_1(P) - 2f_2(P) + 3f_3(P) - 4f_4(P) \ldots = |P \cap \operatorname{int}\operatorname{conv} P|.$$

These two linear relations are very useful in our construction since there $f_1(\Lambda^*) = n$, $f_2(\Lambda^*) = \binom{n}{2}$ and $f_k(\Lambda^*) = 0$ when $k > 6$. So out of the remaining four quantities $f_i(\Lambda^*)$, $i = 3, 4, 5, 6$, only 2 have to be determined. Our choice is to compute f_3 and f_6, which means that out of the 20 entries of the following table, we only compute 10.

Empty: $[\times(3/\pi^2)n^2]$	triangles	quadrilaterals	pentagons	hexagons
3-line	1/24	1/8	1/8	1/24
2-line prime	10/3	29/7	16/7	10/21
2-line non-prime	2/3	54/49	24/49	8/147
1-line 2^s-prime	4/9	9/49	4/49	4/441
other 1-line	$\frac{16}{3}\alpha - \frac{16}{3}\beta$	$8\alpha - 16\beta + \frac{16}{3}\gamma$	$\frac{16}{3}\alpha - 16\beta + \frac{32}{3}\gamma$	$\frac{4}{3}\alpha - \frac{16}{3}\beta + \frac{16}{3}\gamma$

Each entry in the table must be multiplied by $(3/\pi^2)n^2$ to obtain a $\approx$-approximation of the correspoding quantity. E.g., the entry 10/3 in the second row and first column means that Λ^* contains $(10/3)\cdot(3/\pi^2)n^2 + o(n^2)$ 2-line prime triangles. It is easily verified that the 10 entries in the first and last column and the above equations on the f-vector give Theorem 1.

In fact we have computed all entries of the above table. The method is to fix the base, IJ of the k-gon in question, then compute the expectation of the empty k-gons with base IJ, and then sum for all possible bases. This is fairly straightforward although lengthy in all five cases except the 2-line prime k-gons where we need a more detailed analysis.

4 Auxiliary statements

In this section we collect several simple facts (and prove some of them) that will be needed later. Most of them are quite easy.

We say that a segment I^*J^*, $i \neq j$, in Λ^* is *open up* if K^* lies below the line I^*J^* for any lattice point K in the relative interior of IJ. Similarly, we say that a segment I^*J^*, $i \neq j$, is *open down* if K^* lies above the line I^*J^* for any lattice point K in the relative interior of IJ. If I^*J^* is open up or down, then we also say that the segment IJ is *open up* or *down*, respectively.

Clearly, each prime segment is open up and down, and each 2-prime segment is open either up or down. Here is a more general lemma:

Lemma 5 *Let IJ be an s-prime segment in Λ. Then:*

(i) *If s is a power of 2, then the segment IJ is open up (down, respectively) with probability $1/s$.*

(ii) *If s is not a power of 2, then the segment IJ is open neither up nor down.* $\square$

Observation 6 *If $I^*J^*K^*$ is an empty triangle in Λ^*, $i \neq j$, and K lies strictly above the line IJ, then IJ is open up. Analogously, if $I^*J^*K^*$ is an empty triangle in Λ^*, $i \neq j$, and K lies strictly below the line IJ, then IJ is open down.* $\square$

Let $f(n), g(n)$ be two real functions defined for any $n = m^2, m \in \mathbb{N}$. We write $f(n) \approx g(n)$ (and say that $f(n)$ equals $\approx g(n)$), if

$$\lim_{m\to\infty} \frac{f(m^2)}{g(m^2)} = 1.$$

We denote the set of prime segments in the $\sqrt{n} \times \sqrt{n}$ grid Λ by $\mathcal{P}$, and its size by $p_n = |\mathcal{P}|$. It is well-known (see for instance [HW79]) that

$$p_n \approx \frac{6}{\pi^2}\binom{n}{2} \approx \frac{3}{\pi^2}n^2.$$

Lemma 7 *(i) For any $r \geq 2$, the number of r-prime segments in Λ is $\approx \frac{p_n}{r^2}$.*

(ii) For any $r \geq 2$ and $n \geq 1$, the number of r-prime segments in Λ is at most $\frac{8n^2}{r^2}$.

Proof. We first suppose that $\sqrt{n}$ is divisible by r. Consider the mapping $f : \Lambda \to \{1, \ldots, \frac{\sqrt{n}}{r}\} \times \{1, \ldots, \frac{\sqrt{n}}{r}\}$ defined by $f(I) = (\lceil \frac{i}{r} \rceil, \left\lceil \frac{y(I)}{r} \right\rceil)$ for $I \in \Lambda$. Each r-prime segment is mapped to a lattice segment of the same direction and $1/r$ of its original length. Thus, each r-prime segment is mapped to a prime segment. Moreover, each prime segment KL in $\{1, \ldots, \frac{\sqrt{n}}{r}\} \times \{1, \ldots, \frac{\sqrt{n}}{r}\}$ is the image of exactly r^2 r-prime segments in Λ, namely it is the image of the r-prime segments $(r \cdot K + (\alpha, \beta), r \cdot L + (\alpha, \beta))$, where $\alpha, \beta \in \{0, 1, \ldots, r-1\}$.

It follows that if $\sqrt{n}$ is divisible by r then Λ determines $r^2 \cdot p_{n/r^2}$ r-prime segments. This yields (i): for any $n = m^2$, Λ determines at least $r^2 \cdot p_{\lfloor \sqrt{n}/r \rfloor^2} \approx \frac{p_n}{r^2}$ and at most $r^2 \cdot p_{\lceil \sqrt{n}/r \rceil^2} \approx \frac{p_n}{r^2}$ r-prime segments.

If $r \geq \sqrt{n}$ then Λ determines no r-prime segments and (ii) clearly holds. Otherwise Λ determines at most $r^2 \cdot p_{\lceil \sqrt{n}/r \rceil^2} \leq r^2 \cdot \binom{\lceil \sqrt{n}/r \rceil^2}{2} \leq r^2 \cdot \frac{(4n/r^2)^2}{2} = \frac{8n^2}{r^2}$ r-prime segments, as required in (ii). $\square$

Lemma 8 *Let H, H' be two Horton sets, let H lie far below H', and let $H = \{h_0, \ldots, h_z\}$, $H' = \{h'_0, \ldots, h'_{z'}\}$. Further, let $P \subseteq H \cup H'$ be the vertex set of an empty polygon in $H \cup H'$, and let $P \cap H \neq \emptyset$ and $P \cap H' \neq \emptyset$. Then $|P \cap H| \leq 3$ and $|P \cap H'| \leq 3$. Moreover, if $|P \cap H|=3$ then $P \cap H = \{h_i, h_{\frac{i+j}{2}}, h_j\}$, where $j - i$ is a power of 2. Analogously, if $|P \cap H'|=3$ then $P \cap H' = \{h_k, h_{\frac{k+l}{2}}, h_l\}$, where $l - k$ is a power of 2.* □

Let H be a Horton set with vertices denoted as usual. Then we say that a segment $h_i h_j, j > i$, in H is *open down*, if all points $h_k, i < k < j$, lie above it. Similarly, we say that $h_i h_j$ is *open up*, if all points $h_k, i < k < j$, lie below it.

Lemma 9 *(i) Any Horton set of size 2^s determines $2^{s+1} - (s + 2)$ open down segments.*

(ii) If $H = \{h_0, \ldots, h_{2^s-1}\}$ is a Horton set of size 2^s, where the points are listed according to the increasing x-coordinate, then H determines $2^s - (s + 1)$ open down segments $h_i h_j$ with $j > i + 1$.

(iii) In (i) and (ii), "open down" can be replaced by "open up".

Proof. We proceed by induction on s. The lemma clearly holds for $s = 0, 1$. Suppose now that $H = \{h_0, h_1, \ldots, h_{2^s-1}\}$ is a Horton set of size $2^s, s \geq 2$. Let H' be the lower of the sets $H_{0,2}, H_{1,2}$. By the inductive assumption, H' determines $2^s - (s + 1)$ open down segments. The set H determines the following two types of open down segments:

(T1) $2^s - 1$ segments $h_i h_{i+1}$,

(T2) $2^s - (s + 1)$ open down segments determined by H'.

Thus, H determines $(2^s - 1) + (2^s - (s + 1)) = 2^{s+1} - (s + 2)$ open down segments. This gives (i). The open down segments $h_i h_j$ with $j > i + 1$ are just the segments of type (T2). This gives (ii).

(iii) follows from the symmetry. □

Observation 10 *For each $s \in \mathbb{N}$, let $f_s(n), g_s(n)$ be two functions satisfying $f_s(n) \approx g_s(n)$. Moreover, suppose that for each $\varepsilon > 0$ there is a $t \in \mathbb{N}$ such that, for any $n \in \mathbb{N}$,*

$$\sum_{s=t+1}^{\infty} f_s(n) \leq \varepsilon n^2 \quad \text{and} \quad \sum_{s=t+1}^{\infty} g_s(n) \leq \varepsilon n^2.$$

Then

$$\sum_{s=1}^{\infty} f_s(n) \approx \sum_{s=1}^{\infty} g_s(n) + o(n^2).$$

$\square$

5 3-line triangles and hexagons

5.1 The parity of the coordinates of lattice prime segments

Here we estimate the number of 2-prime segments IJ, $I, J \in \Lambda$, such that $\frac{j-i}{2}$ is even. The standard method from [HW79] showing that $p_n \approx \frac{3}{\pi^2} n^2$ gives easily the following.

Lemma 11 *The number of prime segments IJ with $j-i$ even is*

$$\approx \frac{p_n}{3}.$$

$\square$

Lemma 12 *(i) Λ determines $\approx \frac{p_n}{12}$ 2-prime segments IJ with $\frac{j-i}{2}$ even.*
(ii) Λ determines $\approx \frac{p_n}{12}$ 2-prime segments IJ with $\frac{j-i}{2} \neq 0$ even.

Proof. Certainly, it suffices to prove the lemma for $\sqrt{n}$ even.

Consider the mapping $f : \Lambda \to \{1, \dots, \frac{\sqrt{n}}{2}\} \times \{1, \dots, \frac{\sqrt{n}}{2}\}$ defined by $f(I) = (\lceil \frac{i}{2} \rceil, \lceil \frac{y(I)}{2} \rceil)$, as in the proof of Lemma 7 (for $r = 2$). Each 2-prime segment IJ in Λ is mapped to a prime segment, and each prime segment KL in $\{1, \dots, \frac{\sqrt{n}}{2}\} \times \{1, \dots, \frac{\sqrt{n}}{2}\}$ is the image of exactly 4 2-prime segments in Λ. Moreover, for a 2-prime segment IJ in Λ, $\frac{j-i}{2}$ is even if and only if $l - k$ $(= \frac{j-i}{2})$ is even, where k, l are the x-coordinates of the points $K = f(I), L = f(J)$, respectively.

Thus, by Lemma 11, Λ determines

$$\approx 4 \cdot \frac{p_{n/4}}{3} \approx \frac{1}{12} p_n$$

2-prime segments IJ with $\frac{j-i}{2}$ even. This gives (i).

Since there are only $O(n)$ 2-prime segments IJ with $j - i = 0$, (ii) follows from (i) and from $p_n = \Theta(n^2)$. $\square$

5.2 3-line triangles

Let IJK be an IJ-triangle with all three sides 2-prime and no lattice point in the interior. We now find the probability that IJK is empty. Set $R = \frac{I+J}{2}, S = \frac{I+K}{2}, T = \frac{J+K}{2}$ (see Fig. 1).

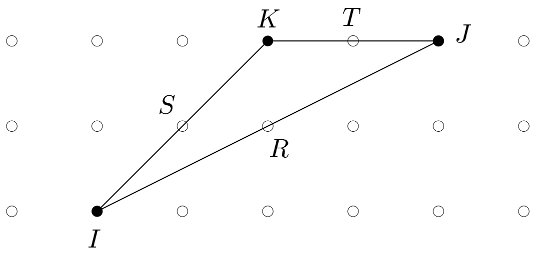

Figure 1: The points R, S, T.

If $\frac{j-i}{2}$ is odd, then (exactly) one of the numbers $\frac{k-i}{2}$, $\frac{j-k}{2}$ is also odd. Without loss of generality, let $\frac{k-i}{2}$ be odd. Then $i \equiv j \equiv k \not\equiv r \equiv s (\text{mod } 2)$. Consequently, R^* and S^* lie either both below or both above the segments I^*J^* and I^*K^*, respectively. Thus, (exactly) one of the points R^*, S^* lies inside the triangle $I^*J^*K^*$. We conclude that IJK is not an empty triangle in this case.

Suppose now that $\frac{j-i}{2}$ is even. If both numbers $\frac{k-i}{2}$, $\frac{j-k}{2}$ are even, then in the triangle IRS the y-components of a side are of the same parity, and then the midpoint of this side is a lattice point. Consequently, one of the sides of the original triangle IJK is not 2-prime.

Thus $\frac{j-i}{2}$ is even and both $\frac{k-i}{2}$, $\frac{j-k}{2}$ are odd. Consequently, $i \equiv j \equiv k \equiv r \not\equiv s \equiv t (\text{mod } 2)$. With probability $1/2$, both points S^*, T^* lie inside the triangle $I^*J^*K^*$. Independently and also with probability $1/2$, the point R^* lies inside the triangle $I^*J^*K^*$. Thus, if $\frac{j-i}{2}$ is even then $I^*J^*K^*$ is empty with probability $1/4$.

For a 2-prime segment $IJ \in \mathcal{P}$ with $\frac{j-i}{2} > 0$ even, there are exactly two lattice points K, $i \leq k \leq j$, such that IJK is a 3-line triangle: One such placement of K is on the line $(IJ^+)^+$ (in which case the points S, T are the two points on IJ^+ satisfying $i \leq s < t < j$), the other placement of K is on the line $(IJ^-)^-$ (in which case the points S, T are the two points on IJ^- satisfying $i < s < t \leq j$), see Fig. 2. It now follows from Lemma 12(ii) that the expected number of empty 3-line triangles is $\approx 2 \cdot \frac{1}{4} \cdot \frac{p_n}{12} = \frac{p_n}{24}$.

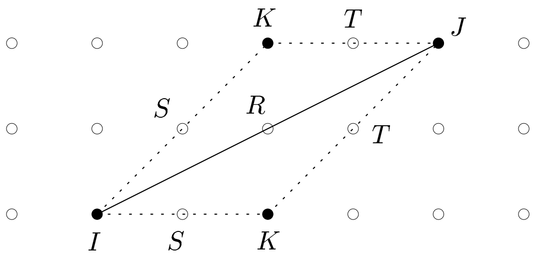

Figure 2: Two possible placements of K.

5.3 3-line hexagons

Each empty 3-line triangle IJK corresponds to the empty 3-line hexagon $I\frac{I+J}{2}J\frac{J+K}{2}K\frac{K+I}{2}$, and vice versa. Thus, the number of empty 3-line hexagons equals the number of empty 3-line triangles.

6 2-line prime triangles and hexagons

6.1 Some lattice properties

Given a non-vertical prime segment $IJ \in \mathcal{P}$, there is a unique $K \in (IJ)^+$ with $i \leq k < j$. We let $q^+(IJ)$ denote this lattice point K. Assume $J - I = (m, t)$ with $0 < t < m$ and let $K - I = (x, y)$. Then $ym + x(-t) = 1$ as one can readily check. Thus x is the inverse of $-t(\text{mod } m)$. We will use a theorem of Balog and Deshoulliers [BD99] saying that x is "uniformly distributed" in $[0, m)$.

Theorem 13 ([BD99]) *Assume m is a positive integer. Then for any $\alpha \in (0, 1]$, and any $\eta > 0$, the number of pairs (t, x) with $t \in \{1, \ldots, m\}$ and $x \in \{1, \ldots, \lfloor \alpha m \rfloor\}$ where $xt \equiv -1(\text{mod } m)$ is*

$$\alpha\varphi(m) + O(m^{1/2+\eta})$$

where the implied constant depends at most on η.

Actually, the original result of Balog and Deshoulliers is more general and is stated in a slightly different form.

For $r \in \mathbb{N}$, we define a subset $\mathcal{P}_r$ of $\mathcal{P}$ as the set of non-vertical prime segments $IJ \in \mathcal{P}$ such that the x-coordinate of $q^+(IJ)$ lies in the interval $[i, i + \frac{j-i}{2^r})$.

Lemma 14 *(i) For any $r \geq 1$,*

$$|\mathcal{P}_r| \approx \frac{|\mathcal{P}|}{2^r},$$

(ii) For any $r, n \geq 1$,

$$|\mathcal{P}_r| \leq \frac{20}{2^r} n^2.$$

Proof. (i) is a direct corollary of Theorem 13.

To prove (ii), suppose that $I \in \Lambda$ and that $t \in \{1, 2, \ldots, \lfloor \sqrt{2n} \rfloor\}$. The number of lattice points $K \in \Lambda, k > i$, with $t \leq ||K - I|| < t + 1$ is approximately πt — certainly smaller than $10t$ (say). Now, let K be one of these points. If IK is non-prime then there is no lattice point J with $K = q^+(IJ)$. Otherwise the lattice points J with $K = q^+(IJ)$ lie on the lattice half-line $IK^- \cap \{(x, y) \in \mathbb{R}^2 : x \geq k\}$ (see Fig. 3). The half-line

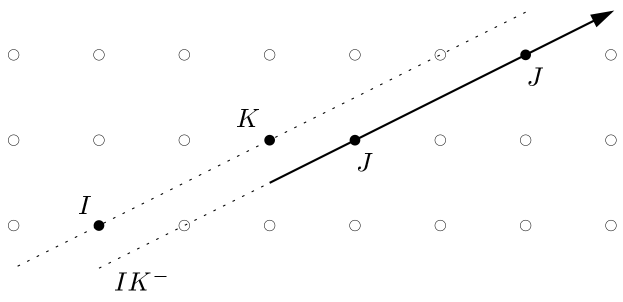

Figure 3: The lattice points J with $K = q^+(IJ)$.

$IK^- \cap \{(x, y) \in \mathbb{R}^2 : x \geq k\}$ contains at most $\frac{\sqrt{2n}}{||K-I||} \leq \frac{\sqrt{2n}}{t}$ lattice points $J \in \Lambda$. It follows that for each I and $t \in \{1, 2, \ldots, \lfloor \sqrt{2n} \rfloor\}$ there are at most $10t \cdot \frac{\sqrt{2n}}{t} = \sqrt{200n}$ lattice points J with $t \leq ||q^+(IJ) - I|| < t + 1$. If $IJ \in \mathcal{P}_r$ then $||q^+(IJ) - I|| < \frac{\sqrt{2n}}{2^r}$. It follows that for each I there are at most $\sum_{t=1}^{\left\lfloor \frac{\sqrt{2}n}{2^r} \right\rfloor} \sqrt{200n} \leq \frac{20}{2^r} n$ lattice points $J \in \Lambda$ with $IJ \in \mathcal{P}_r$. Consequently,

$$|\mathcal{P}_r| \leq \frac{20}{2^r} n^2.$$

□

We denote the lattice points on the open halfline $\overrightarrow{Iq^+(IJ)}$ by $K_1 = q^+(IJ)$, K_2, $K_3, \ldots$, so that $K_t = I + t(q^+(IJ) - I)$ for each $t \in \mathbb{N}$. See Fig. 4. Similarly, we denote the lattice points on the open halfline

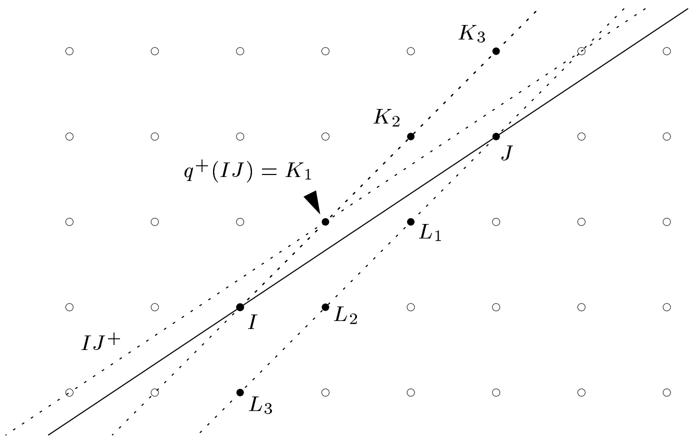

Figure 4: The lattice points K_i, L_i.

$\overrightarrow{J(J - (q^+(IJ) - I))}$ by L_1, L_2, $L_3, \ldots$, so that $L_t = J - t(q^+(IJ) - I)$ for each $t \in \mathbb{N}$. We remark that $\mathcal{P}_r$ is the set of segments $IJ \in \mathcal{P}$ such that $i \leq k_{2^r} < j$, where k_{2^r} is the x-coordinate of the point $K_{2^r} = I + 2^r(q^+(IJ) - I)$.

For $IJ \in \mathcal{P}_1$ and $s \geq 0$, we define two events $\mathrm{E}_s^+ = \mathrm{E}_s^+(IJ)$ and $\mathrm{E}_s^- = \mathrm{E}_s^-(IJ)$ as follows:
$\mathrm{E}_s^+ = \mathrm{E}_s^+(IJ)$: the segment IK_{2^s} is open down,
$\mathrm{E}_s^- = \mathrm{E}_s^-(IJ)$: the segment JL_{2^s} is open up.

Clearly, K_{2^s} lies in some empty 2-line IJ-polygons if and only if $IJ \in \mathcal{P}_s$ and E_s^+ is satisfied. Similarly, L_{2^s} lies in some empty 2-line IJ-polygons if and only if $IJ \in \mathcal{P}_s$ and E_s^- is satisfied.

The following observation follows from Lemma 5(i):

Observation 15 *For any $s \geq 0$ and $IJ \in \mathcal{P}_s$,*

$$\mathrm{Prob}(\mathrm{E}_s^+) = \frac{1}{2^s}, \qquad \mathrm{Prob}(\mathrm{E}_s^-) = \frac{1}{2^s}.$$

□

The following lemma shows that the events E_s^+ and $\mathrm{E}_{s'}^-$ are almost independent if IJ is taken uniformly from $\mathcal{P}_r$.

Lemma 16 *Let $r \in \mathbb{N}$ and $0 \leq s, s' \leq r$. Then*

$$\sum_{IJ \in \mathcal{P}_r} \mathrm{Prob}(\mathrm{E}_s^+ \wedge \mathrm{E}_{s'}^-) \approx \frac{p_n}{2^r} \cdot \frac{1}{2^{s+s'}}.$$

Proof. If $s = 0$ then, by Observation 15 and by Lemma 14(i),

$$\sum_{IJ \in \mathcal{P}_r} \mathrm{Prob}(\mathrm{E}_s^+ \wedge \mathrm{E}_{s'}^-) = \sum_{IJ \in \mathcal{P}_r} \mathrm{Prob}(\mathrm{E}_{s'}^-) \approx \frac{p_n}{2^r} \cdot \frac{1}{2^{s'}},$$

as required. Analogously, the lemma holds also for $s' = 0$. We further suppose that $s, s' \geq 1$.

Let $IJ \in \mathcal{P}_r$ and let $K = K_1 = q^+(IJ)$. Not all three numbers $i + j$, $i + k$, $j + k$ are even, since in that case one of the points $\frac{I+J}{2}$, $\frac{I+K}{2}$, $\frac{J+K}{2}$ (corresponding to an even of the numbers $y(I) + y(J)$, $y(I) + y(K)$, $y(J) + y(K)$) would be a lattice point. Consequently, by a parity argument, exactly one of the numbers $i + j$, $i + k$, $j + k$ is even.

By Lemma 11, there are $\approx \frac{|\mathcal{P}|}{3}$ segments $IJ \in \mathcal{P}$ with $i + j$ even. Consequently, by Theorem 13, there are $\approx \frac{|\mathcal{P}_r|}{3}$ segments $IJ \in \mathcal{P}_r$ with $i + j$ even.

By Lemma 11, there are $\approx \frac{2|\mathcal{P}|}{3}$ segments $IJ \in \mathcal{P}$ with $i + j$ odd, which is the same as $j - i$ odd. Thus, by symmetry, there are $\approx \frac{|\mathcal{P}|}{3}$ segments $IJ \in \mathcal{P}$ with $k - i$ even and also $\approx \frac{|\mathcal{P}|}{3}$ segments $IJ \in \mathcal{P}$ with $k - i$ odd.

We want to use now Theorem 13 with $J - I = (m, t)$ and $x \in [0, m2^{-r})$, with the extra condition that $x = k - i$ is even (resp. odd). When x is even and lies in $[0, m2^{-r})$ then $x/2$ is an integer in $[0, m2^{-r-1})$ for which $(2t)(x/2) \equiv -1 (\mathrm{mod}\ m)$ and $2t$ runs through the reduced residue classes $(\mathrm{mod}\ m)$. The number of such pairs $(2t, x/2)$ is then $2^{-r-1}\varphi(m) +$

$O(m^{1/2+\eta})$, which implies that there are $\approx \frac{|\mathcal{P}_r|}{3}$ segments $IJ \in \mathcal{P}_r$ with $j-i$ odd and $k-i$ even. Then the complementary set of segments with $j-i$ odd and $k-i$ odd is also of size $\approx \frac{|\mathcal{P}_r|}{3}$.

Let $IJ \in \mathcal{P}_r$. If $i+k$ is even (and $i+j, j+k$ are odd), then the x-coordinates of $I, K_1, K_2, \ldots$ have the other parity than the x-coordinates of $J, L_1, L_2, \ldots$. Consequently, the events E_s^+ and $\mathrm{E}_{s'}^-$ are independent and

$$\mathrm{Prob}(\mathrm{E}_s^+ \wedge \mathrm{E}_{s'}^-) = \mathrm{Prob}(\mathrm{E}_s^+) \cdot \mathrm{Prob}(\mathrm{E}_{s'}^-) = \frac{1}{2^{s+s'}}$$

in this case.

If $i+j$ is even, then the x-coordinates of $I, K_2, K_4, \ldots, J, L_2, L_4, \ldots$ have the other parity than the x-coordinates of $K_1, K_3, \ldots, L_1, L_3, \ldots$. Consequently, either K_1^* lies below the line $I^*K_{2^s}^*$ or L_1^* lies above the line $J^*L_{2^{s'}}^*$. Thus,

$$\mathrm{Prob}(\mathrm{E}_s^+ \wedge \mathrm{E}_{s'}^-) = 0$$

in this case (provided $s, s' \geq 1$).

If $j+k$ is even, then the x-coordinates of $K_1, K_3, \ldots, J, L_2, L_4, \ldots$ have the other parity than the x-coordinates of $I, K_2, K_4, \ldots, L_1, L_3, \ldots$. The following two conditions are necessary for $\mathrm{E}_s^+ \wedge \mathrm{E}_{s'}^-$:

C_1 : $\{I^*, K_2^*, K_4^*, \ldots\}$ lies far below $\{K_1^*, K_3^*, \ldots\}$,

C_2 : $\{L_1^*, L_3^*, \ldots\}$ lies far below $\{J^*, L_2^*, L_4^*, \ldots\}$.

Clearly, C_1 is satisfied if and only if C_2 is satisfied. Thus,

$$\mathrm{Prob}(\mathrm{C}_1 \wedge \mathrm{C}_2) = \mathrm{Prob}(\mathrm{C}_1) = \mathrm{Prob}(\mathrm{C}_2) = \frac{1}{2}.$$

Suppose that $\mathrm{C}_1 \wedge \mathrm{C}_2$ is satisfied. Then E_s^+ is satisfied if and only if $I^*K_{2^s}^*$ is open down in the (random) Horton set $\{I^*, K_2^*, K_4^*, \ldots, K_{2^s}^*\}$, i.e., with probability $\frac{1}{2^{s-1}}$. Analogously, $\mathrm{E}_{s'}^-$ is satisfied if and only if $J^*L_{2^{s'}}^*$ is open up in the (random) Horton set $\{J^*, L_2^*, L_4^*, \ldots, L_{2^{s'}}^*\}$, i.e., with probability $\frac{1}{2^{s'-1}}$. Moreover, E_s^+ and $\mathrm{E}_{s'}^-$ are independent (provided $j+k$ is even and $\mathrm{C}_1 \wedge \mathrm{C}_2$ is satisfied), since the x-coordinates of $I, K_2, K_4, \ldots, K_{2^s}$ have the other parity than the x-coordinates of $J, L_2, L_4, \ldots, L_{2^{s'}}$. Thus, if $j+k$ is even then

$$\begin{aligned}
\mathrm{Prob}(\mathrm{E}_s^+ \wedge \mathrm{E}_{s'}^-) &= \mathrm{Prob}(\mathrm{C}_1 \wedge \mathrm{C}_2) \cdot \mathrm{Prob}(E_s^+ | \mathrm{C}_1 \wedge \mathrm{C}_2) \cdot \mathrm{Prob}(E_{s'}^- | \mathrm{C}_1 \wedge \mathrm{C}_2) \\
&= \frac{1}{2} \cdot \frac{1}{2^{s-1}} \cdot \frac{1}{2^{s'-1}} \\
&= \frac{1}{2^{s+s'-1}}.
\end{aligned}$$

Altogether,

$$\sum_{IJ\in\mathcal{P}_r} \mathrm{Prob}(\mathrm{E}_s^+ \wedge \mathrm{E}_{s'}^-) \approx \frac{\mathcal{P}_r}{3}\cdot\frac{1}{2^{s+s'}} + \frac{\mathcal{P}_r}{3}\cdot 0 + \frac{\mathcal{P}_r}{3}\cdot\frac{1}{2^{s+s'-1}} = \frac{p_n}{2^r}\cdot\frac{1}{2^{s+s'}}.$$

□

6.2 2-line prime triangles

The expected number of empty 2-line IJ-triangles with $IJ \in \mathcal{P}_r \setminus \mathcal{P}_{r+1}$ is

$$\begin{aligned}
&\sum_{IJ\in\mathcal{P}_r\setminus\mathcal{P}_{r+1}} \left(\sum_{s=0}^{r}\mathrm{Prob}(\mathrm{E}_s^+) + \sum_{s'=0}^{r}\mathrm{Prob}(\mathrm{E}_{s'}^-)\right)\\
&= \sum_{IJ\in\mathcal{P}_r\setminus\mathcal{P}_{r+1}} \left(\sum_{s=0}^{r}\frac{1}{2^s} + \sum_{s'=0}^{r}\frac{1}{2^{s'}}\right)\\
&= \left(4-\frac{2}{2^r}\right)|\mathcal{P}_r \setminus \mathcal{P}_{r+1}|,
\end{aligned}$$

and thus the expected number of empty 2-line IJ-triangles with $IJ \in \mathcal{P}_1 = \bigcup_{r=1}^{\infty}(\mathcal{P}_r \setminus \mathcal{P}_{r+1})$ is

$$\sum_{r=1}^{\infty}\left(4-\frac{2}{2^r}\right)|\mathcal{P}_r \setminus \mathcal{P}_{r+1}|. \tag{1}$$

By Lemma 14(i),

$$|\mathcal{P}_r \setminus \mathcal{P}_{r+1}| \approx \frac{p_n}{2^{r+1}}. \tag{2}$$

For every $\varepsilon > 0$, there is a $t \in \mathbb{N}$ such that, by Lemma 14(ii), the sum of the terms in (1) with $r \geq t+1$ can be bounded from above by

$$\sum_{r=t+1}^{\infty} 4\cdot\frac{20}{2^r}n^2 = \frac{80}{2^t}n^2 < \varepsilon n^2.$$

Thus, by Observation 10 and by (2),

$$\begin{aligned}(1) &\approx \sum_{r=1}^{\infty}\left(4-\frac{2}{2^r}\right)\frac{p_n}{2^{r+1}}\\ &= \left(2-\frac{1}{3}\right)p_n\\ &= \frac{5}{3}\cdot p_n. \qquad (3)\end{aligned}$$

Similarly, define $\mathcal{P}'_r$ as the set of non-vertical prime segments $IJ \in \mathcal{P}$ such that the x-coordinate of $q^+(IJ)$ lies in the interval $[j-\frac{j-i}{2^r}, j)$. An analogue of the above proof shows that the expected number of empty 2-line IJ-triangles with $IJ \in \mathcal{P}'_1$ is also

$$\approx \frac{5}{3}\cdot p_n. \qquad (4)$$

Consequently, the expected number of empty 2-line prime triangles is the sum of (3) and (4), that is,

$$\approx \frac{10}{3}\cdot p_n.$$

6.3 2-line prime hexagons

We first estimate the number of empty 2-line IJ-hexagons with $IJ \in \mathcal{P}_r \setminus \mathcal{P}_{r+1}$. Each of them is of form $JK_tK_{t/2}IL_{t'}L_{t'/2}$. Thus, their expected number is

$$\sum_{IJ\in\mathcal{P}_r\setminus\mathcal{P}_{r+1}}\sum_{s=1}^{r}\sum_{s'=1}^{r}\mathrm{Prob}(\mathrm{E}_s^+\wedge\mathrm{E}_{s'}^-) = \sum_{s=1}^{r}\sum_{s'=1}^{r}\sum_{IJ\in\mathcal{P}_r\setminus\mathcal{P}_{r+1}}\mathrm{Prob}(\mathrm{E}_s^+\wedge\mathrm{E}_{s'}^-).$$

It follows that the expected number of empty 2-line IJ-hexagons with $IJ \in \mathcal{P}_1 = \bigcup_{r=1}^{\infty}(\mathcal{P}_r \setminus \mathcal{P}_{r+1})$ is

$$\sum_{r=1}^{\infty}\sum_{s=1}^{r}\sum_{s'=1}^{r}\sum_{IJ\in\mathcal{P}_r\setminus\mathcal{P}_{r+1}}\mathrm{Prob}(\mathrm{E}_s^+\wedge\mathrm{E}_{s'}^-). \qquad (5)$$

Lemma 16 and the inclusion $\mathcal{P}_{r+1} \subseteq \mathcal{P}_r$ imply that

$$\begin{aligned}\sum_{s=1}^{r}\sum_{s'=1}^{r}\sum_{IJ\in\mathcal{P}_r\setminus\mathcal{P}_{r+1}} \mathrm{Prob}(\mathrm{E}_s^+ \wedge \mathrm{E}_{s'}^-) &\approx \sum_{s=1}^{r}\sum_{s'=1}^{r}\left(\frac{p_n}{2^r}-\frac{p_n}{2^{r+1}}\right)\cdot\frac{1}{2^{s+s'}} \\ &= \left(1-\frac{1}{2^r}\right)^2\frac{p_n}{2^{r+1}} \qquad (6)\end{aligned}$$

For every $\varepsilon > 0$, there is a $t \in \mathbb{N}$ such that, by Lemma 14(ii), the terms in (5) with $r \geq t+1$ can be bounded from above by

$$\sum_{r=t+1}^{\infty}\sum_{s=1}^{r}\sum_{s'=1}^{r}\sum_{IJ\in\mathcal{P}_r\setminus\mathcal{P}_{r+1}} 1 \leq \sum_{r=t+1}^{\infty} r^2|\mathcal{P}_r\setminus\mathcal{P}_{r+1}| \leq \sum_{r=t+1}^{\infty} r^2\frac{20}{2^r}n^2 < \varepsilon n^2.$$

Thus, by Observation 10 and by (6),

$$\begin{aligned}(5) &\approx \sum_{r=1}^{\infty}\left(1-\frac{1}{2^r}\right)^2\frac{p_n}{2^{r+1}} \\ &= \sum_{r=1}^{\infty}\left(\frac{1}{2^{r+1}}-\frac{1}{2^{2r}}+\frac{1}{2^{3r+1}}\right)p_n \\ &= \left(\frac{1}{2}-\frac{1}{3}+\frac{1}{14}\right)p_n \\ &= \frac{5}{21}\cdot p_n. \qquad (7)\end{aligned}$$

A similar argument as at the end of Paragraph 6.2 shows that the expected number of empty 2-line prime hexagons is twice as much as (7), i.e.,

$$\approx \frac{10}{21}\cdot p_n.$$

7 2-line non-prime triangles and hexagons

7.1 2-line non-prime triangles

If there is an empty 2-line non-prime IJ-triangle, then IJ must be open up or down and thus IJ must be 2^s-prime for some $s \in \mathbb{N}$.

Let $s \in \mathbb{N}$ and let $IJ \in \mathcal{P}$ be a non-vertical 2^s-prime segment with $j > i$. The line IJ^+ contains 2^s points $K \in \Lambda$ with $i \leq k < j$ (unless

$j = i + 2, y(I) = \sqrt{n}$). Each of these points determines an empty 2-line IJ-triangle IJK if and only if IJ is open up, i.e., with probability $\frac{1}{2^s}$. Thus, the expected number of empty 2-line IJ-triangles with one vertex on the line IJ^+ is equal to $\frac{2^s}{2^s} = 1$. By symmetry, the expected number of empty 2-line IJ-triangles with one vertex on the line IJ^- is also 1 (unless $j = i + 2, y(I) = 1$). Thus, the expected number of empty 2-line non-prime triangles is

$$\approx \sum_{s=1}^{\infty} \sum_{(IJ \text{ is } 2^s-\text{prime})} 2. \tag{8}$$

The "$\approx$" appears in (8) since the expected number of empty 2-line IJ-triangles is 0 for vertical 2^s-prime segments $IJ \in \mathcal{P}$ and is smaller than 2 for 2^s-prime segments $IJ \in \mathcal{P}, j = i + 2$, with $y(I) = \sqrt{n}$ or $y(J) = 1$.

It follows from Lemma 7(ii) that the first sum in (8) satisfies the assumptions of Observation 10, and thus (8) can be estimated by

$$\approx \sum_{s=1}^{\infty} \frac{p_n}{4^s} \cdot 2 = \frac{2}{3} \cdot p_n.$$

7.2 2-line non-prime hexagons

If there is an empty 2-line non-prime IJ-hexagon, then IJ must be open up or down and thus IJ must be 2^s-prime for some $s \in \mathbb{N}$.

Let $s \in \mathbb{N}$ and let IJ be a non-vertical 2^s-prime segment with $j > i$. The line IJ^+ contains 2^s points K with $i \leq k < j$ (unless $j = i + 2, y(I) = \sqrt{n}$), forming a (random) Horton set, which we denote by H. By Lemma 9(ii), H determines $2^s - (s + 1)$ open down segments KK' with $\frac{K+K'}{2} \in H$. Each of these segments determines an empty 2-line IJ-hexagon $I\frac{I+J}{2}JK'\frac{K'+K}{2}K$ if and only if IJ is open up, i.e., with probability $\frac{1}{2^s}$. Thus, the expected number of empty 2-line IJ-hexagons with all vertices on the lines IJ and IJ^+ is equal to $\frac{2^s-(s+1)}{2^s} = 1 - \frac{s+1}{2^s}$ (unless $j = i + 2, y(I) = \sqrt{n}$).

Altogether, the expected number of two-line non-prime hexagons is

$$\approx 2 \cdot \sum_{s=1}^{\infty} \sum_{(IJ \text{ is } 2^s-\text{prime})} \left(1 - \frac{s+1}{2^s}\right). \tag{9}$$

The "$\approx$" appears in (9) for analogous reasons as in (8).

It follows from Lemma 7(ii) that the first sum in (9) satisfies the assumptions of Observation 10, and thus (9) can be estimated by

$$\begin{aligned}
&\approx \quad 2 \cdot \sum_{s=1}^{\infty} \frac{p_n}{4^s} \left(1 - \frac{s+1}{2^s}\right) \\
&= \quad 2 \cdot \left(\frac{1}{3} - \frac{8}{49} - \frac{7}{49}\right) \cdot p_n \\
&= \quad \frac{8}{147} \cdot p_n.
\end{aligned}$$

8 1-line 2^s-prime triangles and hexagons

For $k \in \mathbb{N}$ and $s \geq 0$, we define $V_k(s)$ as the expected number of those empty k-gons in a random Horton set H of size $2^s + 1$, which contain both the leftmost point and the rightmost point of H.

Lemma 17 *For any $s \geq 0$,*

$$V_3(s) = s, \qquad V_6(s) = s - 4 + \frac{s+2}{2^{s-1}}.$$

Proof. Let $h_0, h_1, \ldots, h_{2^s}$ be the points of a Horton set H listed according to the increasing x-coordinate. For $i = 0, \ldots, s-1$, we define a 2^i-element subset $H(i)$ of H by

$$H(i) = H_{2^{s-i-1}, 2^{s-i}} = \{h_j \in H : j \equiv 2^{s-i-1} (\mathrm{mod}\ 2^{s-i})\}.$$

Observe that $H \setminus \{h_0, h_{2^s}\}$ is a disjoint union of the sets $H(i), i = 1, \ldots, s-1$ and that each $H(i) = H_{2^{s-i-1}, 2^{s-i}}$ lies far above or far below the set $H_{0,2^s} = \{h_0, h_{2^s}\} \subseteq H_{0,2^{s-i}}$. In particular, each $H(i)$ lies either below or above the line $h_0 h_{2^s}$.

We distinguish s combinatorial cases $\mathrm{C}_1, \mathrm{C}_2, \ldots, \mathrm{C}_s$ defined for $i = 1, 2, \ldots, s-1$ by

C_i: The line $h_0 h_{2^s}$ separates $H(0) \cup H(1) \cup \ldots \cup H(i-1)$ from $H(i)$.

The remaining case C_s is defined by

C_s: The whole set $H(0) \cup \ldots \cup H(s-1)$ lies on one side of the line $h_0 h_{2^s}$.

Clearly,

$$\mathrm{Prob}(C_i) = \begin{cases} \frac{1}{2^i}, & \text{for } i = 1, 2 \ldots, s-1, \\ \frac{1}{2^{s-1}}, & \text{for } i = s. \end{cases}$$

The triangle $h_0 h_{2^{s-1}} h_{2^s}$ is always empty. Moreover, in case C_i ($1 \leq i < s$) there are 2^i empty triangles $h_0 h_{2^s} p$, $p \in H(i)$. It is easy to see that there are no other empty triangles with the two vertices h_0, h_{2^s}. Thus,

$$V_3(s) = 1 + \sum_{i=1}^{s-1} \mathrm{Prob}(\mathrm{C}_i) \cdot 2^i = 1 + \sum_{i=1}^{s-1} 1 = s.$$

It remains to compute $V_6(s)$. Without loss of generality, let $H(0) = \{h_{2^{s-1}}\}$ lie under the line $h_0 h_{2^s}$. By Lemma 9(ii), in case C_i ($1 \leq i < s$) there are $2^i - (i+1)$ empty hexagons $h_0 h_{2^{s-1}} h_{2^s} h_v h_{\frac{v+w}{2}} h_w$ corresponding to the $2^i - (i+1)$ open down segments $h_w h_v$, $v > w+1$, in $H(i)$. By Lemma 8, there are no other empty hexagons with the two vertices h_0, h_{2^s}. Thus,

$$V_6(s) = \sum_{i=1}^{s-1} \frac{1}{2^i} \cdot (2^i - (i+1)) = s - 4 + \frac{s+2}{2^{s-1}}.$$

□

Lemma 18 *Let $k \geq 3$. If $V_k(s) = O(s)$, then the expected number of empty 1-line 2^s-prime k-gons ($s \in \mathbb{N}$) in Λ is*

$$\approx \sum_{s=1}^{\infty} \frac{V_k(s)}{4^s} \cdot p_n.$$

Proof. Let $k \geq 3$. The expected number of empty 1-line 2^s-prime k-gons ($s \in \mathbb{N}$) is

$$\sum_{s=1}^{\infty} \sum_{(IJ \text{ is } 2^s-\text{prime})} V_k(s). \tag{10}$$

We may apply Observation 10, since, by Lemma 7(ii), for any $\varepsilon > 0$ and for any sufficiently large $t = t(\varepsilon)$,

$$\sum_{s=t+1}^{\infty} \sum_{(IJ \text{ is } 2^s-\text{prime})} V_k(s) \quad \leq \quad \sum_{s=t+1}^{\infty} \frac{8n^2}{4^s} V_k(s)$$

$$
\begin{aligned}
&\leq \sum_{s=t+1}^{\infty} O\left(\frac{s}{4^s}\right) \cdot n^2 \\
&< \varepsilon n^2.
\end{aligned}
$$

The lemma now follows from (10), Observation 10, and Lemma 7(i). $\square$

We are ready to estimate the number of empty 1-line 2^s-prime triangles and hexagons. By Lemmas 17 and 18, the expected number of 1-line 2^s-prime triangles is

$$\approx \sum_{s=1}^{\infty} \frac{s}{4^s} \cdot p_n = \frac{4}{9} \cdot p_n,$$

and the expected number of empty 1-line 2^s-prime hexagons is

$$\approx \sum_{s=1}^{\infty} \frac{s - 4 + \frac{s+2}{2^{s-1}}}{4^s} \cdot p_n = \left(\frac{4}{9} - \frac{4}{3} + \frac{16}{49} + \frac{4}{7}\right) \cdot p_n = \frac{4}{441} \cdot p_n.$$

9 1-line r-prime triangles and hexagons ($r \neq 2^s$)

For $k \in \mathbb{N}$ and odd $z \geq 3$, we define $W_k(z)$ as the expected number of those empty k-gons in a random Horton set H of size $z + 1$, which contain both the leftmost point and the rightmost point of H.

Lemma 19 *For any odd $z \geq 3$,*

$$W_3(z) = 4 - \frac{4}{2^\omega}, \qquad W_6(z) = 1 - \frac{4}{2^\omega} + \frac{4}{4^\omega},$$

where $\omega = \lfloor \log_2 z \rfloor$.

Proof. Let $z \geq 3$ be odd and let $H = \{h_0, \ldots, h_z\}$ be a Horton set with vertices listed according to the increasing x-coordinate. For $i = 1, 2, \ldots, \omega = \lfloor \log_2 z \rfloor$, we put $K_i = h_{2^i}$ and $L_i = h_{z-2^i}$. Clearly, only the points h_0, h_z, K_i, L_i ($1 \leq i \leq \omega$) may be vertices of empty polygons with the two vertices h_0, h_z. Without loss of generality, let $\{h_0, K_1, K_2, \ldots, K_\omega\} \subseteq H_{0,2} = \{h_0, h_2, \ldots, h_{z-1}\}$ lie far below $\{L_\omega, L_{\omega-1}, \ldots, L_1, h_z\} \subseteq H_{1,2} = \{h_1, h_3, \ldots, h_z\}$.

By Lemma 5, for any $i = 1, \dots, \omega$, the segment $h_0 K_i$ is open up in $H_{0,2}$ with probability $\frac{1}{2^{i-1}}$. Analogously, $L_i h_z$ is open down in $H_{1,2}$ also with probability $\frac{1}{2^{i-1}}$. Thus, each of the triangles $h_0 K_i h_z$ and $h_0 L_i h_z$ is empty with probability $\frac{1}{2^{i-1}}$, and

$$W_3(z) = 2 \cdot \sum_{i=1}^{\omega} \frac{1}{2^{i-1}} = 4 - \frac{4}{2^{\omega}}.$$

Any two empty triangles $h_0 K_i h_z$ and $h_0 L_j h_z$ $(i, j \geq 2)$ give rise to an empty hexagon $h_0 K_{i-1} K_i h_z L_{j-1} L_j$. Thus,

$$W_6(z) = \sum_{i=2}^{\omega} \sum_{j=2}^{\omega} \frac{1}{2^{i-1}} \cdot \frac{1}{2^{j-1}} = \left(\sum_{i=2}^{\omega} \frac{1}{2^{i-1}} \right)^2 = \left(1 - \frac{1}{2^{\omega-1}} \right)^2 = 1 - \frac{4}{2^{\omega}} + \frac{4}{4^{\omega}}.$$

□

Observation 20 *For any odd $z \geq 3$ and any $k, s \in \mathbb{N}$, the expected number of empty k-gons in a random Horton set H of size $2^s z + 1$ containing both the leftmost point and the rightmost point of H is equal to $W_k(z)$.*

Proof. We denote the points of H as above. The set $H_{0,2^s} = \{h_0, h_{2^s}, \dots, h_{2^s z}\}$ is a random Horton set of size $z + 1$. Its convex hull contains no other points of H. Thus, $H_{0,2^s}$ determines, in expectation, $W_k(z)$ empty k-gons with the two vertices $h_0, h_{2^s z}$. There are no other empty k-gons with the two vertices $h_0, h_{2^s z}$, since the interior of every triangle $h_0 h_{2^s z} h_i$, $h_i \in H \setminus H_{0,2^s}$, contains one of the points $h_{2^s}, h_{2^s (z-1)}$. □

Here is an analogue of Lemma 18:

Lemma 21 *Let $k \in \mathbb{N}$. If $W_k(z) = O(1)$, then the expected number of empty 1-line r-prime k-gons ($r \neq 2^s$) in Λ is*

$$\approx \frac{4}{3} \sum_{z \geq 3 \text{ odd}} \frac{W_k(z)}{z^2} \cdot p_n.$$

Proof. Let $k \in \mathbb{N}$. By Observation 20, the expected number of empty 1-line 2^s-prime k-gons is

$$\sum_{z \geq 3 \text{ odd}} \sum_{s=0}^{\infty} \sum_{(IJ \text{ is } 2^s z-\text{prime})} W_k(z). \tag{11}$$

It follows from Lemma 7 and from two applications of Observation 10 that (11) can be estimated by

$$\approx \sum_{z\geq 3 \text{ odd}} \sum_{s=0}^{\infty} \frac{p_n}{4^s z^2} W_k(z) = \frac{4}{3} \sum_{z\geq 3 \text{ odd}} \frac{W_k(z)}{z^2} \cdot p_n.$$

□

We are ready to estimate the number of empty 1-line r-prime triangles and hexagons ($r \neq 2^s$). By Lemmas 19 and 21, the expected number of 1-line r-prime triangles ($r \neq 2^s$) is

$$\approx \frac{4}{3} \sum_{z\geq 3 \text{ odd}} \frac{4 - \frac{4}{2^\omega}}{z^2} \cdot p_n = \left(\frac{16}{3}\alpha - \frac{16}{3}\beta \right) \cdot p_n,$$

and the expected number of empty 1-line r-prime hexagons ($r \neq 2^s$) is

$$\approx \frac{4}{3} \sum_{z\geq 3 \text{ odd}} \frac{1 - \frac{4}{2^\omega} + \frac{4}{4^\omega}}{z^2} \cdot p_n = \left(\frac{4}{3}\alpha - \frac{16}{3}\beta + \frac{16}{3}\gamma \right) \cdot p_n.$$

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