THE UNIT DISTANCE PROBLEM ON SPHERES

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ABSTRACT. For any $D > 1$ and for any $n \geq 2$ we construct a set of n points on a sphere in $\mathbb{R}^3$ of diameter D determining at least $cn\sqrt{\log n}$ unit distances. This improves a previous lower bound of Erdős, Hickerson and Pach (1989). We also construct a set of n points in the plane not containing collinear triples or the vertices of a parallelogram and determining at least $cn\sqrt{\log n}$ unit distances.

1. INTRODUCTION

For any finite set P of points let $u(P)$ denote the maximum number of unit distances occurring between points of P:

$$u(P) = \#\{\boldsymbol{pq} \in \tbinom{P}{2} : |\boldsymbol{pq}| = 1\},$$

where throughout the paper $|\boldsymbol{pq}|$ denotes the Euclidean distance between $\boldsymbol{p}$ and $\boldsymbol{q}$, and $\#S$ denotes the number of points in the finite set S. For $n \geq 1$, let $u(n) = \max\{u(P) : P \subset \mathbb{R}^2, \#P = n\}$. Erdős [3] initiated the investigation of the function $u(n)$. Currently the best known asymptotic bounds on $u(n)$ are

$$n^{1+\frac{c}{\log\log n}} < u(n) < c'n^{4/3}.$$

The lower bound can be obtained from the (properly scaled) $\sqrt{n} \times \sqrt{n}$ square grid, as shown in [3]. The upper bound was obtained by Spencer, Szemerédi and Trotter [13], and received many different proofs [2, 12], with the simplest being that of Székely [14]. Erdős [3] conjectured that $u(n) < n^{1+o(1)}$ and in many later papers also $u(n) < n^{1+\frac{c}{\log\log n}}$ (for example [4, 5, 6, 7]). In this

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paper we study an analogous problem on a sphere in $\mathbb{R}^3$. Let $\mathbb{S}_D^2$ denote the sphere in $\mathbb{R}^3$ of diameter $D > 1$ centered at the origin $\boldsymbol{o}$:

$$\mathbb{S}_D^2 = \{\boldsymbol{p} \in \mathbb{R}^3 : |\boldsymbol{op}| = D/2\}.$$

For $n \geq 1$, let

$$u_D(n) = \max\{u(P) : P \subset \mathbb{S}_D^2, \#P = n\}.$$

Leo Moser [10] (see also [9, 11]) conjectured that $u_D(n) < cn$ for any $D > 1$. This was disproved by Erdős, Hickerson and Pach [8], who showed that $u_{\sqrt{2}}(n) = \Theta(n^{4/3})$ and $u_D(n) > cn \log^* n$ for all $D > 1$ and $n \geq 2$. Here $\log^* n$ is the *iterated logarithm function*, i.e., the smallest k such that $a_k = 0$, where we define $a_0 = n$ and $a_{k+1} = \log \max\{a_k, 1\}$. Here we improve the $cn \log^* n$ lower bound to $cn\sqrt{\log n}$:

Theorem 1. *There exists $c > 0$ such that for any $D > 1$ and $n \geq 2$,*

$$u_D(n) > cn\sqrt{\log n}.$$

The best known upper bound for $u_D(n)$ is $u_D(n) < cn^{4/3}$. This can be obtained by adapting some of the proofs of $u(n) < cn^{4/3}$ (e.g. Székely's proof [14]) from the plane to the sphere. This bound is asymptotically of the right order if $D = \sqrt{2}$ (see [8]), but for other values of $D > 1$ nothing more is known.

The proof technique giving Theorem 1 can also be applied to construct planar point sets in "general position" with the same lower bound for the number of unit distances. For example, say that P is in *parallelogram-free* if P does not contain three collinear points or the vertex set of a parallelogram. In particular, if P is parallelogram-free then it determines no two parallel unit segments. Let

$$u'(n) = \max\{u(P) : P \subset \mathbb{R}^2, \#P = n, P \text{ is parallelogram-free}\}.$$

For a lower bound we cannot use the grid construction or the Minkowski sum construction (i.e. projecting the 1-skeleton of a hypercube onto the plane) anymore. Brass [1] noticed that a planar analogue of the construction in [8] gives $u'(n) > cn \log^* n$. Using a similar construction as in Theorem 1 we improve this as follows:

Theorem 2. *There exists $c > 0$ such that for any $n \geq 2$,*

$$u'(n) > cn\sqrt{\log n}.$$

We do not know of any upper bound for $u'(n)$ better than $u'(n) \leq u(n) < cn^{4/3}$.

We finally mention that Theorem 1 also holds for the hyperbolic plane of any curvature, and that the proof is virtually identical. This observation is due to Endre Makai, Jr.

2. Proof of Theorem 1

The line through the *poles* $(0, 0, \pm D/2)$ of $\mathbb{S}^2_D$ is called the *main axis* and the great circle $\mathbb{S}^2_D \cap \{(x, y, z) \in \mathbb{R}^3 : z = 0\}$ the *equator*. Let $\mathbb{S}^1$ be the unit circle in the xy-plane. We consider its elements to be angles and we identify an angle $\alpha \in \mathbb{S}^1$ with the rotation around the main axis by α in the counterclockwise direction when looking from the *north pole* $(0, 0, D/2)$. We denote the image of point $\boldsymbol{p}$ after rotation through the angle α by $\boldsymbol{p}_\alpha$, and the image of a set of points A by A_α.

Our construction strongly relies on the fact that the rotations in $\mathbb{S}^1$ are isometries (in particular, unit distances are preserved) and that they form an abelian group.

Let $A \subset \mathbb{S}^2_D$ be a set of $t \geq 1$ points lying in a sufficiently small neighbourhood of some point on the equator. For any ordered pair $(\boldsymbol{p}, \boldsymbol{q}) \in A^2$ let $\beta(\boldsymbol{p}, \boldsymbol{q})$ be the counterclockwise angle $\beta \in \mathbb{S}^1$ such that $|\boldsymbol{p}\boldsymbol{q}_\beta| = 1$. For $S \subseteq A^2$, let $\beta(S) = \sum_{(\boldsymbol{p},\boldsymbol{q}) \in S} \beta(\boldsymbol{p}, \boldsymbol{q})$.

Claim 1. *For any $t \geq 1$ the set A can be chosen in such a way that the 2^{t^2} sets $A_{\beta(S)}$ are mutually disjoint, where S ranges over all subsets of A^2.*

Assuming this claim, let $B = \bigcup_{S \subseteq A^2} A_{\beta(S)}$. Then $\#B = t2^{t^2}$. Consider two subsets $S, S' \subseteq A^2$ such that the symmetric difference $S \Delta S'$ contains exactly one element, say $S = S' \setminus \{(\boldsymbol{p}, \boldsymbol{q})\}$. Then there is at least one unit distance between $A_{\beta(S)}$ and $A_{\beta(S')}$, since

$$|\boldsymbol{p}_{\beta(S)} \boldsymbol{q}_{\beta(S')}| = |\boldsymbol{p}_{\beta(S)} \boldsymbol{q}_{\beta(S) + \beta(\boldsymbol{p},\boldsymbol{q})}| = |\boldsymbol{p}\boldsymbol{q}_{\beta(\boldsymbol{p},\boldsymbol{q})}| = 1.$$

Since there are $\frac{t^2}{2} 2^{t^2}$ unordered pairs of subsets of A^2 with symmetric difference of size 1, the number of unit distances in B is at least $\frac{t^2}{2} 2^{t^2}$, hence $u_D(t2^{t^2}) \geq \frac{t^2}{2} 2^{t^2}$.

Now let $n \geq 2$ be arbitrary. Let $t \geq 1$ be the unique integer satisfying $t2^{t^2} \leq n < (t+1)2^{(t+1)^2}$. Take $k := \lfloor \frac{n}{t2^{t^2}} \rfloor$ disjoint copies of the set B constructed above (e.g. take $B_{i\varepsilon}$, $i = 1, 2, \ldots, k$, where $\varepsilon > 0$ is a sufficiently small angle). Further, add $n - kt2^{t^2}$ arbitrary points to obtain n points in total. Then

$$u(n) \geq k\frac{t^2}{2}2^{t^2} > \frac{n}{2t2^{t^2}}\frac{t^2}{2}2^{t^2} = \frac{nt}{4} > cn\sqrt{\log n}$$

where $c > 0$ is a reasonable constant, e.g. $c = 1/10$ will do if the logarithm is base 2. We mention that this method actually gives

$$u(n) > (1 - o(1))\frac{1}{2}n\sqrt{\log_2 n}.$$

It remains to prove Claim 1. Though Claim 1 is intuitive, its proof is technical.

Suppose that for some set A the sets $A_{\beta(S)}$ are not mutually disjoint. We will show that they are all disjoint after finitely many small perturbations of A.

We may suppose that the following condition holds:

$(*)$ The points of A have distinct distances from the north pole.

We say that that an unordered pair $\{S, S'\}$ of distinct subsets $S, S' \subseteq A^2$ is *bad* if $\beta(S) = \beta(S')$, and *good* if $\beta(S) \neq \beta(S')$. Clearly, if $(*)$ holds then all $A_{\beta(S)}$ are disjoint iff all pairs of distinct subsets of A^2 are good.

Standard continuity arguments give the following observations:

Observation 1. *If $(*)$ holds, then $(*)$ still holds after any sufficiently small perturbation of A.*

Observation 2. *If $\{S, S'\}$ is good, then it is still good after any sufficiently small perturbation of A.*

It remains to show that if $\{S, S'\}$ is bad then there are arbitrarily small perturbations of A changing $\{S, S'\}$ to a good pair. Let $\{S, S'\}$ be a bad pair.

If $S\Delta S' \subseteq \{(\boldsymbol{a}, \boldsymbol{a}) : \boldsymbol{a} \in A\}$, then

$$0 = \beta(S) - \beta(S') = \sum_{(\boldsymbol{a},\boldsymbol{a})\in S\setminus S'} \beta(\boldsymbol{a}, \boldsymbol{a}) - \sum_{(\boldsymbol{a},\boldsymbol{a})\in S'\setminus S} \beta(\boldsymbol{a}, \boldsymbol{a}).$$

We choose any $(\boldsymbol{a}, \boldsymbol{a}) \in S\Delta S'$, and perturb $\boldsymbol{a}$ such that $\beta(\boldsymbol{a}, \boldsymbol{a})$ changes. Then $\{S, S'\}$ becomes a good pair.

Otherwise there exists $(\boldsymbol{a}, \boldsymbol{b}) \in S\Delta S'$ with $\boldsymbol{a} \neq \boldsymbol{b}$. In this case let $\gamma = \beta(\boldsymbol{a}, \boldsymbol{b})$. Let C_1 be the set of points on $\mathbb{S}^2_D$ at distance 1 to $\boldsymbol{b}_\gamma$. Then clearly C_1 is a circle passing through $\boldsymbol{a}$ (see Figure 1).

Observation 3. $\beta(\boldsymbol{a}', \boldsymbol{b}) = \gamma$ *for any* $\boldsymbol{a}' \in C_1$ *with* $|\boldsymbol{a}\boldsymbol{a}'| < \frac{1}{10}$.

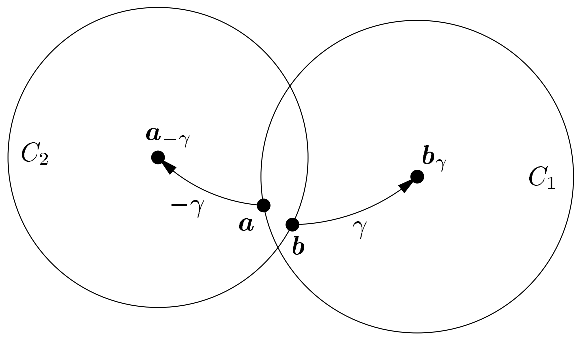

FIGURE 1

Similarly, let C_2 be the set of points on $\mathbb{S}_D^2$ at distance 1 to $\boldsymbol{a}_{-\gamma}$. Then C_2 is a circle passing through $\boldsymbol{b}$ (Figure 1), and

Observation 4. *$\beta(\boldsymbol{a}, \boldsymbol{b}') = \gamma$ for any $\boldsymbol{b}' \in C_2$ with $|\boldsymbol{b}\boldsymbol{b}'| < \frac{1}{10}$.*

Suppose that $\{S, S'\}$ stays bad if we replace $\boldsymbol{a} \in A$ by $\boldsymbol{a}'$, as well as if we replace $\boldsymbol{b}$ by $\boldsymbol{b}'$. A simple calculation then shows that if we simultaneously replace $\boldsymbol{a}$ by $\boldsymbol{a}'$ and $\boldsymbol{b}$ by $\boldsymbol{b}'$, then $\beta(S) - \beta(S')$ is increased by $\beta(\boldsymbol{a}', \boldsymbol{b}') - \beta(\boldsymbol{a}, \boldsymbol{b})$. The following claim shows that we can perturb $\boldsymbol{a}, \boldsymbol{b}$ along the circles C_1, C_2 such that $\beta(\boldsymbol{a}, \boldsymbol{b})$ is changed.

Claim 2. *For any $\varepsilon > 0$ there exist $\boldsymbol{a}' \in C_1, \boldsymbol{b}' \in C_2, |\boldsymbol{a}\boldsymbol{a}'| < \varepsilon, |\boldsymbol{b}\boldsymbol{b}'| < \varepsilon$ such that $\beta(\boldsymbol{a}, \boldsymbol{b}) \neq \beta(\boldsymbol{a}', \boldsymbol{b}')$.*

Proof. Fix any $\boldsymbol{b}' \in C_2$ with $0 < |\boldsymbol{b}\boldsymbol{b}'| < \min\{\frac{1}{10}, \varepsilon\}$. There are at most two points on C_1 at unit distance from $\boldsymbol{b}'_\gamma$ (since two circles intersect in at most two points). Choose for $\boldsymbol{a}' \in C_1$ any other point near $\boldsymbol{a}$. $\square$

Thus we can make all pairs of subsets of A^2 good by at most $\binom{2^{t^2}}{2}$ small perturbations of A. This finishes the proof of Claim 1, and also of Theorem 1. $\square$

3. Proof of Theorem 2

The proof is almost identical to that of Theorem 1. We consider rotations around the point $(0, r)$, $r > 0$, as well as horizontal translations, which may be thought of as rotations around $(0,\infty)$. We denote the image of a point $\boldsymbol{p}$ when rotated in the counterclockwise direction by an angle α around $(0,r)$ by $\boldsymbol{p}^r_\alpha$ (and when translated by the distance α to the right by $\boldsymbol{p}_\alpha$).

Let $A \subseteq \mathbb{R}^2$ be a set of $t \geq 1$ variable points $\{\boldsymbol{p}_1,\dots,\boldsymbol{p}_t\}$ in a $1/10$-neighbourhood of $(0,-1)$. We denote their coordinates by $\boldsymbol{p}_i = (x_i, y_i)$. We assume that $y_i \neq y_j$ for all $1 \leq i < j \leq t$. We define $\beta^r(\boldsymbol{p},\boldsymbol{q})$ to be the counterclockwise angle β such that $|\boldsymbol{p}\boldsymbol{q}^r_\beta| = 1$ (and $\beta(\boldsymbol{p},\boldsymbol{q})$ to be the distance β such that $|\boldsymbol{p}\boldsymbol{q}_\beta| = 1$). Note that $\beta(\boldsymbol{p},\boldsymbol{q})$ and $\beta^r(\boldsymbol{p},\boldsymbol{q})$ are smooth functions of $\boldsymbol{p}$ and $\boldsymbol{q}$ in a bounded open set (for $r > 0$ sufficiently large), and that $\boldsymbol{a}_{\beta(\boldsymbol{p},\boldsymbol{q})} = \lim_{r\to\infty} \boldsymbol{a}_{\beta^r(\boldsymbol{p},\boldsymbol{q})}$ for any point $\boldsymbol{a}$.

We now use the set of unordered pairs $\binom{A}{2}$ instead of A^2 to define B. For any $S \subseteq \binom{A}{2}$, let $\beta(S) = \sum_{\boldsymbol{p}_i\boldsymbol{p}_j \in S} \beta(\boldsymbol{p}_i,\boldsymbol{p}_j)$, where in the sum we always have $i < j$. Let $B = \bigcup_{S\subseteq\binom{A}{2}} A_{\beta(S)}$, where $A_{\beta(S)}$ is defined as before. We also similarly define $\beta^r(S)$, $A_{\beta^r(S)}$ and B_r.

Claim 3. *For any $t \geq 1$ and any sufficiently large $r > 0$ the set A can be chosen in such a way that the sets $A_{\beta^r(S)}$ are mutually disjoint, where S ranges over all subsets of $\binom{A}{2}$, and furthermore such that B is parallelogram-free.*

Assuming this claim, the proof of Theorem 2 can be finished by making the appropriate changes to the remainder of the proof of Theorem 1. In particular, due to the fact that we use unordered pairs instead of ordered pairs, the lower bound obtained now has a slightly worse constant:

$$u'(n) > (1+o(1))\frac{\sqrt{2}}{4} n\sqrt{\log_2 n}.$$

Proof of Claim 3. Again the proof is very technical. To show that there are arbitrarily small perturbations of any A such that $A_{\beta^r(S)}$ are all disjoint for $r > 0$ sufficiently large, the proof of Claim 1 may be repeated almost verbatim (we only have to modify $(*)$ by replacing "north pole" with "the point $(0,r)$"). It remains to show that if $r > 0$ is sufficiently large, B_r does not contain collinear triples nor the vertex set of a parallelogram. Note that as $r \to \infty$, $B_r \to B$. Thus we first prove that A can be chosen in such a way that if B contains a collinear triple or a parallelogram, it must be of a very special type, and then the corresponding points in B_r cannot be a collinear triple or a parallelogram. Then if $r > 0$ is sufficiently large, no new collinear triples or parallelograms are created that was not originally in B, and it follows that B_r is the required set.

First consider a parallelogram

$$(\boldsymbol{p}_i)_{\beta(S)} - (\boldsymbol{p}_j)_{\beta(T)} = (\boldsymbol{p}_k)_{\beta(U)} - (\boldsymbol{p}_\ell)_{\beta(V)}$$

in B. Then $y_i - y_j = y_k - y_\ell$. By making a sufficiently small perturbation of A, we may assume $y_i - y_j \neq y_k - y_\ell$ for all distinct pairs $\{i, j\}$ and $\{k, \ell\}$. It follows that without loss of generality the parallelogram is

$$(\boldsymbol{p}_1)_{\beta(S)} - (\boldsymbol{p}_1)_{\beta(T)} = (\boldsymbol{p}_2)_{\beta(U)} - (\boldsymbol{p}_2)_{\beta(V)}.$$

Thus $\beta(S) - \beta(T) = \beta(U) - \beta(V)$, and we have a linear dependence

$$\sum_{i<j} \lambda_{ij} \beta(\boldsymbol{p}_i, \boldsymbol{p}_j) = 0,$$

where $\lambda_{ij} \in \{0, \pm 1, \pm 2\}$ depends only on which of S, T, U, V contains $\boldsymbol{p}_i \boldsymbol{p}_j$.

Observation 5. *When considered as functions of $x_1, \ldots, x_t, y_1, \ldots, y_t$ ranging over any open set where $x_i \neq x_j, y_i \neq y_j$, $\beta(\boldsymbol{p}_i, \boldsymbol{p}_j)$ $(i < j)$ are linearly independent.*

Proof. Let $F(x_1, \ldots, x_t, y_1, \ldots, y_t) = \sum_{i<j} \lambda_{ij} \beta(\boldsymbol{p}_i, \boldsymbol{p}_j)$. Note that

$$\beta(\boldsymbol{p}_i, \boldsymbol{p}_j) = x_i - x_j + \sqrt{1 - (y_i - y_j)^2}.$$

Assume $F \equiv 0$ and for any fixed $i < j$ take partial derivatives with respect to y_i and y_j:

$$\partial^2 F / \partial y_i \partial y_j = \lambda_{ij} (1 - (y_i - y_j)^2)^{-3/2} \equiv 0,$$

thus $\lambda_{ij} = 0$. $\square$

It follows that all $\lambda_{ij} = 0$, and we have $\beta^r(S) - \beta^r(T) = \beta^r(U) - \beta^r(V)$ for any r. Thus in B_r we have that the segments $(\boldsymbol{p}_1)^r_{\beta^r(S)} (\boldsymbol{p}_1)^r_{\beta^r(T)}$ and $(\boldsymbol{p}_2)^r_{\beta^r(U)} (\boldsymbol{p}_2)^r_{\beta^r(V)}$ are chords spanning the same angle at $(0, r)$. Since we have assumed that $\boldsymbol{p}_1$ and $\boldsymbol{p}_2$ have different y-coordinates, $(\boldsymbol{p}_1)^r_{\beta^r(S)}$ and $(\boldsymbol{p}_2)^r_{\beta^r(U)}$ have different distances to $(0, r)$ for sufficiently large r. It follows that the two chords do not have the same length, hence cannot form a parallelogram.

Next consider a collinear triple $(\boldsymbol{p}_i)_{\beta(S)}, (\boldsymbol{p}_j)_{\beta(T)}, (\boldsymbol{p}_k)_{\beta(U)} \in B$. If the triple is horizontal, $\boldsymbol{p}_i, \boldsymbol{p}_j, \boldsymbol{p}_k$ have the same y-coordinate, which means $\boldsymbol{p}_i = \boldsymbol{p}_j = \boldsymbol{p}_k$ (since we assume that the points in A have distinct y-coordinates). Then the corresponding points in B_r are not collinear, as they are on a circle with centre $(0, r)$. In the remaining case, $\boldsymbol{p}_i, \boldsymbol{p}_j, \boldsymbol{p}_k$ are distinct, and we assume without loss of generality that these points are $\boldsymbol{p}_1, \boldsymbol{p}_2, \boldsymbol{p}_3$.

Assume that we cannot perturb A to make $(\boldsymbol{p}_1)_{\beta(S)}(\boldsymbol{p}_2)_{\beta(T)}(\boldsymbol{p}_3)_{\beta(U)}$ non-collinear. Then it is possible to arrive at a contradiction, as we now sketch. Thus in some open subset of $\mathbb{R}^{2t}$

$$D(x_1,\ldots,x_t,y_1,\ldots,y_t) := \begin{vmatrix} 1 & x_1+\beta(S) & y_1 \\ 1 & x_2+\beta(T) & y_2 \\ 1 & x_3+\beta(U) & y_3 \end{vmatrix} \equiv 0.$$

For any distinct $i,j \geq 4$, if we calculate $\partial^2 D/\partial y_i \partial y_j$ we obtain that $\boldsymbol{p}_i\boldsymbol{p}_j$ must either be in each of S,T,U or in none of S,T,U. Thus S,T,U coincide on $\binom{\{\boldsymbol{p}_4,\ldots,\boldsymbol{p}_t\}}{2}$.

For any $i \geq 4$ if we calculate $\partial^3 D/\partial y_i \partial y_2 \partial y_1$ we obtain that S and U coincide on $\boldsymbol{p}_1\boldsymbol{p}_i$, and T and U on $\boldsymbol{p}_2\boldsymbol{p}_i$. Similarly, by considering $\partial^3 D/\partial y_i \partial y_3 \partial y_1$ and $\partial^3 D/\partial y_i \partial y_3 \partial y_2$ we obtain that S,T,U coincide on $\{\boldsymbol{p}_i\boldsymbol{p}_j : 1 \leq i \leq 3, 4 \leq j \leq t\}$.

It follows that we may replace S by $S' = S \cap \{\boldsymbol{p}_1\boldsymbol{p}_2, \boldsymbol{p}_2\boldsymbol{p}_3, \boldsymbol{p}_1\boldsymbol{p}_3\}$, and T and U by similarly defined T', U' without changing D (just expand the determinant).

We use the following indicator function:

$$\chi(\boldsymbol{pq} \in C) = \begin{cases} 1 & \text{if} \quad \boldsymbol{pq} \in C \\ 0 & \text{if} \quad \boldsymbol{pq} \notin C \end{cases}.$$

By calculating $\partial D/\partial x_1$ we find

$$\begin{aligned} \frac{\partial D}{\partial x_1} = -y_3 + y_2 \; & +(y_2-y_3)(\chi(\boldsymbol{p}_1\boldsymbol{p}_2 \in S) + \chi(\boldsymbol{p}_1\boldsymbol{p}_3 \in S)) \\ & +(y_3-y_1)(\chi(\boldsymbol{p}_1\boldsymbol{p}_2 \in T) + \chi(\boldsymbol{p}_1\boldsymbol{p}_3 \in T)) \\ & +(y_1-y_2)(\chi(\boldsymbol{p}_1\boldsymbol{p}_2 \in U) + \chi(\boldsymbol{p}_1\boldsymbol{p}_3 \in U)) \equiv 0. \end{aligned}$$

It follows that

$$\begin{aligned} & 1 + \chi(\boldsymbol{p}_1\boldsymbol{p}_2 \in S) + \chi(\boldsymbol{p}_1\boldsymbol{p}_3 \in S) \\ = \; & \chi(\boldsymbol{p}_1\boldsymbol{p}_2 \in T) + \chi(\boldsymbol{p}_1\boldsymbol{p}_3 \in T) \\ = \; & \chi(\boldsymbol{p}_1\boldsymbol{p}_2 \in U) + \chi(\boldsymbol{p}_1\boldsymbol{p}_3 \in U). \end{aligned} \tag{1}$$

By similarly calculating $\partial D/\partial x_3$ we obtain

$$\begin{aligned} & \chi(\boldsymbol{p}_2\boldsymbol{p}_3 \in S) + \chi(\boldsymbol{p}_1\boldsymbol{p}_3 \in S) \\ = \; & \chi(\boldsymbol{p}_2\boldsymbol{p}_3 \in T) + \chi(\boldsymbol{p}_1\boldsymbol{p}_3 \in T) \\ = \; & -1 + \chi(\boldsymbol{p}_2\boldsymbol{p}_3 \in U) + \chi(\boldsymbol{p}_1\boldsymbol{p}_3 \in U). \end{aligned} \tag{2}$$

Finally, by calculating

$$\frac{\partial^3 D}{\partial y_3 \partial y_2 \partial y_1} = \frac{-\chi(\boldsymbol{p}_1\boldsymbol{p}_2 \in S) + \chi(\boldsymbol{p}_1\boldsymbol{p}_2 \in T)}{(1-(y_1-y_2)^2)^{3/2}} + \frac{-\chi(\boldsymbol{p}_2\boldsymbol{p}_3 \in T) + \chi(\boldsymbol{p}_2\boldsymbol{p}_3 \in U)}{(1-(y_2-y_3)^2)^{3/2}} + \frac{-\chi(\boldsymbol{p}_1\boldsymbol{p}_3 \in U) + \chi(\boldsymbol{p}_1\boldsymbol{p}_3 \in S)}{(1-(y_1-y_3)^2)^{3/2}} \equiv 0,$$

we obtain

$$\chi(\boldsymbol{p}_1\boldsymbol{p}_2 \in S) = \chi(\boldsymbol{p}_1\boldsymbol{p}_2 \in T) \quad \text{and} \quad \chi(\boldsymbol{p}_2\boldsymbol{p}_3 \in T) = \chi(\boldsymbol{p}_2\boldsymbol{p}_3 \in U). \tag{3}$$

From the first inequalities in (1), (2) we obtain $1+\chi(\boldsymbol{p}_1\boldsymbol{p}_3 \in S) = \chi(\boldsymbol{p}_1\boldsymbol{p}_3 \in T)$, and from the latter inequalities in (2) and (3) we obtain $\chi(\boldsymbol{p}_1\boldsymbol{p}_3 \in T) = -1 + \chi(\boldsymbol{p}_1\boldsymbol{p}_3 \in U)$. Thus $2 + \chi(\boldsymbol{p}_1\boldsymbol{p}_3 \in S) = \chi(\boldsymbol{p}_1\boldsymbol{p}_3 \in U)$, a contradiction.

This finishes the proof of Claim 3, and also of Theorem 2. □

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