The Erdős-Szekeres theorem: upper bounds and related results

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Abstract

Let $ES(n)$ denote the least integer such that among any $ES(n)$ points in general position in the plane there are always n in convex position. In 1935, P. Erdős and G. Szekeres showed that $ES(n)$ exists and $ES(n) \leq \binom{2n-4}{n-2} + 1$. About 62 years later, the upper bound has been slightly improved by Chung and Graham, a few months later it was further improved by Kleitman and Pachter, and another few months later it was further improved by the present authors. Here we review the original proof of Erdős and Szekeres, the improvements, and finally we combine the methods of the first and third improvements to obtain yet another tiny improvement.

We also briefly review some of the numerous results and problems related to the Erdős–Szekeres theorem.

1 Introduction

In 1933, Esther Klein raised the following question. Is it true that for every n there is a least number $ES(n)$ such that among any $ES(n)$ points in general position in the plane there are always n in convex position?

[*]Supported by OTKA-T-038397
[†]Supported by project LN00A056 of The Ministry of Education of the Czech Republic.

This question was answered in the affirmative in a classical paper of Erdős and Szekeres [ES35]. In fact, they showed [ES35, ES60] that

$$2^{n-2} + 1 \leq ES(n) \leq \binom{2n-4}{n-2} + 1.$$

The lower bound, $2^{n-2} + 1$, is sharp for $n = 2, 3, 4, 5$ and has been conjectured to be sharp for all n. However, the upper bound, $\binom{2n-4}{n-2} + 1 \approx c\frac{4^n}{\sqrt{n}}$, was not improved for 60 years. Recently, Chung and Graham [CG98] managed to improve it by 1. Shortly after, Kleitman and Pachter [KP978] showed that $ES(n) \leq \binom{2n-4}{n-2} + 7 - 2n$. A few months later the present authors [TV98] proved that $ES(n) \leq \binom{2n-5}{n-2} + 2$, which is a further improvement, roughly by a factor of 2.

In this note we review the original proof of Erdős and Szekeres, all three improvements, and then we combine the ideas of the first and third improvements to obtain the following result, which is a further improvement by 1.

Theorem 1 *For $n \geq 5$, any set of $\binom{2n-5}{n-2} + 1$ points in general position in the plane contains n points in convex position. That is, $ES(n) \leq \binom{2n-5}{n-2} + 1$.*

Next section contains a brief review of some of the numerous results and problems related to the Erdős–Szekeres theorem.

2 Some related results

Many researchers have been motivated by the Erdős–Szekeres theorem. Here we mention only a small part of the research related to the Erdős–Szekeres theorem. See [MS00] and [BMP04] for the latest survey.

2.1 Empty polygons

A famous open problem related to the Erdős–Szekeres theorem is the *empty–hexagon problem*. Let P be a finite set of points in general position in the plane. A subset $Q \subset P, |Q| = n$, is called an *n-hole* (or an *empty convex n-gon*) *in P*, if it is in convex position and its convex hull contains no further points of P. Let $g(n)$ be the smallest positive integer such that any P, $|P| \geq g(n)$, in general position contains an n-hole. It is easy to see that

$g(3) = 3$, $g(4) = 5$. Harborth [H78] proved $g(5) = 10$. Horton [H83] gave a construction showing that no finite $g(7)$ exists.

The empty–hexagon problem: *Is there a finite* $g(6)$*?*

Using a computer search, Overmars [O03] found a set of 29 points in general position having no empty hexagon. Thus, if $g(6)$ exists then $g(6) \geq 30$.

Let $X_k(P)$ be the number of empty k-gons in an n-element point set P in general position, for $k \geq 0$ (every subset of P of size at most 2 is considered as an empty polygon; thus $X_0(P) = 1$, $X_1(P) = n$, $X_2(P) = \binom{n}{2}$). There are several equalities and inequalities involving these parameters. Ahrens et al. [AGM99] proved general results giving the following interesting equalities on the numbers $X_k(P)$:

$$\sum_{k \geq 0} (-1)^k X_k(P) = 0,$$

$$\sum_{k \geq 1} (-1)^k k X_k(P) = -|P \cap \mathrm{Int}(P)|,$$

where $|P \cap \mathrm{Int}(P)|$ is the number of interior points of P. Pinchasi et al. [PRS04] proved the above two equalities by a simple argument ("continuous motion" method) and gave also some other equalities and inequalities, e.g.

$$X_4(P) \geq X_3(P) - \frac{n^2}{2} - O(n),$$

$$X_5(P) \geq X_3(P) - n^2 - O(n).$$

Let $Y_k(n) = \min_{|P|=n} X_k(P)$, that is, the minimum number of empty convex k-gons in a set of n points. By the construction of Horton, $Y_k(n) = 0$ for $k \geq 7$. For $k \leq 6$, the best known bounds are the following.

$$\begin{aligned}
n^2 - 5n + 10 &\leq Y_3(n) \leq 1.6195...n^2 + o(n^2),\\
\binom{n-3}{2} + 6 &\leq Y_4(n) \leq 1.9396...n^2 + o(n^2),\\
3\left\lfloor \frac{n}{12} \right\rfloor &\leq Y_5(n) \leq 1.0206...n^2 + o(n^2),\\
0 &\leq Y_6(n) \leq 0.2005...n^2 + o(n^2).
\end{aligned}$$

The lower bounds are given in [D87], the upper bounds in [BV04].

2.2 Convex bodies

Bisztriczky, Fejes Tóth, Pach, and Tóth [BF89], [BF90], [PT98], [T00] extended the Erdős-Szekeres theorem to families of pairwise disjoint convex sets, instead of points.

A family of pairwise disjoint convex sets is said to be in *convex position* if none of its members is contained in the convex hull of the union of the others.

It is easy to construct an arbitrarily large family of pairwise disjoint convex sets such that no three or more of them are in convex position. So, without any additional condition on the family, we cannot generalize the Erdős-Szekeres theorem.

For points we had the condition "no three points are on a line", that is, "any three points are in convex position". Therefore, the most natural condition to try for families of convex sets is "any three convex sets are in convex position".

Bisztriczky and Fejes Tóth [BF89] proved that there exists a function $P_3(n)$ such that if a family $\mathcal{F}$ of pairwise disjoint convex sets has more than $P_3(n)$ members, and any *three* members of $\mathcal{F}$ are in convex position, then $\mathcal{F}$ has n members in convex position. In [BF90] they showed that this statement is true with a function $P_3(n)$, triply exponential in n. Pach and Tóth [PT98] further improved the upper bound on $P_3(n)$ to a simply exponential function. The best known lower bound for $P_3(n)$ is the classical lower bound for the original Erdős-Szekeres theorem, $2^{n-2} \leq P_3(n)$.

In the case of points, if we have a stronger condition that every *four* points are in convex position, then the problem becomes uninteresting; in this case all points are in convex position.

In case of convex sets, the condition "every four are in convex position" does not make the problem uninteresting, but it still turns out to be a rather strong condition. Let $\mathcal{F}$ be a family of pairwise disjoint convex sets. If any k members of $\mathcal{F}$ are in convex position, then we say that $\mathcal{F}$ satisfies *property* P_k. If no n members of $\mathcal{F}$ are in convex position, then we say that $\mathcal{F}$ satisfies *property* P^n. Property P_k^n means that both P_k and P^n are satisfied. Using these notions, the above cited result of Pach and Tóth states that if a family $\mathcal{F}$ satisfies property P_3^n, then $|\mathcal{F}| \leq \binom{2n-4}{n-2}^2$.

Bisztriczky and Fejes Tóth [BF90] raised the following more general question. What is the maximum size $P_k(n)$ of a family $\mathcal{F}$ satisfying property P_k^n? Some of their bounds were later improved in [PT98] and [T00]. The best known bounds are the following:

$$
\begin{array}{rlll}
2^{n-2} & \leq P_3(n) \leq & \binom{2n-4}{n-2}^2 & \text{[ES60], [PT98]} \\
2\left\lfloor \frac{n+1}{4} \right\rfloor^2 & \leq P_4(n) \leq & n^3 & \text{[PT98]} \\
n-1+\left\lfloor \frac{n-1}{k-2} \right\rfloor & \leq P_5(n) \leq & 6n-12 & \text{[BF90], [T00]} \\
n-1+\left\lfloor \frac{n-1}{k-2} \right\rfloor & \leq P_k(n) \leq & n+\frac{1}{k-5}n \quad \text{for } k \geq 6 & \text{[BF90], [T00]}
\end{array}
$$

Pach and Tóth [PT00] investigated the case when the sets are not necessarily disjoint.

2.3 The partitioned version

It follows from the exponential upper bound on the number $ES(n)$ by a simple counting argument that for a given n every "huge" set of points in general position in the plane contains "many" n-point subsets in convex position. However, geometric arguments yield much stronger results.

A *convex n-clustering* is defined as a finite planar point set in general position which can be partitioned into n finite sets $X_1, X_2, \ldots, X_n$ of *equal* size such that $x_1 x_2 \ldots x_n$ is a convex n-gon for each choice $x_1 \in X_1, x_2 \in X_2, \ldots, x_n \in X_n$.

The positive fraction Erdős-Szekeres theorem [BV98] states that for any n any sufficiently large finite set X of points in general position contains a convex n-clustering of size $\geq \varepsilon_n \cdot |X|$, where $\varepsilon_n > 0$ is independent of X. Answering a question of Bárány, Pór and Valtr [P03], [PV02] proved a partitioned version of the Erdős-Szekeres theorem: any finite X in general position can be partitioned into at most c_n convex clusterings and a remaining set of at most c'_n points. The optimal constants $1/\varepsilon_n, c'_n$ are exponential in n, while c_n is known to be at least exponential in n and at most of order $n^{O(n^2)}$ (see [PV02] for details).

The positive fraction Erdős–Szekeres theorem for collections of convex sets can be found in [PS98], and the partitioned Erdős–Szekeres theorem for collections of convex sets can be found in [PV04].

3 The upper bound of Erdős and Szekeres

Definition. The points $(x_1, y_1), (x_2, y_2), \ldots, (x_n, y_n)$, $x_1 < x_2 < \ldots < x_n$, form an n-*cap* if

$$\frac{y_2 - y_1}{x_2 - x_1} > \frac{y_3 - y_2}{x_3 - x_2} > \ldots > \frac{y_n - y_{n-1}}{x_n - x_{n-1}}.$$

Similarly, they form an n-*cup* if

$$\frac{y_2 - y_1}{x_2 - x_1} < \frac{y_3 - y_2}{x_3 - x_2} < \ldots < \frac{y_n - y_{n-1}}{x_n - x_{n-1}}.$$

Theorem 2 (Erdős and Szekeres [ES35]) *Let $f(n, m)$ be the least integer such that any set of $f(n, m)$ points in general position in the plane contains either an n-cap or an m-cup. Then*

$$f(n, m) = \binom{n + m - 4}{n - 2} + 1.$$

The following observation has a key role in the proof of the Erdős–Szekeres theorem.

Observation 1 *If a point v is the rightmost point of a cap and also the leftmost point of a cup then the cap or the cup can be extended to a larger cap or cup, respectively.*

Proof. Let u be the second point of the cap from right, and let w be the second point of the cup from left. Now, depending on the angle uvw, either the cap can be extended by w, or the cup can be extended by u. See Fig. 1. $\square$

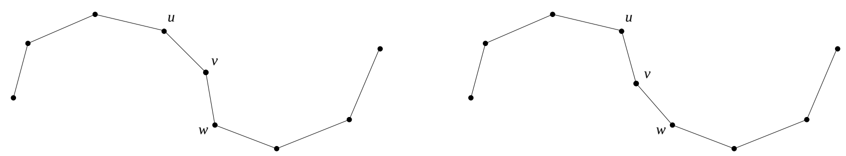

Figure 1: *Either the cap can or the cup can be extended.*

Proof of $f(n,m) \le \binom{n+m-4}{n-2} + 1$. We use double induction on n and m. The statement trivially holds for $n = 2$ and any m, and for $m = 2$ and any n. Let $n, m \ge 3$ and suppose that the statement holds for $(n, m-1)$ and for $(n-1, m)$. Take $\binom{n+m-4}{n-2} + 1$ points in general position. By induction we know that any subset of at least $\binom{n+m-5}{n-3} + 1$ points contains either an $n-1$-cap or an m-cup. In the latter case we are done, so we can assume that any subset of at least $\binom{n+m-5}{n-3} + 1$ points contains an $n-1$-cap. Take an $n-1$-cap and remove its right endpoint from the point set. Since we still have at least $\binom{n+m-5}{n-3} + 1$ points, we have another $n-1$-cap, remove its right endpoint again, and continue until we have $\binom{n+m-5}{n-3}$ points left. We have removed $\binom{n+m-4}{n-2} + 1 - \binom{n+m-5}{n-3} = \binom{n+m-5}{m-3} + 1$ points, each of them a right endpoint of some $n-1$-cap. But the set of these points, by induction, contains either an n-cap or an $m-1$-cup. In the first case we are done. In the second case we have an $m-1$-cup whose left endpoint v is the right endpoint of some $n-1$-cap. Observation 1 then finishes the induction step. $\square$

Proof of the Erdős-Szekeres theorem. Since $ES(n) \le f(n,n)$, we have $ES(n) \le \binom{2n-4}{n-2} + 1$. $\square$

Erdős and Szekeres [ES35] also proved that the bound $f(n,m) \le \binom{n+m-4}{n-2} + 1$ is tight for any n, m. But it does not imply that the bound for $ES(n)$ is tight as well. The best known lower bound is $2^{n-2} + 1 \le ES(n)$ [ES60] and in fact it is conjectured to be the truth.

4 Three improvements

Theorem 3 (Chung and Graham [CG98]) *For $n \ge 4$,*

$$ES(n) \le \binom{2n-4}{n-2}.$$

Proof. Take $\binom{2n-4}{n-2}$ points in general position. Let A be the set of those points which are right endpoints of some $n-1$-cap. Just like above, we can argue that $|A| \ge \binom{2n-4}{n-2} - \binom{2n-5}{n-3} = \binom{2n-5}{n-3}$. If $|A| > \binom{2n-5}{n-3}$, then A contains either an n-cap or an $n-1$-cup. In the first case we are done immediately, in the second we have an $n-1$-cup whose left endpoint is also a right endpoint of some $n-1$-cap and we are done like in the previous proof. So we can assume that $|A| = \binom{2n-5}{n-3}$. Let B be the set of the other points, clearly

$|B| = \binom{2n-5}{n-3}$. Let $b \in B$. The set $\{b\} \cup A$ has size $\binom{2n-5}{n-3} + 1$ so again it contains either an n-cap or an $n-1$-cup. In the case of n-cap we are done, so we can assume that it an $n-1$-cup for any choice of b. If the left endpoint of this $n-1$-cup is an element of A, we are done by Observation 1, since we have an $n-1$-cup whose left endpoint is also a right endpoint of some $n-1$-cap. So, the left endpoint of this $n-1$-cup is b. Therefore, any $b \in B$ is the left endpoint of an $n-1$-cup whose right endpoint is in A. We can argue analogously, that any $a \in A$ is the right endpoint of an $n-1$-cap whose left endpoint is in B. Let S be the set of all segments ab, where $a \in A$, $b \in B$, and there is an $n-1$-cup or $n-1$-cap whose right endpoint is a and left endpoint is b. Let ab be the element of S with the largest slope. Suppose that ab represents an $n-1$-cup, the other case is analogous. We know that there is an $n-1$-cap whose right endpoint is a and left endpoint is b'. Now it is easy to see that either the $n-1$-cup and b', or the $n-1$-cap and b determine a convex n-gon. This concludes the proof, see Fig. 2. $\square$

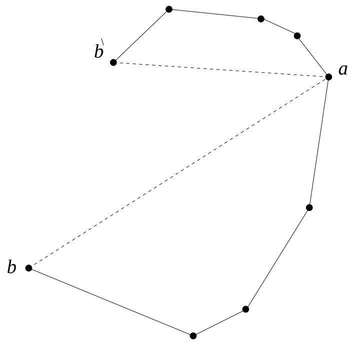

Figure 2: *Either b can be added to the cup, or b' to the cap.*

Theorem 4 (Kleitman and Pachter [KP98]) *For $n \geq 4$,*

$$ES(n) \leq \binom{2n-4}{n-2} - 2n + 7.$$

Proof. We say that a point set is *vertical* if its two leftmost points have the same x-coordinate. Observe, that any point set can be made vertical by an appropriate rotation. We define caps and cups for vertical sets just like for any set of points, the only difference is that now the vertical edge determined by the two leftmost points is allowed to be the leftmost edge of a cup or a cap, see Fig. 3.

Let $f_v(n,m)$ be the least integer such that any *vertical* set of $f_v(n,m)$ points in general position contains either an n-cap or an m-cup. Take $f_v(n,m)-1$ points in a vertical point set with no n-caps and m-cups. Let a and b be the two leftmost points such that a is above b. Let A be the set of those points which are right endpoints of some $n-1$-cap, and B be the set of the other points. Since the two leftmost points do not belong to A, B is a vertical point set. If $|B| \geq f_v(n-1,m)$ then B has an $n-1$-cap or an m-cup. The first case contradicts the definition of A, the second case contradicts the assumption that we do not have an m-cup. So, $|B| \leq f_v(n-1,m)-1$. Now consider the set $A' = A \cup \{b\}$ and suppose that $|A'| \geq f(n,m-1)$. Then A' has an n-cap or an $m-1$-cup. The first case is a contradiction immediately, in the second case consider the left endpoint of that $m-1$-cup. If it is b, then it can be extended to an m-cup by a, a contradiction. If it is in A, then the usual argument works, we have an $n-1$-cup whose left endpoint is also a right endpoint of some $n-1$-cap and one of them can be extended by Observation 1. So $|A| = |A'|-1 \leq f(n,m-1)-2$. Combining the two inequalities we get that

$$f_v(n,m) \leq f_v(n-1,m) + f(n,m-1) - 2,$$

and an analogous argument shows that

$$f_v(n,m) \leq f_v(n,m-1) + f(n-1,m) - 2.$$

Using the known values of $f(n,m)$, and that $f_v(n,3) = f_v(3,n) = n$, we get that $f_v(n,m) \leq \binom{n+m-4}{n-2} + 7 - n - m$, and the result follows. In fact, the inequality obtained for $f_v(n,m)$ is sharp [KP98]. □

Theorem 5 (Tóth and Valtr [TV98]) *For $n \geq 3$,*

$$ES(n) \leq \binom{2n-5}{n-2} + 2.$$

Proof. Take $\binom{2n-5}{n-3}$ points in general position. Suppose that the set P does not contain n points in convex position. Let x be a vertex of the convex hull of P. Let y be a point outside the convex hull of P such that none

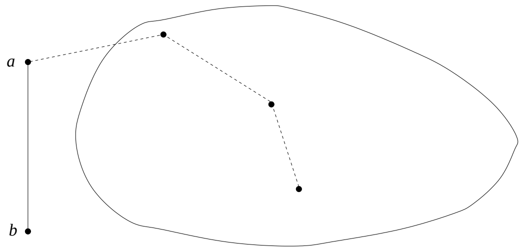

Figure 3: *A vertical point set with a* 5*-cap.*

of the lines determined by the points of $P \setminus \{x\}$ intersects the segment $\overline{xy}$. Finally, let ℓ be a line through y which avoids the convex hull of P.

Consider a projective transformation T which maps the line ℓ to the line at infinity, and maps the segment $\overline{xy}$ to the vertical half-line $v^-(x')$, emanating downwards from $x' = T(x)$. We get a point set $P' = T(P)$ from P. Since ℓ avoided the convex hull of P, the transformation T does not change convexity on the points of P, that is, any subset of P is in convex position if and only if the corresponding points of P' are in convex position. So the assumption holds also for P', no n points of P' are in convex position. By the choice of the point y, none of the lines determined by the points of $P' \setminus \{x'\}$ intersects $v^-(x')$. Therefore, any m-cap in the set $Q' = P' \setminus \{x'\}$ can be extended by x' to a convex $(m+1)$-gon.

Since no n points of P' are in convex position, Q' cannot contain any n-cup or $(n-1)$-cap. Therefore, by the Lemma,

$$|Q'| \leq f(n-1,n) - 1 = \binom{2n-5}{n-2}, \quad |P| \leq \binom{2n-5}{n-2} + 1,$$

and the theorem follows. □

5 A combination of two methods

We now prove Theorem 1.

Proof of Theorem 1. Suppose that the set P does not contain n points in convex position and $|P| = \binom{2n-5}{n-2} + 1$. Let x be a vertex of the convex hull of P and y be a point outside the convex hull of P so close to x that

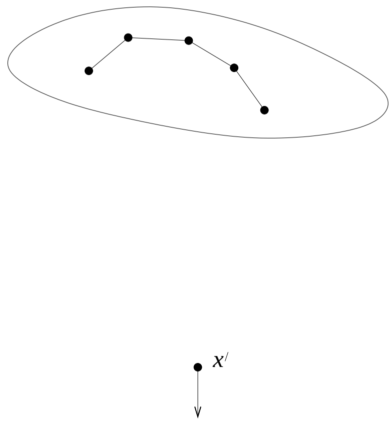

Figure 4: *Any $n-1$-cap can be extended by x' to a convex n-gon.*

none of the lines determined by the points of $P \setminus \{x\}$ intersects the segment $\overline{xy}$. Finally, let ℓ be a line through y which avoids the convex hull of P.

Consider a projective transformation T which maps the line ℓ to the line at infinity, and maps the segment $\overline{xy}$ to the vertical half-line $v^-(x')$, emanating downwards from $x' = T(x)$. We get a point set P' from P. Just like in the previous proof, T does not change convexity on the points of P. Let $P'' = P' \setminus \{x'\}$. By the assumption, P'' does not contain any $n-1$-cap or n-cup.

Let A be the set of those points of P'' which are right endpoints of some $n-2$-cap, and let $B = P'' \setminus A$. If $|A| > \binom{2n-6}{n-3}$ then A contains either an $n-1$-cap or an $n-1$-cup. The first case contradicts the assumption, in the second case we have an $n-1$-cup whose left endpoint is also a right endpoint of some $n-2$-cap, so, either the $n-1$-cup or the $n-2$-cap can be extended by one point and we get a contradiction. So, $|A| \leq \binom{2n-6}{n-3}$. If $|B| > \binom{2n-6}{n-2}$, then B contains either an $n-2$-cap or an n-cup. The first case contradicts the definition of A, since we find a right endpoint of some $n-2$-cap in B, the second case contradicts the assumption. So $|B| \leq \binom{2n-6}{n-2}$. But then $|P''| = |A| + |B| \leq \binom{2n-6}{n-3} + \binom{2n-6}{n-2} = \binom{2n-5}{n-2} = |P''|$, therefore, $|A| = \binom{2n-6}{n-3}$ and $|B| = \binom{2n-6}{n-2}$.

Let $b \in B$. The set $\{b\} \cup A$ has size $\binom{2n-6}{n-3} + 1$ so again it contains either

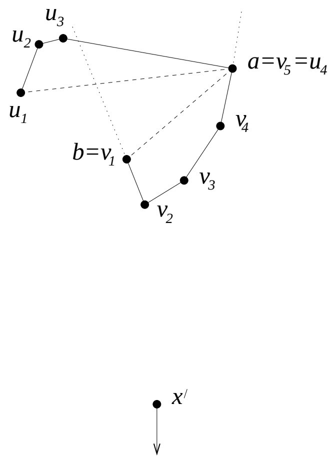

Figure 5: $x', u_1, u_2, u_3, v_1, v_2$ *determine a convex hexagon.*

an $n-1$-cap or an $n-1$-cup. In the case of $n-1$-cap we are done, so we can assume that it is an $n-1$-cup for any choice of b. If the left endpoint of this $n-1$-cup is an element of A, we have an $n-1$-cup whose left endpoint is also a right endpoint of some $n-2$-cap, so, either the $n-1$-cup or the $n-2$-cap can be extended by one point and we get a contradiction again. Hence the left endpoint of the $n-1$-cup is b. Therefore, any $b \in B$ is the left endpoint of an $n-1$-cup whose right endpoint is in A. We can argue analogously, considering the sets $\{a\} \cup B$, that any $a \in A$ is the right endpoint of an $n-2$-cap whose left endpoint is in B.

Let S be the set of all segments ab, where $a \in A$, $b \in B$, and there is either an $n-1$-cup or $n-2$-cap whose right endpoint is a and left endpoint is b. Let ab be the element of S with the largest slope. Suppose that ab represents an $n-1$-cup. The argument in the other case is analogous. Let $b = v_1, v_2, \ldots, v_{n-1} = a$ be the points of the $n-1$-cup from left to right. We know that there is also an $n-2$-cap whose right endpoint is a and left endpoint in B. Let $u_1, u_2, \ldots, u_{n-2} = a$ be its points from left to right. If u_{n-3} lies above the line v_1v_2, then $u_j, v_1, v_2, \ldots, v_{n-1}$ determine a convex n-gon and we are done. Otherwise $x', u_1, u_2, \ldots, u_{n-3}, v_1, v_2$ determine a convex n-gon, see Fig. 5. This concludes the proof. $\square$

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