Eulerian tours in graphs with forbidden transitions and bounded degree

Zdeněk Dvořák
rakdver@kam.ms.mff.cuni.cz

Abstract

We study a problem of finding an Eulerian tour in a graph with forbidden transitions. This problem was proved to be NP-complete by [MN02], but their construction requires the degrees of the graph to be unbounded. We prove that the problem is NP-complete even for graphs with degree at most four. We also investigate some cases in that the problem can be solved in a polynomial time.

Introduction

We consider a complexity of a problem of finding an Eulerian tour in graphs with forbidden transitions. It is easy to prove that the problem is NP-complete both in oriented and unoriented graphs ([MN02]), but the reduction requires the degrees of vertices to be unbounded. We investigate the complexity of the question for graphs with bounded degree. We prove that the problem is NP-complete for oriented graphs with indegree and outdegree bounded by four (since the indegrees and outdegrees of all vertices must be the same if the problem is to be interesting, we call this value simply "degree"), and that it is NP-complete for unoriented graphs with degree bounded by 8.

On the other hand, we show that the problem is polynomial for $\Delta \leq 2$ in the oriented case and $\Delta \leq 4$ in the unoriented one. This still leaves the gap where we are so far unable to decide the complexity – the cases of degrees at most 3 (6, respectively).

The problem is formulated as follows. We are given a graph G together with graphs of forbidden transitions assigned to each of its vertices. We identify the vertices of a graph of forbidden transitions assigned to v with the halfedges incident to v. We want to find a closed Eulerian tour of G such that for every two edges e_1 and e_2 consecutive in it, e_1e_2 is not an edge in the graph of forbidden transitions assigned to the vertex through that it passes. Since most of the graphs we are going to consider have only few transitions allowed, we draw their complements (graphs of allowed transitions) instead.

We allow all graphs in our constructions to contain loops and parallel edges. Note that it does not change the complexity of a question, as we may simply split each edge with two vertices with indegree and outdegree one (or degree two in the unoriented case) and the transitions allowed. The resulting graph is simple and it has an allowed Eulerian tour if and only if the original graph had one.

1 NP-completeness

Since the problems are obviously in NP, the interesting part is to show their NP-hardness. It is easy to see that it suffices to prove NP-completeness of the oriented version of the problem:

Theorem 1 *If the problem of finding an Eulerian tour in an oriented graph with indegree and outdegree bounded by d is NP-complete, the problem of finding an Eulerian tour in an unoriented graph with degree bounded by $2d$ is also NP-complete.*

Proof Suppose we are given an instance G of the former problem. We construct an instance G' of the latter problem as follows: We cancel the orientation of edges and for each vertex v we add the forbidden transitions between any two incoming edges, as well as between any two outgoing ones. Clearly this can be done in a polynomial time and the allowed Eulerian tours in G and G' are composed of the corresponding edges. ∎

Note that the Eulerian tour can be described by assigning each vertex a perfect matching in its graph of allowed transitions. The corresponding Eulerian tour can be reconstructed by starting at arbitrary edge and repeatedly proceeding to the edge that is connected to the current edge in the matching assigned to its end vertex. Of course not every assignment of matchings corresponds to an Eulerian tour, as the resulting tour does not have to be connected.

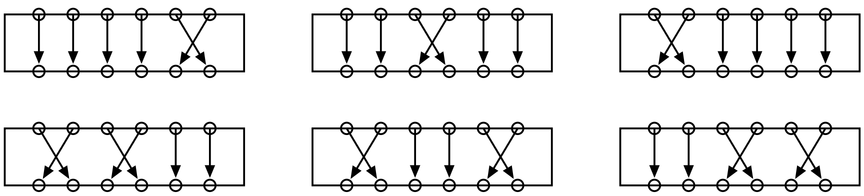


Figure 1: NAE_3 gadget

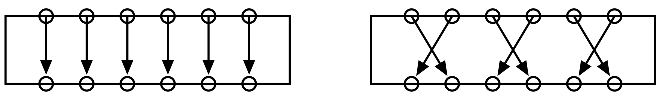


Figure 2: $COPY_3$ gadget

In the construction in the proof of NP-hardness we use "gadgets" defined as follows. Each gadget is a graph with some k inputs (free incoming half-edges) and k outputs (free outgoing half-edges). For gadget X they are denoted by $i_X^1, \ldots, i_X^k$ and $o_X^1, \ldots, o_X^k$. Each gadget is characterized by the set of allowed matchings between inputs and outputs – members of the set are the matchings $\pi : [k] \to [k]$ such that there are k mutually edge-disjoint tours, j-th of them leading from i_X^j to $o_X^{\pi(j)}$, such that they cover all edges in the gadget X. In other words, the gadget behaves as a vertex at that we are not given allowed transitions, but directly allowed matchings. We specify the matchings as the corresponding permutations.

For the proof of the theorem we need the following gadgets:

- NAE_3 (figure 1) has degree 6 and the following allowed matchings: 213456, 124356, 123465, 214356, 213465 and 124365. I.e. of the three pairs not all, but at least one, is crossed.
- $COPY_n$ (figure 2) has degree $2n$. It has two allowed matchings, $1234\ldots(2n-1)(2n)$ and $2143\ldots(2n)(2n-1)$. I.e. either all pairs are crossed or none is.

Theorem 2 *Suppose that NAE_3 and $COPY_n$ gadgets consisting of vertices of degree at most four exist and $COPY_n$ gadget can be constructed in a time*

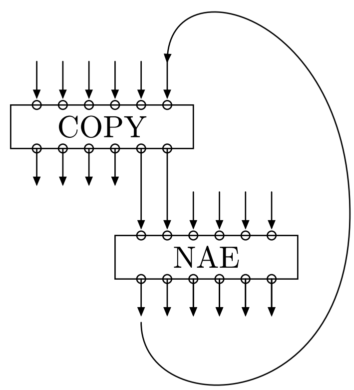


Figure 3: Linkage for one occurrence

polynomial in n. Then the problem of finding an Eulerian tour in a graph with forbidden transitions and degrees bounded by four is NP-complete.

Proof We describe a reduction from the problem $3-NAE-SAT$ without negations that is NP-complete due to [GJ79]. The instance of the problem is a formula with clauses of size exactly 3 and all variable occurrences positive. The question is whether there exists an assignment such that both it and its negation satisfies the formula, it is iff it assigns true to at least one but not all variables in every clause.

Given such an instance, we construct the instance of our problem in the following manner. For each variable x we add a $COPY_n^x$ gadget, where n is the number of occurrences of the variable. For each clause c we add a NAE_3^c gadget. The idea behind their linkage is this (see figure 3): For each variable occurrence we have a pair of edges that is first lead to the copy gadget for the variable and then through NAE_3 gadget. The variable is true if the matching in the copy gadget is the crossing one, we therefore just need to ensure that it is only legal to come out in non-crossed state (either because there was no crossing or two crossings, i.e. the value of occurence is copied into the NAE_3 gadget). This is done by connecting one of the exits to the entry of the other one – if the pair would come out in the crossed state, this link would form a cycle and the tour could not be connected. Then we link the remaining entries and exits of the pairs so that they form a single cycle, therefore if there is no crossing, they indeed form an Eulerian tour.

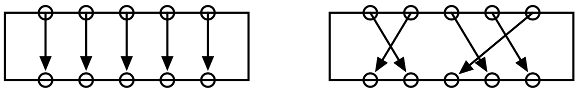
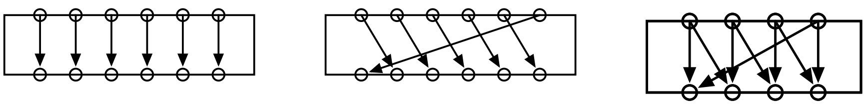


Figure 4: $INCMOD3$ gadget



(a) (b)

Figure 5: $SHIFT_n$ gadget

More precisely, let x_j be the j-th occurrence (the numbering is arbitrary) of variable x. Mark $i^{2j-1}_{COPY^x_n}$ and $i^{2j}_{COPY^x_n}$ by $i^{x_j}_1$ and $i^{x_j}_2$. Connect $o^{2j-1}_{COPY^x_n}$ to $i^{2k-1}_{NAE^c_3}$ and $o^{2j}_{COPY^x_n}$ to $i^{2k}_{NAE^c_3}$, where c is the clause where the j-th occurrence of x occurs on the k-th place, and mark $o^{2k-1}_{NAE^c_3}$ and $o^{2k}_{NAE^c_3}$ by $o^{x_j}_1$ and $o^{x_j}_2$. Connect $o^{x_j}_1$ to $i^{x_j}_2$. Then order the variable occurrences in a cyclical ordering $y_0, y_1, \ldots, y_{m-1}$ and for each l, connect $o^{y_l}_2$ to $i^{y_{l+1}}_1$, where $l+1$ is computed modulo m.

According to the idea above, there is an allowed Eulerian tour in this graph if and only if there is a valid assignment of variables for the $3-NAE-SAT$ instance. Since the $COPY_n$ gadget can be constructed in polynomial time, the whole reduction is polynomial. ∎

2 Gadgets

Now we just need to construct the mentioned gadgets. We construct a few auxiliary gadgets first, concretely:

- $INCMOD3$ gadget (figure 4) has degree 5 and allowed matchings 12345 and 21453.
- $SHIFT_n$ gadget (figure 5 (a)) has degree n and allowed matchings $1234\ldots(n-1)n$ and $2345\ldots n1$. Note that $SHIFT_4$ gadget is simply

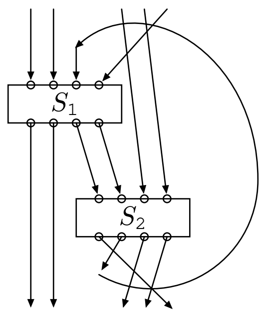


Figure 6: $SHIFT_5$ gadget

the vertex of degree four with allowed edges as drawn in figure 5 (b).

Lemma 3 *There exists a $SHIFT_5$ gadget with degrees of vertices bounded by four.*

Proof See figure 6. Take two $SHIFT_4$ gadgets S_1 and S_2. Join $o^3_{S_1}$ to $i^1_{S_2}$, $o^4_{S_1}$ to $i^2_{S_2}$ and $o^2_{S_2}$ to $i^3_{S_1}$. Let $i^1_{SHIFT_5}$ be $i^1_{S_1}$, $i^2_{SHIFT_5}$ be $i^2_{S_1}$, $i^3_{SHIFT_5}$ be $i^3_{S_2}$, $i^4_{SHIFT_5}$ be $i^4_{S_2}$, $i^5_{SHIFT_5}$ be $i^3_{S_1}$, $o^1_{SHIFT_5}$ be $o^1_{S_1}$, $o^2_{SHIFT_5}$ be $o^2_{S_1}$, $o^3_{SHIFT_5}$ be $o^3_{S_2}$, $o^4_{SHIFT_5}$ be $o^4_{S_2}$ and $o^5_{SHIFT_5}$ be $o^1_{S_2}$.

Due to the linkage, matching in S_1 is identity iff the matching in S_2 is, as otherwise there is a cycle disconnected from the rest of the tour. This leaves two matchings that are the matchings of the $SHIFT_5$ gadget, as we may easily verify. ∎

For sake of briefness the similar simple deductions as well as detailed descriptions of the connections that is clear from the picture is left out in the following lemmae.

Lemma 4 *There exists a $COPY_2$ gadget with degrees of vertices bounded by four.*

Proof Take two $SHIFT_5$ gadgets and connect them as in figure 7. ∎

Lemma 5 *There exists a $INCMOD3$ gadget with degrees of vertices bounded by four.*

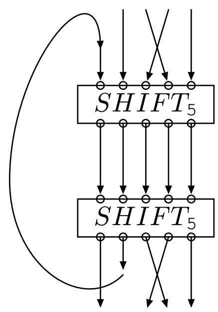


Figure 7: $COPY_2$ gadget

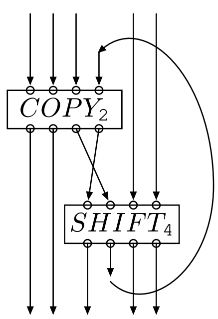


Figure 8: $INCMOD3$ gadget

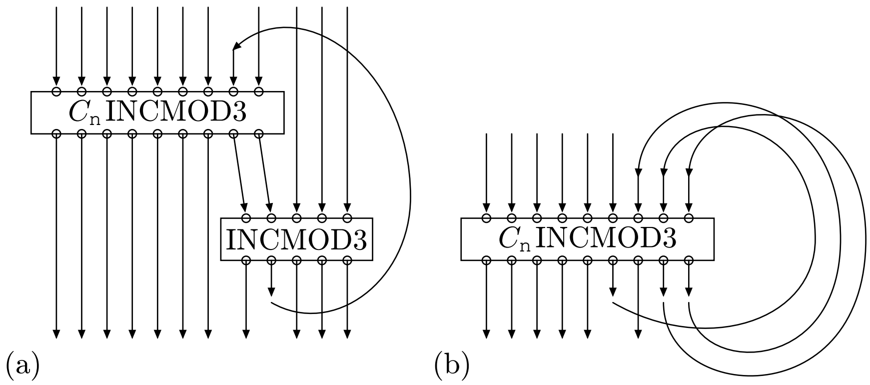


Figure 9: $COPY_n$ gadget

Proof Take $COPY_2$ and $SHIFT_4$ gadgets and connect them as in figure 8. ∎

Lemma 6 *There exists a $COPY_n$ gadget with degrees of vertices bounded by four that can be created in polynomial time.*

Proof We first construct an auxiliary gadget $C_nINCMOD3$ with degree $2n+3$ and allowed matchings $1234\ldots(2n-1)(2n)(2n+1)(2n+2)(2n+3)$ and $2143\ldots(2n)(2n-1)(2n+2)(2n+3)(2n+1)$. $C_1INCMOD3$ is just $INCMOD3$ gadget, the rest is done by induction – we create $C_{n+1}INCMOD3$ gadget by linking $C_nINCMOD3$ gadget with $INCMOD3$ gadget as in figure 9 (a).

$COPY_n$ gadget is then constructed by eliminating the $INCMOD3$ part from $C_nINCMOD3$ as in figure 9 (b). ∎

Lemma 7 *There exists a NAE_3 gadget with degrees of vertices bounded by four.*

Proof Take three $INCMOD3$ gadgets and link them as in figure 10. ∎

This together with theorem 2 gives us the main result:

Corollary 8 *The problem of finding an Eulerian tour in an oriented graph with forbidden transitions and degrees of vertices bounded by 4 is NP-complete.*

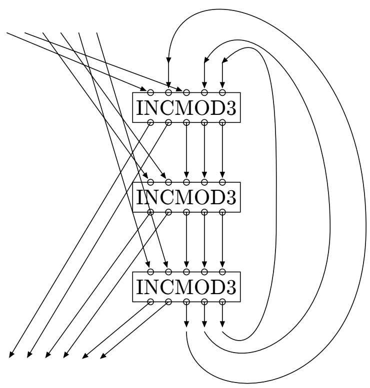


Figure 10: NAE_3 gadget

3 Polynomial-time algorithms

On the other hand, in some cases we may prove that the problem is polynomial. First there is a closure property analogical to Bondy – Chvátal closure for Hamiltonian cycle ([BC76]):

Theorem 9 *Let G be an unoriented graph with forbidden transitions and denote by H_v a graph of allowed transitions associated with v. Let G' be a graph obtained from G by repeating the following operation while it is possible:*

If e_1 and e_2 are two non-adjacent vertices of H_v and sum of their degrees is at least $\frac{3}{2}\deg v - 2$, add an allowed transition between e_1 and e_2.

Then G has an Eulerian tour respecting the forbidden transitions if and only if G' has one.

Proof The implication from left to right is obvious. For the other implication suppose that G' has such an Eulerian tour. It clearly suffices to show that it holds when just one step of the closure is performed. Let v be the vertex at that it is done, e_1 and e_2 the vertices of H_v between that the allowed transition was added.

The Eulerian tour of G' is split into parts by vertex v, these parts are connected together through a matching in H'_v. If this matching does not contain the edge e_1e_2, we are done. Otherwise consider a graph obtained from H'_v this way: For each pair of vertices e' and e'' that is connected by a part of the Eulerian tour, we add a vertex e''' and connect it both to e' and to e''. The Eulerian tour then corresponds to the Hamiltonian cycle in this graph. We see that removing an edge e_1e_2 is a reverse to Bondy – Chvátal closure ([BC76]) operation, so the graph after the removal has a Hamiltonian cycle as well. By replacing the added vertices by the corresponding parts of the Eulerian tour we then obtain an Eulerian tour in G. ∎

Using this theorem, we easily get

Theorem 10 *Let G be an unoriented graph with forbidden transitions and degrees of vertices bounded by four. Then the problem of finding an Eulerian tour respecting the transitions can be solved in polynomial time.*

Proof First we eliminate the vertices of degree 2 – if their transition is forbidden, the Eulerian tour does not exist, otherwise we may contract them.

Only the vertices of degree 4 remain. If there is a vertex v such that H_v contains vertex of degree zero, the Eulerian tour does not exist. If it contains a vertex of degree one, this transition is forced and we may split the vertex into two vertices of degree 2 and eliminate them as described before. So we may assume that degrees of all vertices in any H_v are at least 2. Then we may apply the previous theorem; since $2 + 2 \geq \frac{3}{2}4 - 2$, we may add all the transitions and thus reduce the problem to finding ordinary Eulerian tour. All of this can be done in time polynomial in size of G – in fact in a linear time. ∎

Similarly we may proceed for oriented graphs. Unlike the oriented case, we do not need the closure property for proof of polynomiality and we were also unable to prove it. For the sake of completeness let us state a bit simpler result:

Theorem 11 *Let G be an Eulerian oriented graph with forbidden transitions, such that if H_v is an oriented graph of allowed transitions for vertex v, then $\delta^+(H_v) + \delta^-(H_v) \geq \deg v + 2$. Then G has an Eulerian tour respecting the allowed transitions.*

Proof Because of the degrees every H_v has a perfect matching: Choose any matching between vertices of H_v (it does not have to consist of edges), merge the corresponding vertices and remove the loops. We end up with an oriented graph with $\delta^+ + \delta^- \geq \deg v$, $\deg v$ is a number of its vertices. Thus due to [W72], it contains an oriented Hamiltonian cycle, that after splitting the vertices back becomes a perfect matching.

Choose a perfect matching at each vertex. This determines some tour in G but this tour does not necessarily have to be connected. We pass over the vertices and "glue" the disconnected parts together: suppose we are processing vertex v. It splits our current tour into parts that connect pairs of edges coming in and out of v. This determines a matching between vertices of H_v. We merge the matched vertices and use again [W72] to obtain an oriented Hamiltonian cycle in this graph. After splitting the vertices again, we obtain a matching in H_v that makes the parts of the tour that passes through v to be connected. We replace the tour with this new one and proceed with other vertices. Each such replacement decreases the number of components of the tour, so we eventually end up with the Eulerian tour. ∎

Theorem 12 *Let G be an oriented graph with forbidden transitions and degrees of vertices bounded by four. Then the problem of finding an Eulerian tour respecting the transitions can be solved in polynomial time.*

Proof If there is some vertex in some H_v of degree 0, there is no Eulerian tour. If it has degree one, the transition is forced to be in the Eulerian tour we seek, so we may replace it with edge and decrease the size of the vertex. Otherwise all H_v are complete bipartite graphs which reduces the problem to the one for ordinary graphs. That is solvable in linear time. ∎

4 Conclusion

As already noted, whether the problem is polynomial or NP-complete for degrees bounded by 6 in unoriented and 3 in oriented case is still unsure. Note that all vertices in the construction for degrees bounded by four are of the same type – $SHIFT_4$. It would therefore suffice to somehow simulate this gadget, but we were unable to do it so far.

It might also be interesting to investigate the boundary between NP-complete and polynomial complexity in cases when all vertices have assigned the same graph of allowed transitions. $SHIFT_4$ and many other then fall to NP-complete side, whereas for all graphs with sufficiently large minimal degree it is polynomial due to theorem 11, but the exact characterization would be interesting.

References

[BC76] J.A.Bondy, V. Chvátal: A method in graph theory. *Discrete Math* **15** (1976), 111–135

[GJ79] M. R. Garey and D. S. Johnson. Computers and Intractability: A Guide to the Theory of NP-Completeness. W.H. Freeman, New York, 1979.

[MN02] J. Maxova, J. Nesetril: Complexity of Compatible Decomposition of Eulerian Graphs and Their Transformations. *Lecture Notes In Computer Science//0.415* **2461** (2002), 711–722

[W72] D.R.Woodall: Sufficient conditions for cycles in digraphs. *Proc. London Math. Soc* **24** (1972), 739–755.