Minimal Universal and Dense Minor Closed Classes

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Abstract

We study the homomorphism (coloring) order induced on minor-closed classes. In [7], the minor-closed class $\mathcal{P}$ of directed paths is shown to be universal and in [8], $\mathcal{P}$ is shown to contain a dense subset. In this note we prove $\mathcal{P}$ is a unique minimal class of oriented graphs which is both universal and dense. Moreover, we show a dichotomy result for any minor-closed class $\mathcal{K}$ of directed trees: $\mathcal{K}$ is either universal or it is wqo. We also prove structure theorems about K_4-minor-free undirected graphs, in an attempt to determine the minimal universal and dense minor-closed classes of undirected graphs.

1 Introduction

In this paper we consider minor-closed classes of directed trees and finite series-parallel graphs (SPG). (Recall that a graph G is a series-parallel graph if and only if K_4 is not a minor of G.) We consider minor-closed subclasses of SPG and prove results from the point of view of universality and density of related partial order. We introduce some relevant notions first:

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Let G, G' be graphs. A *homomorphism* from G to G' is a mapping $f{:}V(G) \to V(G')$ which preserves adjacency, i.e., $f(u)f(v) \in E(G')$ whenever $uv \in E(G)$. We write $G \leq G'$ if there is a homomorphism from G to G'. The notation $G < G'$ means $G \leq G'$ and $G' \not\leq G$, whereas $G \sim G'$ means $G \leq G'$ and $G' \leq G$. If $G \sim G'$, we say G and G' are *hom-equivalent.* The smallest graph H for which $G \sim H$ is called the *core* of G. For finite graphs, the core is uniquely determined up to an isomorphism. It can also be seen that H is an induced subgraph of G, ($H \subseteq G$, induced subgraph inclusion will be denoted by '$\subseteq$'.)

We say a graph G is a *minor* of G', written $G \preceq G'$, if G can be obtained from G' by deleting and contracting edges of G'. The class of graphs that is closed under the minor inequality is called *a minor-closed class.* It is well known that graphs are well-quasi-ordered (wqo) under the minor relation $\preceq$, [10]. However, this is not true for the homomorphism relation $\leq$. On the contrary, it is known that even simple classes such as the class of all directed paths, or for the undirected case, a proper subclass of SPG induces, under $\leq$, a 'universal' partial order. A class $\mathcal{K}$ of finite graphs is said to induce a *universal* partial order, [1], [2], [3], if $\mathcal{K}$ together with the homomorphism order $\leq$ represents any countable partial order.

It is indeed surprising that such simple classes of graphs can be used to represent universality (or density). A related natural question arises: Does there exist yet simpler minor-closed class of graphs which is universal (or dense)? (i.e. could we find a proper minor-closed subclass or a minor-closed class not containing the class given in [6], [7] and still preserve universality and density?) In the next section we give a negative answer to these questions for directed trees.
In the consecutive sections, we study undirected K_4-minor-free graphs. In Section 3, we exhibit the class of graphs that forbids K_4 and the graph we call "3-pentagon" (see Figure 1) induces a linear order. Particularly, we show the core of every graph G forbidding the two graphs as a minor is its smallest odd-cycle. In Section 4, we study a class of graphs we call "ear-faces", that are basic structural elements of graphs in SPG.
We prove ear-faces do not form infinite anti-chain. By doing so, we are able to deduce that a universal minor closed class consisting of ear-faces does not exist.
In Section 5, we show every 2-connected graph in SPG is a finite 'ear-face recursion'. In Section 6 we present discussion and problems.

2 Unique minimal universal minor-closed class of directed trees

We study directed trees in this section. For any minor-closed class $\mathcal{K}$ of directed trees, a sufficient condition for universality is to contain all finite paths $\mathcal{P}$, [7]. $\mathcal{P}$ is also known to be dense by the result of [8]. We prove here that the condition $\mathcal{P} \subseteq \mathcal{K}$ is also necessary: If $\mathcal{P} \not\subseteq \mathcal{K}$, then $\mathcal{K}$ is neither universal nor it contains a dense subset. Moreover, we show if $\mathcal{K}$ forbids any path, then $\mathcal{K}$ is wqo by the homomorphism order. Such a dichotomy result perhaps does not hold for the case of graphs in general.
The main result of this section in Theorem 3 is proved using the following two easy (and perhaps well known) lemmas.

Lemma 1 *The set $\mathcal{T}_h$ of all directed trees of height at most $h \geq 0$ is wqo in the subtree order $\subseteq$.*

Proof. We prove a stronger result by assuming trees are rooted. Clearly, there is no infinite strictly descending chain of finite subtrees by subtree inclusion. We show there is no infinite anti-chain either. For $h = 0$, we only have a root vertex and the inclusion $\subseteq$ preserves root. We proceed by induction. Let $h > 0$ such that $X = \{T_i\}_{i=1}^{\infty}$ is an infinite anti-chain in $\mathcal{T}_h$. For each i, delete the root of T_i and let $F_i = [(T_1^i, x_1^i), (T_2^i, x_2^i), \ldots, (T_{k_i}^i, x_{k_i}^i)]$ be the forest of rooted trees T_j^i we obtain in $\mathcal{T}_{h-1}$ and $x_j^i \in \{in, out\}$ indicating whether an in or out edge connects the root of T_i with the root of T_j^i. By induction assumption and by Higman's lemma [5], there exist $i, j, i < j$, such that $F_i \subseteq^{<\omega} F_j$ (i.e. for some $i < j$, there is an injective function $f : \{1, 2, \ldots, k_i\} \to \{1, 2, \ldots, k_j\}$ such that for all $k = 1, 2, \ldots, k_i$, $T_k^i \subseteq T_{f(k)}^j$ and $x_k^i = x_{f(k)}^j$.) This implies $T_i \subseteq T_j$, a contradiction. □

Lemma 2 *The set $\mathcal{T}_h$ of all directed trees of height at most $h \geq 0$ contains no infinite strictly increasing sequence in the homomorphism order $\leq$.*

Proof. Assume the contrary and let $Y = \{T_i\}_{i=1}^{\infty}$ be an infinite strictly increasing sequence of core trees. let $F_i = [(T_1^i, x_1^i), (T_2^i, x_2^i), \ldots, (T_{k_i}^i, x_{k_i}^i)]$ as denoted in proof of Lemma 1. We may assume Y is monotone under $\subseteq$, by Lemma 1. Hence, in particular for all i, j, $x_k^i = x_k^j$, $1 \leq k \leq k_i$. Moreover by induction, there is no infinite strictly increasing sequence under $\leq$ in

$\{T_i'\}_{i=1}^{\infty}$, where $T_i' \in F_i$. Hence, for all $i \geq 1$ and $k = 1, 2, \ldots, k_i$, T_k^i is isomorphic to T_k^j, for all $j > i$, since T_j is a core. It follows that for all $j > 1$ and for $k > k_{(j-1)}$, $T_k^j \not\leq T_l^i$ or $x_k^j \neq x_l^i$ if $i < j, 1 \leq l \leq k_i$, for otherwise $T_k^j \leq T_l^j$ and $x_k^j = x_l^j$, and so T_j maps to it proper subtree, contrary to T_j being a core. Note that, by assumption $\{k_i\}_{i=1}^{\infty}$ is strictly increasing. Let $Z = \{T_{k_i}^i, x_{k_i}^i\}_{i=1}^{\infty}$. By Lemma 1, $\{T_{k_i}^i\} = \mathbb{T}$ can be seen monotone under $\subseteq$. Then $\mathbb{T}$ is also monotone under $\leq$. We may also choose a subsequence such that $x_{k_i}^i = x_{k_j}^j$ for all $i \neq j$. By induction, $\mathbb{T}$ can not be strictly increasing under $\leq$. Hence we find $i < j$ such that $T_{k_i}^i \sim T_{k_j}^j$ (i.e. $T_{k_j}^j \leq T_{k_i}^j$) and $x_{k_i}^i = x_{k_j}^j$ (i.e. $x_{k_j}^j = x_{k_i}^j$), a contradiction. $\square$

We note that without the condition of bounded height, the sequence we obtain from the set of all finite paths with all edges directed forward is an infinite strictly increasing sequence under $\leq$ inclusion. It is also not hard to construct infinite anti-chains under $\subseteq$ and even under $\leq$ if we have unbounded height tree sequence.

We give a formal definition of a directed path and a 'somewhere dense' set before proving the next theorem. We say P is a *directed path* if P is a directed graph (V, E) with $V = \{v_0, v_1, \ldots, v_m\}$ where for every $i = 1, 2, \ldots, m$ either $v_{i-1}v_i \in E$ or $v_iv_{i-1} \in E$ (but not both), and there are no other edges. We say an edge $v_{i-1}v_i$ is *directed forward* and v_iv_{i-1} is *directed backward*. The *length* of P is the number of edges in P. For any directed tree T, let $height(T)$ denote the maximum length of a directed path-subtree in T.
A class of graphs $\mathcal{K}$ that does not consist of only one vertex, is said to be *somewhere dense* if there exists a set $\mathcal{S} \subseteq \mathcal{K}$ such that for a graph $G \in \mathcal{S}$ and for any $G' \in \mathcal{S}$, $G' < G$ there exists a $G'' \in \mathcal{S}$ such that $G' < G'' < G$. Note that by definition if $\mathcal{K}$ is somewhere dense, then it is infinite. If $\mathcal{K}$ is not somewhere dense we say it is *nowhere dense.*

Theorem 3 *Let $\mathcal{K}$ be a minor closed class of directed trees and let $\mathcal{P}$ be the set of all directed paths. Then, the following are equivalent:*
(i) $\mathcal{P} \subseteq \mathcal{K}$
(ii) $\mathcal{K}$ is universal
(iii) $\mathcal{K}$ is somewhere dense
(iv) $\mathcal{K}$ is not wqo

Proof. We note that $(i) \to (ii)$ and $(i) \to (iii)$ follow from [7] and [8],

respectively. $(ii) \to (iv)$ is obvious. Hence (i) implies (ii), (iii) and (iv). Conversely, assume $\mathcal{P} \not\subseteq \mathcal{K}$. Let P_r be a directed path of length $r \geq 0$ such that $P_r \notin \mathcal{K}$. We show each statement (ii), (iii) and (iv) fails.
To prove that $\mathcal{K}$ is not universal it suffices to show $\mathcal{K}$ contains no infinite anti-chain of directed trees under $\leq$. Suppose $X = \{T_i\}_{i=1}^{\infty}$, is an infinite anti-chain of directed trees. Since $T_1 \not\leq T_i$, for all $i > 1$, there exists a natural number M, $1 \leq M < height(T_1)$ such that any path-subtree P in T_i with all edges directed forward or all directed backward has length at most M. (For otherwise, for some i there is a path-subtree P of T_i with more than $height(T_1)$ one-direction edges, and we get $T_1 \leq P \leq T_i$, a contradiction.) Also there exists a natural number N, $1 \leq N < 2r$ such that, for all i, any path-subtree P in T_i has number of changes in direction of its edges (i.e. from forward to backward and vice versa) at most N times. Otherwise by contracting P at most r times, we obtain $P_r \preceq P \preceq T_i$, contrary to assumption that $P_r \notin \mathcal{K}$. Hence, for all i, we have $height(T_i) \leq MN$. By Lemma 1, $T_i \subseteq T_j$ for some $i < j$. This implies $T_i \leq T_j$, a contradiction. Hence $\mathcal{K}$ is not universal.

Next, we show the stronger implication that $\mathcal{K}$ is wqo, by showing there is no infinite strictly decreasing sequence either. Indeed, if we have an infinite strictly descending chain $Y = \{T_i\}_{i=1}^{\infty}$, then we deduce Y is a bounded height tree sequence because $T_1 \not\leq T_i$ for all $i > 1$ and $P_r \notin \mathcal{K}$. By Lemma 1, Y is wqo by $\subseteq$. Hence, we find $i < j$, such that $T_i \sim T_j$, a contradiction.

To see that $\mathcal{K}$ is nowhere dense, assume the contrary and for some $\mathcal{S} \subseteq \mathcal{K}$ pick $T, T' \in \mathcal{S}$, where $T' < T$. Then we can find an infinite chain $T' < T_1 < T_2 < \ldots < T$. Since $T \not\leq T_i$ for all $i > 1$ and since $P_r \notin \mathcal{K}$, we deduce similarly we have a bounded height tree sequence, contrary to Lemma 2. $\square$

3 A total order by forbidding $3\mathbb{P}$ as a minor

For the remaining part of this paper, we examine undirected graphs of SPG. Several universal classes that are proper subclasses of SPG were found in [6] using K_1-concatenations of core graphs. The core-graphs are constructed by doubling the edges of K_3 and by subdividing the resulting graph a finite number of times. There is exactly one minor-minimal core-graph which we call *3-pentagon* (see Figure 1) that is constructed by this procedure. Since we are interested in finding the minimal universal minor-closed classes, we

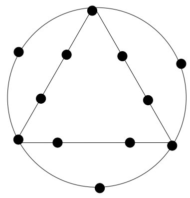

Figure 1: The 3-Pentagon

consider in particular this unique minor-minimal graph. Interestingly any minor-closed class in SPG that forbids it, has a linear order by homomorphism. We prove this fact in this section. The *3-pentagon*, denoted by $3\mathbb{P}$, is defined below.

Hence any subclass of SPG forbidding $3\mathbb{P}$ is non-universal. However, we can use clique-concatenations of $3\mathbb{P}$, to construct a universal class. In Section 6 we describe in greater detail how a universal class of undirected graphs is obtained by clique concatenations of core-graphs such us $3\mathbb{P}$. One may ask that, if $3\mathbb{P}$ is a unique minor-minimal core-graph that attains universality, why can not we obtain a unique minimal universal class of SPG, as in the case of directed trees. The reason we do not have unique minimal universal class is due to the fact that there is more than one way of concatenating $3\mathbb{P}$ and each one of them give us a universal class which is incomparable to the others by the subset order. Therefore we know we have several minimal universal classes.

Are there minimal universal classes other than classes obtained by clique-concatenation? The result of this and the next sections are basic structural results that can be used as tools to answer this question.
For convenience, we introduce a few more definition and notation. Let $\mathcal{K}$ be a minor closed class of graphs, and let F be a finite set of graphs. We write $\mathcal{K}$/F to denote the set of all graphs in $\mathcal{K}$ not containing any element of F as a minor. We denote the set of all graphs by $\mathcal{G}$. The following is a restatement of a well known fact that all SPG graphs are 3-colorable.

Theorem 4 *If* $G \in \mathcal{G}/\{K_4\}$*, then* $G \leq K_3$

Thus, any graph in SPG that is not K_3-free is hom-equivalent to K_3. If C_k is

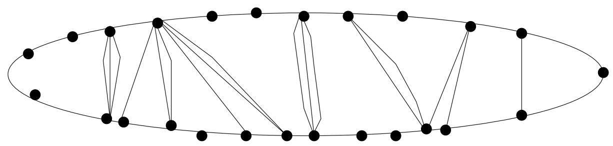

Figure 2: A Hamiltonian Outer-planar graph is a ladder

a cycle of length $k \geq 2$, we use a notation $2C_k$, to denote the graph obtained from C_k by doubling at most k edges of C_k. We will first prove $SPG/\{2C_3\}$ is totally ordered. Later we will show the same holds for $SPG/\{3\mathbb{P}\}$ which is a superset of SPG/$\{2C_3\}$.

Call a graph L a *ladder* if L is Hamiltonian and outer-planar, as shown in Figure 2. Note that the set of outer-planar graphs is a subset of SPG. We call an edge that is not on the Hamilton cycle of a ladder *a step*. Let G be a 2-connected planar graph with an outer cycle C and a facial cycle C' in a planar embedding of G. If $E(C) \cap E(C') \neq \emptyset$ and if contracting all the edges of C' to a vertex results a 2-connected graph G', we call C' an *ear* of G. For a ladder L, if P is a path on the outer cycle of L and $e \in E(P)$, we call e a *foot* of L if both end vertices of P are adjacent to one step. For instance the ladder in Figure 2, has 6 foot edges .

We can start with a finite set of ladders and by identifying a foot of a ladder to a step of another ladder recursively, we can obtain a graph which has a structure of branching ladders. It is not hard to see that if a graph G forbids K_4 and $2C_3$, then G has a structure of a tree of ladders. Note that, in particular the proper subgraphs of such trees of ladders, which have no clique-cuts can also be constructed by first constructing the tree of ladders and then by deleting the K_2-cuts from the resulting graph.

Theorem 5 *If $G \in SPG/\{2C_3\}$ then G is hom-equivalent to its smallest odd cycle.*

Proof. Let $G \in SPG/\{2C_3\}$ be a minimal counterexample graph which is a core and that contains a smallest odd cycle $C = C_{2k+1}, k > 0$, as a proper subgraph. If G contains exactly one odd cycle C_{2k+1}, the result is trivial. Note also that, since G is minimal, G can not be separated by a

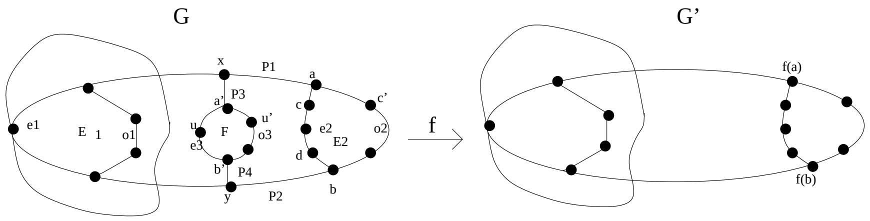

Figure 3: A homomorphism from G to G' contradicts the choice of G

clique, into smaller graphs G_1 and G_2. Otherwise the smallest odd cycle C belongs to either of the smaller graphs. Then by the choice of G each G_i maps to its smallest odd cycle C_i. By identifying the separation clique (i.e. K_1 or K_2), then G would map to the shorter of these odd cycles, which is C, a contradiction.

Since G can not be separated by K_2, G has no steps. Hence if G is a ladder then G is a cycle and we are done. Otherwise let us consider a planar embedding of G (see Figure 3).

Note that G contains no $2C_3$ as a minor and so G consists of at most two ears E_1 and E_2. We now prove a minimal counterexample G can not exist by reducing G to a graph $G' \in SPG/\{2C_3\}$ with less vertices, $G \leq G'$, and that has the same odd-girth. Then, we have $G \leq G' \leq C$ and by transitivity of $\leq$, we obtain a contradiction.

First we note that every odd cycle C' that contains the paths P_i depicted in Figure 3 is strictly longer than C, for otherwise some edges of E_2 or F can be identified with edges of C' to get G'. Also we have $|E_1| = |E_2| = |F| = |C|$. Otherwise we let G' be the graph obtained by shortening any of the longest of the three cycles and clearly $G \leq G'$, a contradiction.

Let $s = \Sigma_{i=1}^{4} |P_i|$. If $s = 0$, then clearly F and E_2 can be identified to one cycle. If $s = 1$, then the odd-girth constraint $e_2 + 1 + e_3 \geq o_3 + e_3$ (i.e. $e_2 + 1 \geq o_3$), implies we can identify an edge (if $e_2 + 1 > o_3$ from the e_2 edges of E_2, and if $e_2 + 1 = o_3$ from the o_3 edges of F) with the edge of P_i, where $|P_i| = 1$ and the remaining e_2 (or e_2-1) edges with o_3 (or $o_3 - 1$) edges of F to get G'. If $s = 2$, let $s' = |P_i| + |P_j|$, $(i \cong j)$ modulo 2. If

$s' = 2$ then we may identify a with a' (or b with b'). If $s' = 1$, then either $o_2 = o_3$, (then we identify a with u and b with v) or $|o_3 - o_2| \geq 2$, which also reduces to G' by identifying the longer one from o_2, o_3, with the two edges of P_i and the remaining edges with the shorter odd path length. The case $s = 3$ or 4 implies $s' = 2$, and so we are done. □

Theorem 6 *If $G \in SPG/\{3\mathbb{P}\}$ then G is hom-equivalent to its smallest odd cycle.*

Proof. Let $G \in SPG/\{3\mathbb{P}\}$ be a minimal counterexample which is a core. If the smallest odd cycle of G is C_3 or if $2C_3 \not\preceq G$, then the result follows from Theorem 4 or Theorem 5, respectively. Hence G is K_3-free and $2C_3 \preceq G$. By the same argument as in proof of Theorem 5, G is not separable by K_1 nor by K_2. Let $|C|$ be the odd-girth of G. Consider the planar embedding of G. By Theorem 5, we note that G has at least 3 odd-length ears. The set of minor minimal graphs with three ears and odd-girth of at least five are shown below in Figure 4. Only the first graph $3\mathbb{P}$ is without a K_2 separation. It is also easy to check that the $3\mathbb{P}$ is a core-graph.

□

Corollary 7 *Let $\mathcal{K} \subseteq SPG$ be a minor-closed class. If $\mathcal{K}$ forbids any minor of $3\mathbb{P}$, then $\mathcal{K}$ is totally ordered and hence non-universal.*

4 Anti-chains of ear-faces

We call a graph G, a *k-ear-face* if $k \geq 2$ and G is obtained by subdividing the edges of $2C_k$ a finite number of times. Recall that $2C_k$ is obtained by doubling k' edges of C_k, $1 \leq k' \leq k$. When we do not specify the number of ears we say G is an *ear-face*. If E is an ear of an ear-face G, we say the *even-side* and the *odd-side* of E when we refer to the even and odd-length paths in E, which have both end vertices having degree larger than two.

One reason we study ear-faces first is their simplicity. Ear-faces are the simplest form of graphs in SPG where we find incomparable graphs under homomorphism. A second and a fundamental reason is that, any 2-connected graph in SPG is in fact a 'recursive ear-face'. We give a formal description of this concept in the next section.

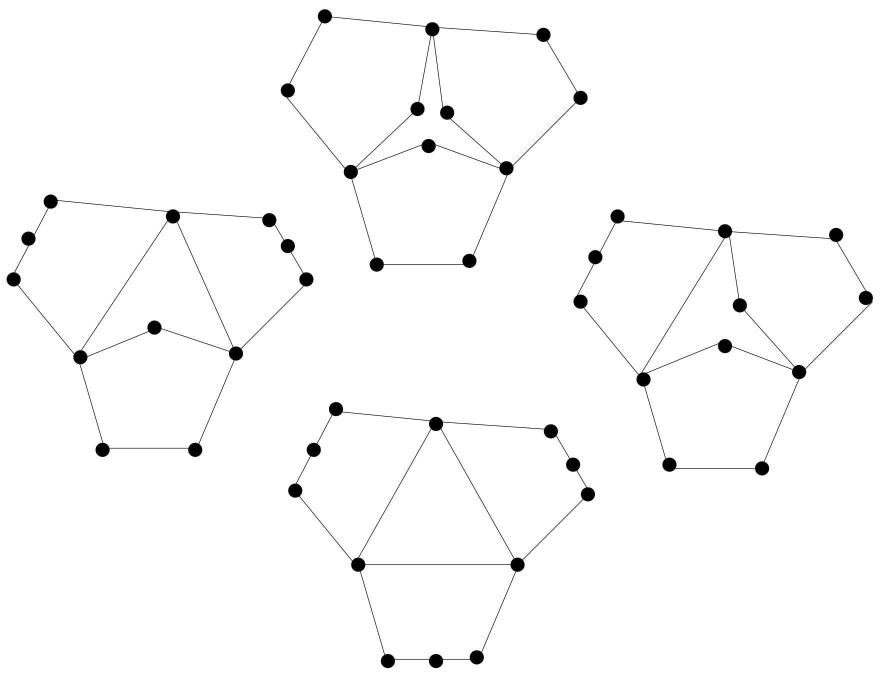

Figure 4: Only the 3-Pentagon is without K_2-separation

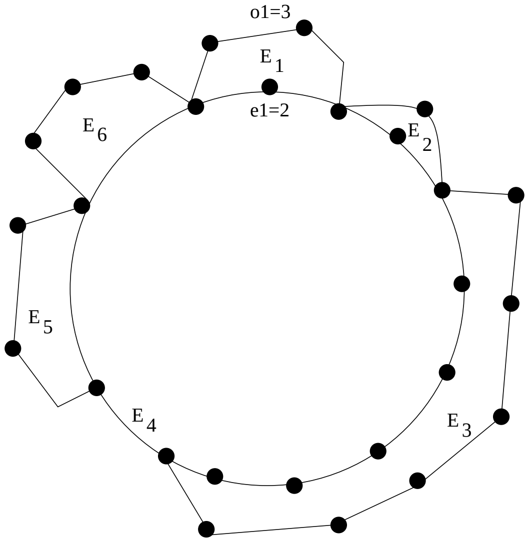

Figure 5: A 6-ear-face with five ears and one diminished-ear

We also investigate the algorithmic aspect of cores of ear-face graphs. Recognizing a core of an arbitrary graph is shown to be an NP-complete problem in [4]. For an ear-face G, however, we show there is a linear time algorithm deciding if G is a core or not.
An example of a 6-ear-face G is shown below in Figure 5. The ears E_1, E_3 and E_6 have even and odd-sides, whereas E_2, E_4 and E_5 do not. For instance deleting the length-3 path of E_5 we obtain a 5-ear-face $G' \sim G$. Hence the graph in Figure 5 is not a core. By definition, E_4 is not an ear. However, we note that E_4 can be replaced by double edges to construct an ear without losing hom-equivalence. Hence we define a path $P \subseteq G$ to be a *diminished-ear* of an ear-face G, if P meets exactly two distinct vertices of ears of G only by its two end vertices.

We can represent the graph in Figure 5, by six pairs of integers: ([2,3], [2,∞], [6,7], [∞,1], [∞,1], [4,1]). In general, for any $k \geq 2$, we may represent a k-ear-face G by $(E_1, E_2, \ldots, E_k)$, $E_i = [\epsilon_i, \delta_i], v_{i(i+1)} \in V(E_i) \cap V(E_{i+1}), 1, \leq i \leq k$ (where k+1=1), and ϵ_i is either the minimum even distance between $v_{(i-1)i}$ and $v_{i(i+1)}$ or ∞ (i.e. $\epsilon_i = \infty$ if E_i is a diminished-ear of odd length as in case of E_4 in Figure 5 or if E_i is an even cycle consisting of two odd-length parallel paths as in the case of E_5). The integer δ_i is defined similarly for odd length.

Lemma 8 *For any integer $N \geq 2$, and odd composite g, $g \neq 9$, there exist an anti-chain of length 2^N, consisting of odd-girth g ear-faces.*

Proof. Let $g = pq$ be an odd composite, $3 \leq p \leq q$. If $g \neq 9$, then $g - p \neq 2p$. Let $X = G_1, G_2, \ldots, G_{2^{N+1}}$, where for all i, the vector representation of G_i is $V_i = (E_1^i, E_2^i, \ldots, E_{N+2}^i)$, $E_1^i = [\infty, 1]$ and $E_j^i \in \{[g-p, p], [2p, g-2p]\}, j = 2, 3, \ldots, N+2$. Note that under any homomorphism a cycle is mapped to a cycle and so if $G_i \leq G_j$, then no edge of an ear of G_i is mapped to the edge of E_1^j and so the diminished-ear E_1^i must be mapped to the diminished-ear E_1^j. Each G_i and G_j are core $(N+2)$-ear-faces, and so the ears of G_i must injectively map to ears of G_j. Thus, either $V_i = V_j$ or the reverse order of V_j. Hence we have a $(2^{N+1})/2$ length anti-chain. □

Indeed, it is also possible to find an anti-chain of length α^N, for any constant α, if we have large enough g. Despite this result we show ear-faces form no infinite anti-chain. It is easy to see this if the number of ears is bounded, as the following lemma shows.

Lemma 9 *Let $X = G_1, G_2, \ldots$ be an infinite sequence of ear-faces such that, for all i, G_i is a k_i-ear-face, and $\{k_i\}_{i=1}^{\infty}$ is bounded. Then $G_i \leq G_j$, for some $i \neq j$.*

Proof. Let $V = \{V_i\}_{i=1}^{\infty}$, where $V_i = ([\epsilon_1^i, \delta_1^i], [\epsilon_2^i, \delta_2^i], \ldots, [\epsilon_k^i, \delta_{k_i}^i])$ is the vector representation of G_i. Since $\{k_i\}$ is bounded, by taking subsequence, we may assume $k = k_i$, for all i. Integers are wqo under the usual integer inequality and so is the finite Cartesian product of integers and the k-Cartesian product of the products, ordered by co-ordinatewise corresponding inequality. Hence for some i, j where $i < j$, we have $V_i \leq V_j$. This implies $G_j \leq G_i$, as needed. □

Non-existence of infinite anti-chain holds even if the ear-sequence is unbounded. We show first, in any anti-chain of ear-faces the odd-girth can be taken constant, using the next lemma.

Lemma 10 (Ear reduction lemma) *Let G be a k-ear-face of odd-girth $g > 3$ and $k > 3$. Then there exists a $(k-1)$-ear-face G' of same odd-girth such that $G \leq G'$.*

Proof. Let $G = ([\epsilon_1, \delta_1], [\epsilon_2, \delta_2], \ldots, [\epsilon_k, \delta_k])$ be a minimal counterexample k-ear-face of odd-girth g, with ears $E_i = [\epsilon_i, \delta_i], k \geq 4$. Then by the choice of G, we deduce G has no diminished ears and that $\epsilon_i + \delta_i = g$, for all i. Suppose that for some i_0, (without loss of generality, we may choose δ_{i_0} or ϵ_{i_0}) $\epsilon_{i_0} \leq \gamma_i$ for all i, where $\gamma_i \in \{\epsilon_i, \delta_i\}$ for $i = 1, 2, \ldots, k$. In addition we assume that among the minimal even lengths, ϵ_{i_0} is chosen so that either ϵ_{i_0+1} or ϵ_{i_0-1} is as large as possible. Say $i_0 = 1$, by relabeling if necessary. Taking the largest from $\delta_2, \epsilon_2, \delta_k$, and ϵ_k, we have by symmetry two cases:
case 1: δ_2 is largest. Then let $G' = ([\epsilon_2+\epsilon_1, \delta_2-\epsilon_1], [\epsilon_3, \delta_3], [\epsilon_4, \delta_4] \ldots, [\epsilon_k, \delta_k])$.
case 2: ϵ_2 is largest. Then let $G'' = ([\epsilon_2-\epsilon_1, \delta_2+\epsilon_1], [\epsilon_3, \delta_3], [\epsilon_4, \delta_4], \ldots, [\epsilon_k, \delta_k])$.
To see that $G \leq G'$, identify v_{k1} to the vertex $v \in V(E_2)$ that is of distance ϵ_1 from v_{12} and of distance $\delta_2 - \epsilon_1$ from v_{23}. To see that odd-girth(G')=g, we observer every ear of G' has length g and that (since $k \geq 4$ and $\epsilon_1 \leq \gamma_3$), $\delta_2 - \epsilon_1 + \gamma_3 + \gamma_4 + \ldots + \gamma_k \geq \delta_2 + \gamma_k \geq \bar{\gamma}_k + \gamma_k = g$. Similar proof works for G''. Note also by the choice of ϵ_1, if case 2 holds, then we have $\epsilon 2 - \epsilon_1 > 0$. Hence both G' and G'' are (k-1)-ear-faces with the desired property. $\square$

Corollary 11 *Let G be an ear-face of odd-girth $g > 3$. Then $G \leq C_{g'}$, where g' is odd and $g' \geq (g+1)/2$.*

Proof. By Ear-reduction lemma, we may assume $G = ([\epsilon_1, \delta_1], [\epsilon_2, \delta_2], [\epsilon_3, \delta_3])$. If $\epsilon_i \leq (g-1)/2$ for some i, say $i = 1$, then $\delta_1 \geq (g+1)/2$. Let G' be the graph obtained by identifying v_{k1} with v_{12}. Clearly, $G \leq G'$, and from the odd-girth constraint for G we have odd-girth(G') $= g' = min(\delta_1, \epsilon_2 + \delta_3, \delta_2 + \epsilon_3) \geq (g+1)/2$. On the other hand, suppose $\epsilon_i > (g-1)/2$, for all i. Say $\delta_1 \leq \delta_3$. We have $G \leq G'$ where $G' = ([\delta_1 + \delta_2, \epsilon_2 - \delta_1], [\epsilon_3, \delta_3])$. Note that $g' = \epsilon_2 - \delta_1 + \delta_3 \geq \epsilon_2 \geq (g+1)/2$. By Theorem 5, $G' \sim C_{g'}$. $\square$

It follows from Corollary 11, every anti-chain has a bounded odd-girth sequence. Another interesting corollary of ear-reduction lemma is that $3\mathbb{P}$ is a maximum of the class of all core-ear-faces. Note that a 3-ear-face that has K_2-cut is not a core, by Theorem 6.

Corollary 12 *$G \leq 3\mathbb{P}$, for any core-ear-face G.*

We now give a necessary and sufficient condition for ear-faces to be a core. The lemma allows us to get a fast algorithm for deciding and even for computing the core of any ear-face.

Lemma 13 *Let G be a k-ear-face of odd-girth g. Then G is a core if and only if*

(i) every ear of G is an odd cycle, and
(ii) $G \not\leq C_g$.

Proof. If G is a core, then clearly both (i) and (ii) must hold. Conversely, suppose $G' \subseteq G$ and $G \leq G'$ and assume G' is a core. By assumption (ii) and by transitivity of $\leq$, we deduce $G' \not\leq C_g$, and so G' is a k'-ear-face, $3 \leq k' \leq k$. Since G' is an induced subgraph, it follows that $k = k'$ and so the ears of G are mapped to ears of G' injectively. Since every ear of G is an odd cycle, the homomorphism from G to G' restricted to each ear is an isomorphism, for otherwise G' would have a shorter ear, which is not an induced subgraph. It follows that $G = G'$. $\square$

Using Lemma 13 and the following observations, we give a linear time "Labeling" algorithm *Core-Ear-Face* (CEF) to decide whether a k-ear-face is core or not. If g is an odd-girth of a core-ear-face of G then:
(*) for each ear E of G, $|E| \geq g$, and for some ear we have equality. Note that either ϵ or δ is larger or equal to $(g+1)/2$
(**) for any ear $E = [\epsilon, \delta]$ of G at least one of ϵ or δ is less than g (if $\epsilon \geq g$ and $\delta \geq g$, then $G \leq C_g$.)
(***) for any non-ear odd-cycle C, we have $|C| > g$.

Algorithm 14 Core-Ear-Face (CEF)
Input: k-ear-face $G = ([\epsilon_1, \delta_1], [\epsilon_2, \delta_2], \ldots, [\epsilon_k, \delta_k])$, $k \geq 3$, $\epsilon_1 + \delta_1 = g$, $\epsilon_i + \delta_i \neq 0$ for all i:
1. verify (*) and (**)
2. verify (***) in two ways:
(2.1) let $s = \Sigma_{i=1}^{k} \gamma_i$, where $\gamma_i = min(\epsilon_i, \delta_i)$. If s is odd and $s \leq g$ output NOT-CORE
(2.2) for $i = 1, 2, \ldots, k$ let $s_i = \zeta_i + \Sigma_{j \neq i} \gamma_i$, where $\zeta_i = \max(\epsilon_i, \delta_i)$ and $\gamma_j = min(\epsilon_j, \delta_j)$. If s_i is odd and $s_i \leq g$, output NOT-CORE. (Note that for any two maximums $\zeta_i, \zeta_j, i \neq j$, we have $\zeta_i + \zeta_j + \gamma_l > g$, any $l, 1 \leq l \leq k$
3. label v_{k1} by $L_0 = \{0\}$.
4. for $i = 1, \ldots, k-1$, label $v_{i(i+1)}$ by L_i where
$L_i = L_{i-1} + \epsilon_i \cup L_{i-1} + \delta_i$, (where $L + x = \{l + x \text{ (modulo } g) : l \in L\}$)
5. if $0 \in L_k = L_{k-1} + \epsilon_k \cup L_{k-1} + \delta_k$ output NOT-CORE, otherwise output CORE.

The correctness of CEF follows form Lemma 13, and one can easily verify that CEF runs in linear time in the size of G. CEF will also be useful in

proving that no infinite anti-chain of ear-faces exists.
The following lemma is due to A. Pór, which we use in proving non-existence of infinite anti-chains of ear-faces. Let $S_p \subseteq S = \{x_1, x_2, \ldots, x_g\}$ be a p-element subset of a finite set S. Let $W = (w_1, w_2, \ldots, w_n)$ and $W' = (w'_1, w'_2, \ldots, w'_{n'})$ be finite sequences of S_p. Let $W \circ W'$ denote the sequence $(w_1, w_2, \ldots, w_n, w'_1, w'_2, \ldots, w'_{n'})$, the concatenation of W and W'. We write $W \leq_g W'$ if $W' = W'_1 \circ w_1 \circ W'_2 \circ w_2 \circ \ldots W'_n \circ w_n \circ W'_{n+1}$, where for each i and for any $y \in S_p$, y appears in W'_i at least g times or zero times. It can be seen that $\leq_g$ is a quasi-order. Denote by $(\mathbb{F}, \leq_g)$, the set of all finite sets ordered by $\leq_g$.

Lemma 15 *$(\mathbb{F}, \leq_g)$ is wqo.*

Proof. Let an infinite sequence $X = W_1, W_2, \ldots$ of words of $S_p \subseteq \{x_1, x_2, \ldots, x_g\}$ be given. Clearly there is no infinite strictly descending chain in X under $\leq_g$. To prove $W_i \leq_g W_j$, for some $i < j$, we induct on $|S_p|$. Suppose first $p = 1$. Then each $W_i = (x_t, x_t, \ldots, x_t)$, repeated n_i times where $1 \leq t \leq g$. We can find $i < j$, such that $W_i = W_j$ if $\{n_i\}$ is bounded , and $n_j - n_i \geq g$ if unbounded. We have $W_i \leq_g W_j$.
Suppose next $p > 1$. Partition each word $W_i = P^i_1 \circ x_{i_1} P^i_2 \circ x_{i_2} \circ \ldots P^i_{m_i} \circ x_{i_{m_i}} \circ P^i_{r_i}$, such that each $y \in S_p$ appears at least g times, x_{i_j} appears exactly $g-1$ times and there exists $y \in S_p$ such that y appears in $P^i_{r_i}$ less than g times. If $m_i \geq 2m_1 + 2$ for some i, then $W_1 \leq_g W_i$. Hence assume $\{m_i\}$ is bounded and by taking subsequence let $m_i = m$ for all i.
Further partition, P^i_j as $P^i_j = P^i_{j_1} \circ x_{i_j} \circ P^i_{j_2} \circ x_{i_j} \ldots \circ x_{i_j} \circ P^i_{j_g}$, where $j = 1, 2, \ldots, m$. Each $P^i_{j_l}$ is an S_{p-1}-element word and hence by induction we can recursively take subsequence to construct a monotone subsequence of X. □

Theorem 16 *There is no infinite anti-chain of ear-faces.*

Proof. Suppose $X = G_1, G_2, \ldots$ is an infinite anti-chain of core-ear-faces. For each $i \geq 1$, let $V_i = ([\epsilon^i_1, \delta^i_1], [\epsilon^i_2, \delta^i_2], \ldots, [\epsilon^i_{k_i}, \delta^i_{k_i}])$ be the vector representation of G_i and let g_i be the odd-girth of G_i. By Corollary 11, $\{g_i\}$ is bounded and so we may assume $g_i = g$, for all i. By Lemma 9, if $K = \{k_i\}$ is bounded we are done, so assume K is a strictly increasing sequence.
Let d_i be the number of diminished ears of G_i. Then $d_i < g - 1$, or else $G_i \leq C_g$, contrary to G_i being a core. Let c_i be the number of ears of G_i with length bigger than g. Once more $c_i < g - 1$, or else $G_i \leq C_g$. Hence $\{c_i\}$ and $\{d_i\}$ are bounded. Let c and d denote the common values by taking

subsequence.
Partition each G_i in to $c+d+1$ parts $P_1, P_2, \ldots, P_{c+d+1}$, such that between each part there is a diminished-ear or a long-ear. There are finitely many ways of arranging these finite 'irregular-ears' and hence may assume, by taking subsequence that for all i, if we contract all P_j^i of G_i to a vertex, then we obtain G_i', such that $G_1', G_2', \ldots$ are all isomorphic.
Now consider P_j^i. Each ear is of length g, hence we may represent each ear by $\epsilon_{j_l}^i$ since $\delta_{j_l}^i = g - \epsilon_{j_l}^i$. So P_j^i is a finite set of integers. Apply Lemma 15, for each $\{P_j^i\}_{i=1}^{\infty}$ $j = 1, 2, \ldots, c+d = 1$ to obtain inclusion by $\leq_g$.
Here we use a crucial property of addition of integers modulo g, which is commutativity and associativity. Indeed it suffices that each number appears in P_j^i, either zero or more than $g-1$ times. We have the ears of some G_i injectively contained, by a map $f : V(G_i) \to V(G_j)$, as a subsequence in some G_j. Between the images $E_{f(i)}^j$ and $E_{f(i+1)}^j$, every ear that appears has length g represented by integer $m < g$ appearing more than g-1 times. We map all these ears to C_g such that the vertices $v_{f(i)(f(i)+1)}$ and $v_{(f(i+1)-1)f(i+1)}$ are identified to a vertex. It follows $G_j \leq G_i$. □

Corollary 17 *Let $\mathcal{K} \subseteq SPG$ be a minor-closed class consisting of ear-faces and their minors. Then $\mathcal{K}$ is non-universal.*

5 2-connected SPG core-graphs are 'Recursive Ear-faces'

We prove in this section that every 2-connected graph G in SPG can be constructed by a simple recursive procedure. Assume $G_{-1} = K_2$. Starting from a cycle G_0, we recursively replace some edges of the outer-cycle of G_i in $E(G_i) \backslash E(G_{i-1})$ by cycles and obtain G_{i+1}. We show for every 2-connected graph $G \in SPG$, there is a natural number $r_G \geq 0$ such that $G \sim G_{r_G}$ (the case $r_G = 0$ is the trivial case where G is a cycle). We call this constructive procedure *ear-face recursion.* A typical structure of an arbitrary 2-connected graph embedded in a plane is depicted in Figure 6 below. We define a *recursive ear-face* to be a graph $G = G_m, m \geq 1$, that can be obtained from a recursive sequence of graphs $G_0, G_1, \ldots, G_m$, where G_0 is a cycle, and for $i = 1, 2, \ldots, m$, G_i is obtained from G_{i-1} by ear-face-recursion.

We first explain why a planar embedding of G has a structure as in Figure

6. Clearly, for every non-trivial graph $G \in SPG$, there exists a natural number $k \geq 3$ such that $G \notin SPG/2C_k$ and $G \in SPG/2C_{k+1}$. This means $2C_k \preceq G$, and k is the maximum number of ears G can have in any planar embedding of G. Note that if $2C_3 \not\preceq G$, then by Theorem 5, $G \sim C_g$, where g is the odd-girth of G and so G would be trivial.

Subdivide $2C_k$ as many times as needed to obtain the proper k-ear-face H that is a minor of G. By definition of minor relation, we can add edges on H to obtain a graph hom-equivalent to G. (Note that, when H is an ear-face, spitting a vertex is not required to obtain G). Keeping in mind that k is maximal and that $K_4 \not\preceq G$, we make a few more important observations:

(1) Call a vertex u of an ear E of H an *ear-vertex* if the degree$(u) = 2$. Then, when reconstructing G, we need not have any operation on any ear-vertex u of H for two reasons: Suppose first a path P consisting of all newly added edges connects u to a vertex $v \in V(H)$ not in $V(E)$, then we get a K_4 minor, a contradiction. Next if v is a ear-vertex of E on the opposite-parity side, then once more we get a K_4 minor. By Theorem 5 and by maximality of k, we deduce any operation on u is trivial. Hence, we do not make any changes on ear-vertices of H to obtain G.
(2) Suppose $u \in V(H)$ is not an ear-vertex and there is path P consisting of only newly added edges connecting u and $v \in V(H)$. Once more, v is not an ear-vertex (since $K_4 \not\preceq G$), and by Theorem 5 and since k is maximal, we may assume between v and u every path P, with $V(P) \cap V(H) = \{u, v\}$ traverses at least two ears of H, as shown in Figure 6. That is, as labelled in Figure 6, if we have a path between u and v which intersects only one ear (if $e_1 = 1$), by Theorem 5, we can replace the subgraph G_1^2 by a cycle.
(3) By observation (2), we deduce the number k_1 in Figure 6, is at most $k/2$, and k_2 is at most $k_1/2$, and so on. By induction, we note that we may start from $G_0 = C_d$ where $d \geq log(k)$, and arrive at a graph $G' \sim G$. An example is given in Figure 7 to illustrate this procedure.

Theorem 18 *For every 2-connected graph $G \in SPG$, there is a graph $G' \sim G$, that is constructible by ear-face recursion in $O(log(k))$ time, where $2C_k \preceq G$ and k is maximal.*

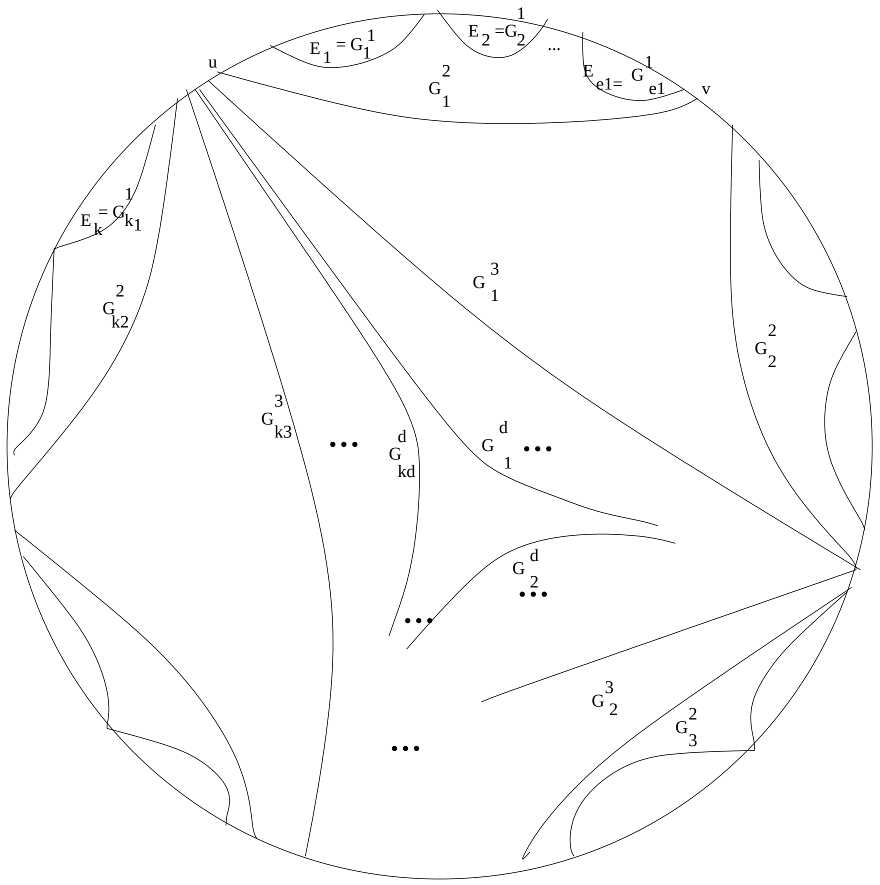

Figure 6: A general structure of a 2-connected graph in SPG

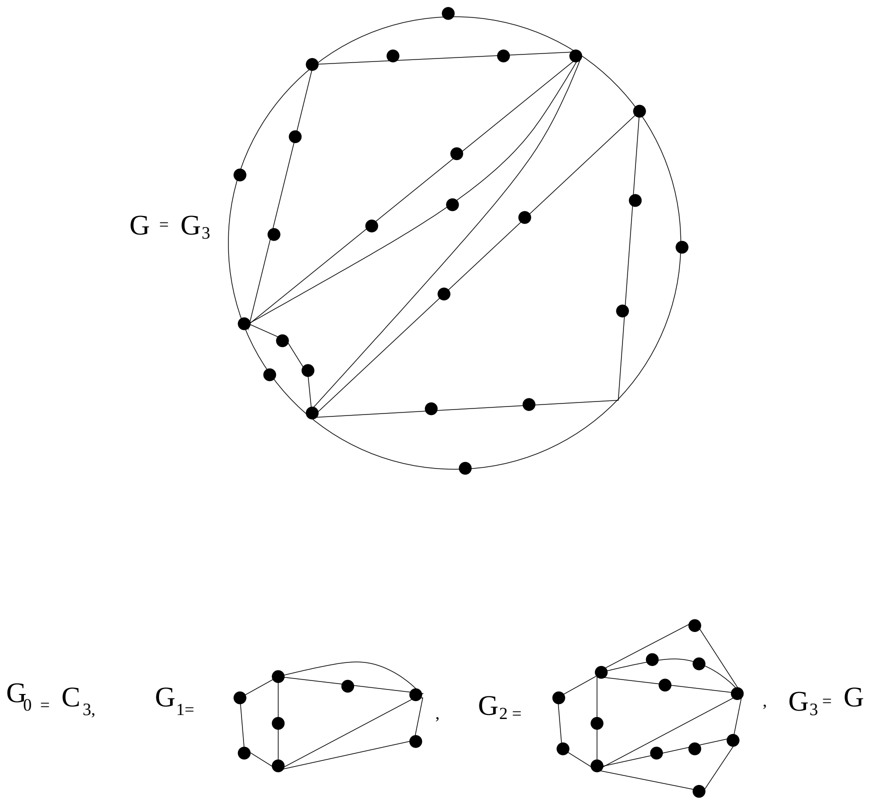

Figure 7: An example of construction by recursion

6 Concluding remarks and problems

Our goal is to determine all minimal universal minor-closed classes in SPG. In Section 4, we showed ear-faces can not form a universal set. We hope the structural description in Section 5, will help us to obtain a larger class of graphs which contains no infinite anti-chain and thus non-universal. Perhaps we may able to prove the necessity of unbounded concatenation using structural arguments. Presently, however, the question is not settled.
Adopting the method in [6], we show in a forthcoming paper that K_1-concatenation of ear-faces (in particular $3\mathbb{P}$), we obtain several minimal universal minor-closed subclasses of SPG. The minimality of such subclasses of SPG is proved using the result of Section 2, and by associating each concatenation of the $3\mathbb{P}$ to a particular oriented path.

Without going in to technical details we give an outline of the association of a K_1-concatenation of an-ear-face G, (for instance $3\mathbb{P}$) to a path P, that is shown [6] as follows: For each oriented path $P \in \mathcal{P}$ of length $n \geq 1$, we define a K_1-concatenation of G (denoted by $P * (G, a, b)$ where $a, b \in V(G)$ as follows: First, we take n isomorphic copies $G_1, G_2, \ldots, G_n$, of G and let a_i, b_i be the vertices of G_i corresponding to a and b. For $i = 1$ and n, we choose either a_i or b_i, depending on the orientation of the the first and the last edges of P. Then according to the orientations of the the i-th edge of P, for $i = 2, 3. \ldots, n-1$ we choose either ,a_i or b_i to be identified with the chosen vertex of G_{i-1} and G_{i+1}, so that such a construction allows us to deduce for any two paths P and P', we have $P \leq P'$ if and only if $P * (G, a, b) \leq P' * (G, a, b)$. Then the universality and minimality of the minor-closed class formed by the concatenation follows from universality and minimality of the class of directed paths $\mathcal{P}$ (see Theorem 3). The property of somewhere density also can be proved similarly. However, we state in general here that there can be several ways of concatenating a given graph G. As a result, we obtain a number of minimal universal classes. Aside from this method, one may also ask is there another way of constructing a universal minor-closed class, which does not contain as a subset, any of the ones we have? Along this work, we present the questions below:

Problem 19 *Can we determine the set of all minimal universal minor-closed classes of SPG?*

Problem 20 *We have shown in Section 3, that forbidding K_4 and $3\mathbb{P}$ as minors induces a total order under homomorphism. Does there exist a finite* *set F of core-graphs such that forbidding $K_5, K_{3,3}$ and elements of F as minors results a total order under homomorphism? (Note that to avoid triviality, we must assume that if $G \in F$ then G is minor-incomparable to K_5 and $K_{3,3}$. We may also assume no element of F is a cycle.)*

Problem 21 *We have given a linear time algorithm that decides of an ear-face is a core not. Can this be extended for any 2-connected graph in SPG? For any G in SPG?*

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