On the new reformulation of Hadwiger's conjecture

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Abstract

Assuming that every minor closed class of graphs contains a maximum with respect to the homomorphism order, we prove that such a maximum must be homomorphically equivalent to a complete graph. This proves that Hadwiger's conjecture is equivalent to say that every minor closed class of graphs contains a maximum with respect to homomorphism order. This later conjecture one is one of two conjectures introduced by J. Nešetřil and P. O. De Mendez in an effort to refine Hadwiger's conjecture. Let $\mathcal{F}$ be a finite set of 2-connected graphs, and let $\mathcal{C}$ be the class of graph with no minor from $\mathcal{F}$. We prove that if $\mathcal{C}$ has a maximum, then any maximum of $\mathcal{C}$ must be homomorphically equivalent to a complete graph. Finally we introduce yet another reformulation of the Hadwiger's conjecture.

1 Introduction

Given two graphs G and H we say H is a minor of G if H can be obtained from G by a series of operations: contracting edges, deleting vertices and deleting edges. Minors are one of the important concepts in graph theory. One of the first theorems in the theory of graph minors is Kuratowski's well known theorem which says a graph G is planar if and only if it does not contain K_5 or $K_{3,3}$ as a minor.

In view of Kuratowski's theorem, H. Hadwiger strengthened the four colour conjecture, to state that every 5-chromatic graph contains a minor of K_5. (This would imply that it can not be planar). He then generalized this as follows:

Hadwiger's conjecture [2] Every k-chromatic graph G contains the complete graph K_k as a minor.

This conjecture is almost trivial for $k = 1, k = 2$ and $k = 3$. For $k = 4$ it is proved by G. A. Dirac in [1]. For $k = 5$, as we mentioned, it is stronger than the four colour theorem. However it was proved by K. Wagner that the case $k = 5$ is actually equivalent to the four colour conjecture [11]. For $k = 6$ it also has been proved by N. Robertson, P. D. Seymour and R. Thomas in [10] to be equivalent to the four colour theorem. It remains open for $k \geq 7$, and has been a fruitful research area. A remarkable partial result is the following theorem of W. Mader:

Theorem 1 [6] *For every positive integer k there exists an integer $h(k)$ such that if the minimum degree of a graph G is at least $h(k)$, then G contains K_k as a minor.*

A graph G is said to be k-degenerate if every subgraph of G contains a vertex of degree at most k. It is a well know theorem of Szekeres and Wilf that the every k-degenerate graph is $(k + 1)$-colorable. Using this theorem one can see that Mader's theorem implies the following theorem, originally introduced by Wagner.

Lemma 2 [11] *For every proper minor closed class $\mathcal{C}$ of graphs, there is an integer k, such that each graph in $\mathcal{C}$ is k-colorable.*

We say a class $\mathcal{C}$ of graphs is *minor closed* if for every graph G in $\mathcal{C}$ and every minor H of G, H is also in C. A minor closed class of graphs which consists of only a graph H and all of its minors is called *principle ideal*, and will be denoted by $[H]$. We say $\mathcal{C}$ is a proper minor closed class of graphs if it is not the class of all graphs. Obviously a minor closed class is proper if and only if it does not contain all complete graphs. By Mader's theorem every proper minor closed class $\mathcal{C}$ of graphs have bounded minimum degree. But since $\mathcal{C}$ is closed under taking subgraphs, every graph in G must have bounded degeneracy number and therefore bounded chromatic number. This type of properties of a minor closed class of graph was studied in [8]. The same authors have also reformulated the Hadwiger's conjecture in [9] using the notation of graph homomorphisms, and homomorphism order.

Given two graphs G and H, a *homomorphism* of G to H is an edge preserving mapping $f : V(G) \to V(H)$, that is to say, for every edge xy of G, $f(x)f(y)$ is an edge of H. The existence of a homomorphism of G to H is denoted by $G \to H$. This notation captures the classical vertex coloring problem of graphs because a graph G is k-colorable if and only if it admits a homomorphism to the complete graph K_k.

Graphs G and H are said to be homomorphically equivalent provided that each admits a homomorphism to the other. If G and H are homomorphically equivalent then we write $G \sim H$. The smallest graph H to which G is homomorphically equivalent is called the *core* of G. It easy to see that core of a graph G must be a subgraph of G. It is also not hard to check that the core of a graph is unique up to isomorphism.

Using the notation of homomorphisms we can define an order on the class of graphs by

$$G \preceq H \quad \text{if and only if} \quad G \to H.$$

This homomorphism order which some times is also called *coloring order* is a quasi order on the class of all graphs. An arbitrary class of graphs, $\mathcal{C}$, is said to be *bounded* by a graph H if for every graph G in the class $\mathcal{C}$, $G \preceq H$. Moreover if H is also in $\mathcal{C}$ then H is called a *maximum* of $\mathcal{C}$.

Now using Mader's theorem one can see that Hadwiger's conjecture is equivalent to the conjunction of the following two conjectures. These two conjectures were introduce by J. Nešetřil and P. O. De Mendez in [9].

Conjecture 3 *Any bounded minor closed class of graphs has a maximum.*

Conjecture 4 *If a minor closed class of graphs, $\mathcal{C}$, admits a maximum then the core of the maximum is a complete graph.*

Here we prove that Conjecture 4 is implied by Conjecture 3, in other words we show that if every minor closed class of graphs contains a maximum then the maximum of every minor closed class of graphs must be homomorphically equivalent to a complete graph. Using this we show that Conjecture 3 is equivalent to the Hadwiger's conjecture. Nest we show that Conjecture 4 can be proved independently for a wide family of minor closed classes of graphs. More precisely we show that if $\mathcal{C}$ is a minor closed class of graphs consisting of all graph with no minor from $\mathcal{F}$ where $\mathcal{F}$ is a finite set of 2-connected graphs then every maximum of the class $\mathcal{C}$ (if exists) must be homomorphically equivalent to a complete graph.

2 On the maximum of a bounded minor closed class

The next theorem shows that Conjecture 4 is implied by Conjecture 3, in fact we prove a some what stronger statement, that a weaker form of Conjecture 3 implies Conjecture 4.

Theorem 5 *Suppose every principle ideal contains a maximum. Let $\mathcal{C}$ be any minor closed class of graphs with a maximum element H. Then H must be homomorphically equivalent to a complete graph.*

Proof. We will prove this by contradiction. Assume this is not true for some minor-closed families. Let $\mathcal{G}/K_k$ be the class of all graphs which do not contain K_k as a minor. By Lemma 2 any proper minor-closed family is contained in some $\mathcal{G}/K_k$. Let k be the smallest integer such that $\mathcal{G}/K_k$ contains a minor-closed subfamily, $\mathcal{C}$, for which the statement of the theorem does not hold. Note that k must be greater than or equal to 7 (because for the smaller values of k Hadwiger's conjecture has been verified).

Let H be a maximum of $\mathcal{C}$. Then the class formed by H and all of its minors is a finite minor-closed family of graphs for which the statement of the theorem also does not hold. Because of this finiteness we may assume $\mathcal{C}$ is a minimal subfamily of $\mathcal{G}/K_k$ with respect to having a maximum, H, which is not homomorphically equivalent to a complete graph. By the minimality, $\mathcal{C}$ must be formed only by the family of all the minors of H.

By the choice of k, we know K_{k-1} must also be in $\mathcal{C}$, otherwise $\mathcal{C} \subseteq \mathcal{G}/K_{k-1}$ and we are done. Since H is a maximum of $\mathcal{C}$, and K_{k-1} is an element of $\mathcal{C}$

we have $K_{k-1} \to H$. Thus K_{k-1} must be also a subgraph of H. Let K be the subgraph of H which is isomorphic to K_{k-1}.

We first claim that every vertex of K must be adjacent to a vertex of H which is not in K. To see this, suppose there is a vertex x of K which is only adjacent to the $k-2$ vertices of $V(K)\backslash x$. By the minimality of $\mathcal{C}$, the graph H_x obtained from H by deleting the vertex x must be $(k-1)$-colourable. Otherwise H_x with all of its minors form another minor closed family for which the maximum is not homomorphically equivalent to a complete graph. This family is properly contained in $\mathcal{C}$, which contradicts the minimality of $\mathcal{C}$. Since x is adjacent to $k-2$ vertices, any $k-1$ colouring of H_x can be extended to a $k-1$ colouring of H. This implies that H must be homomorphically equivalent to K_{k-1}, which is a contradiction.

Our next claim is that the induced subgraph H' of H on $V(H)\backslash V(K)$ is connected. Again by contradiction assume it has parts H'_1 and H'_2 with no edges from H'_1 to H'_2. Then by a similar argument as before each of the subgraphs induced on $V(H'_1) \cup V(K)$ and $V(H'_2) \cup V(K)$ must be $(k-1)$-colourable. But then just a permutation of colours will produce a $k-1$ colouring of H. Thus H must be homomorphically equivalent to K_{k-1}.

To complete the proof, note that because H' is connected, by contracting all the edges in H' we will obtain a single vertex which must be adjacent to all the vertices of K. Therefore K_k is a minor of H, but this contradicts the choice of $\mathcal{C}$ and H. $\square$

This theorem proves that the validity of Hadwiger's conjecture for all graphs is equivalent to the validity of Conjecture 3 for all minor-closed classes. For the sake of completeness we give a proof of this equivalence in the following theorem.

Theorem 6 *The following two statements are equivalent:*

(a) Every graph G with $\chi(G) = k$ contains K_k as a minor.

(b) Every proper minor-closed family of graphs contains a maximum with respect to homomorphism order.

Proof. Suppose (a) is true, and let $\mathcal{C}$ be a proper minor closed class of graphs. Then by Lemma 2 the chromatic number of the graphs in $\mathcal{C}$ is bounded. Let k be the maximum chromatic number of the graphs in $\mathcal{C}$ and let G be a graph in $\mathcal{C}$ with the chromatic number equal to k. Then by (a), G contains K_k as a minor, so K_k is in the class, and therefore C contains a maximum.

For the other side assume (b) is true, and let G be a graph with $\chi(G) = k$. Then $[G]$ has a maximum. By Theorem 5 such a maximum can be chosen to be a complete graph K_r. But since $G \to K_r$, $r \geq \chi(G)$ and therefore G contains K_k as a minor. $\square$

We believe that it should not be very difficult to prove Conjecture 4 independently. In fact we present a proof of this conjecture for a large family of minor closed classes. To prove this we will need a lemma on vertex transitive graphs. This lemma, which seems to be a folklore lemma, was formulated by P. Hell and the proof we are presenting was suggested by N. Robertson.

Lemma 7 *If G is a vertex transitive graph, then G does not contain a clique cut set.*

Proof. By contradiction, let K be a clique cut and B a component of the induced subgraph G_K on $V(G) \backslash V(K)$. Moreover assume B has the smallest size over all possible K.

Let B' another component of G_k, and x be a vertex in K which is adjacent to a vertex in B'. Let b be a vertex in B and t any arbitrary vertex in K. Since G is vertex transitive there is an automorphism φ of G which maps t to b. The image of K under φ is a clique K' containing b, therefore K' can not intersect any component of G_k other than B.

Let $G_{K'}$ be the subgraph induced on $V(G) \backslash V(K')$. We claim that B' is also a component of $G_{K'}$. In fact there are 3 possibilities for a component of $G_{K'}$; (1) it is a subgraph of B, (2) it contains $V(K) \backslash V(K')$, (3) it is also a component of G_K (connecting only to $V(K) \cap V(K')$). The first case can not happen because of the minimality of B. There is only one component of the second type, since GK and $G_{K'}$ have the same number of components every other component of G_k (i.e. a component other than B) must be a component of $G_{K'}$ as well. This proves our claim.

Since B' is a component of $G_{K'}$, the vertex x of K which is connected to a vertex in B must be a vertex of K' as well. This implies in particular that x

is adjacent to b, but since the choice of b was arbitrary, x must be adjacent to every vertex in B. This is a contradiction because x is adjacent to every possible neighbour of b (i.e., all the vertices in K and B) plus at least one more vertex in B'. $\square$

Corollary 8 *Let G be a connected vertex transitive graph which contains K_k as a proper subgraph, then it must contain K_{k+1} as a minor.*

Proof. Let K_k be a subgraph in G, since G is connected there must be an edge connecting a vertex in K_k to a vertex not in K_k, but then because G is vertex transitive every vertex of K_k must be adjacent to a vertex not in K_k. On the other hand by Lemma 7 the subgraph induced on $V(G)\backslash V(K_k)$ is connected, so if we contract all the edges in this subgraph we will obtain K_{k+1} as a minor of G. $\square$

Theorem 9 *Let $\mathcal{F}$ be a set of 2-connected graphs, and $\mathcal{C}$ be the class of all graphs with no minor in $\mathcal{F}$. If $\mathcal{C}$ contains a maximum then every maximum must be homomorphically equivalent to a complete graph.*

Proof. Let H be a maximum of $\mathcal{C}$ with the minimum number of vertices (therefore H is connected). It will be enough to prove that H is a complete graph, because then any other maximum must be homomorphically equivalent to H. We first claim H must be vertex transitive. Let H_1 and H_2 be two disjoint copies of H and let x and y be two distinct vertices of H. Form a new graph H' from H_1 and H_2 by identifying the copy of x in H_1 and copy of y in H_2.

Note that H' is also in $\mathcal{C}$, because it does not contain any member of $\mathcal{F}$ as a minor (otherwise if $F \in \mathcal{F}$ is a minor of H' then since F is a 2-connected graph it must be either a minor of H_1 or minor of H_2). Now since H a maximum of $\mathcal{C}$, there is a homomorphism $f: \quad H' \to H$. By the minimality of H, the restriction f_1 of f on H_1 (and similarly restriction f_2 of f on H_2,) is an isomorphism of H. Obviously $f_1(x) = f_2(y)$, so $f_1^{-1}f_2$ is an isomorphism of H which maps y to x.

Now let k be the size of the largest complete graph in $\mathcal{C}$. Since H is a maximum of $\mathcal{C}$, $K_k \to H$ and therefore H contains K_k as a subgraph. If H is not isomorphic to K_k then it contains K_k as a proper subgraph, but then by Corollary 8 H (and therefore $\mathcal{C}$) contains K_{k+1} as a minor which is a contradiction. $\square$

We should remark that this theorem in particular implies (without using the four colour theorem) that if the class $\mathcal{P}$ of planar graphs contains a maximum then K_4 is a maximum of $\mathcal{P}$. This is an old result of P. Hell and our methods in proving Theorem 9 is an extension of methods of [3].
We would like to end this article by introducing a yet new reformulation of the Hadwiger's conjecture.

Conjecture 10 *Let G be a graph and let H_1 and H_2 be two minors of G, then $\{H_1, H_2\}$ is bounded in $[G]$.*

It is clear that Hadwiger's conjecture implies Conjecture 10. To see that this conjecture also implies Hadwiger's conjecture, note that it assures the existence of a maximum for every principle ideal and therefore we can use Theorem 5.

Remark We have just been informed that Theorem 6, has been discovered independently (with a similar proof) by J. Nešetřil and P. O. De Mendez.

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