Two-factors in Oriented Graphs with Forbidden Transitions

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Abstract

The instance of the problem of finding 2-factors in an oriented graph with forbidden transitions consists of an oriented graph G and for each vertex v of G a graph H_v of forbidden transitions assigned to it. Vertices of H_v are the edges incident to v. An oriented 2-factor F of G is called *legal* if there is not a vertex v such that the two edges of F adjacent to it form an edge in H_v. Deciding whether a legal 2-factor exists in G is NP-complete in general. We investigate the case when the graphs of forbidden transitions are taken from some class C. We provide an exact characterization of classes C for that the problem is NP-complete. Specially we prove a dichotomy for this problem, i.e., that for any class C it is either polynomial or NP-complete.

Introduction

The instance of the studied problem consists of an oriented graph G in that some pairs of edges incident to a vertex are marked as forbidden. The task is to find a 2-factor in G that does not contain such a pair of edges. Since the graphs of forbidden transitions in the graphs we use in the following constructions are usually quite dense, we rather study graphs with **allowed** transitions. We first present some definitions in order to formulate the problem exactly:

Let Γ be a set of a bipartite graphs with vertices explicitly assigned to the two classes. If $H \in \Gamma$, denote the classes by $I(H)$ and $O(H)$. A graphs in this set are used to represent the allowed transitions at a vertex. We do not consider edgeless graphs to be members of this set, since obviously throwing them away does not change complexity of the considered problem.

We work with oriented graphs (in some parts of the paper with parallel edges and loops allowed) with allowed transitions at vertices. Graphs in Γ are used to represent the transitions. We say that the transition is *forbidden* if it is not allowed. In the figures we mark allowed transitions by solid lines and forbidden ones by dotted lines. When the transition is not drawn it is irrelevant for the particular construction and we don't care whether it is allowed or forbidden.

Sometimes we consider the edges to consist of two half-edges. We also allow the half-edges to be "free" and to connect them to the appropriate matching half-edge later in some constructions.

For $C \subseteq \Gamma$ we denote $M(C)$ the family of oriented multigraphs with allowed transitions belonging to the class C, i.e., each of the vertices of $G \in M(C)$ has assigned a graph H of allowed transitions that belongs to C together with matching between incoming edges and $I(H)$ and outcoming edges and $O(H)$. We identify the vertices of H with corresponding (half)edges of G in the rest of a paper. Specially if H is assigned to a vertex v then $|I(H)| = d_v^+$, $|O(H)| = d_v^-$. $S(C)$ is a similar family, but here we require the underlying graph to be simple, i.e., without parallel edges and loops.

In the significant part of the paper we focus our attention to graphs where for each vertex v, $d_v^+ = d_v^-$, and consequently $|I(H)| = |O(H)|$ for each graph of allowed transitions. Let us denote a class of such bipartite graphs Γ_r.

Let us denote the induced subgraph of H on vertices V by $H[V]$. There are some special subsets of Γ. Let $O'(H) = \{v \in O(H) : (\exists e \in E(H))\ e = (uv)\}$, $I'(H) = \{v \in I(H) : (\exists e \in E(H))\ e = (vu)\}$ and $\widehat{H} = H[I'(H) \cup O'(H)]$ for $H \in \Gamma$. Let us denote $D^+ \subset \Gamma$ the set of graphs H such that $|I'(H)| \leq 2$. Analogically $D^- \subset \Gamma$ is the class of graphs H such that $|O'(H)| \leq 2$. Let us denote $B \subset \Gamma$ the set of graphs H such that $\widehat{H}$ is a complete bipartite graph.

We consider a complexity of finding an oriented 2-factor that respects the allowed transitions (we call such a 2-factor *legal*), i.e., if e and f are two consecutive edges on this 2-factor passing through vertex v that has assigned a graph H_v of allowed transitions, then $\{e_v, f_v\} \in H_v$. This question is of course NP-complete in general. We study its complexity when restricted to graphs from $M(C)$ (or $S(C)$) for some fixed C. We prove that if $C \subseteq \Gamma_r$, if $C \subseteq D^+$ or $C \subseteq D^-$ or $C \subseteq B$, then the problem is solvable in a polynomial time, and that it is NP-complete otherwise.

In the general case ($C \nsubseteq \Gamma_r$) it may happen that $M(C) = M(C \cap \Gamma_r)$ in cases when either all $H \in C \setminus \Gamma_r$ have $|I(H)| > |O(H)|$, or all of them have $|I(H)| < |O(H)|$. Then of course the set $C \setminus \Gamma_r$ may contain anything without affecting the complexity of the question, thus forcing us to formulate the characterization more carefully. We call C *simple* if the following conditions are satisfied:

- $C \cap \Gamma_r \subseteq D^+$ or $C \cap \Gamma_r \subseteq D^-$ or $C \cap \Gamma_r \subseteq B$
- If $M(C) \neq M(C \cap \Gamma_r)$, $C \subseteq D^+$ or $C \subseteq D^-$ or $C \subseteq B$.

This problem is inspired by [KP92]. They study the similar problem for unoriented graphs and prove that in this case the problem is polynomial if the graphs of allowed transitions are either complete multipartite ones or subgraphs of the cycle of length four, and NP-complete otherwise.

Analogical modifications of standard problems were already studied: Szeider [S03] proved the dichotomy result for problem of finding a path in graph with forbidden transitions. Interesting structural results on compatible paths and cycles were derived in context of Eulerian graphs, see for example [F90] and [MN02].

1 Polynomial cases

Lemma 1 *If $C \subseteq D^+$ or $C \subseteq D^-$, or $M(C) = M(C \cap \Gamma_r)$ and $C \cap \Gamma_r$ satisfies one of these conditions, the problem of finding a legal 2-factor for a graph $G \in M(C)$ can be solved in time $O(|E(G)|^2)$.*

Proof If $M(C) = M(C \cap \Gamma_r)$, no vertex may have $H \in C \setminus \Gamma_r$ assigned as graph of allowed transitions, so it is enough to solve the problem in the former case. WLOG let $C \subseteq D^+$ (the case $C \subseteq D^-$ is solved symmetrically). We reformulate the problem as a 2-SAT instance of size $O(|E(G)|^2)$, thus obtaining the desired time bound. We create a variable x_e for each edge of G – we want x_e to be true if and only if e belongs to the 2-factor. We add the following clauses:

1. $\neg x_e$ if $e = (uv)$ and $e \notin I'(H_v)$ or $e \notin O'(H_u)$.

2. x_e if $e = (uv)$, $e \in I'(H_v)$ and $|I'(H_v)| = 1$.

3. $x_e \vee x_f$ and $\neg x_e \vee \neg x_f$ if $e \neq f$, $e = (uv)$, $f = (wv)$, $e \in I'(H_v)$ and $f \in I'(H_v)$.

4. $\neg x_e \vee \neg x_f$ if $e = (uv)$, $f = (vw)$, $e \in I'(H_v)$ and $ef \notin H_v$.

5. $\neg x_e \vee \neg x_f$ if $e \neq f$, $e = (uv)$ and $f = (uw)$.

It should be clear that this 2-SAT instance is satisfiable iff there is the sought 2-factor: If there is a legal 2-factor, we may set x_e true exactly for edges of the 2-factor and all the clauses are obviously satisfied. When the formula is satisfied, we may reversely add all edges for that x_e is true to our candidate for 2-factor. Due to 1, 2 and 3 the indegree of every vertex is 1. Due to 5 the outdegree is at most 1, but since sum of in- and outdegrees is the same, it must also be exactly 1 for each vertex. Due to 4 the allowed transitions are respected.

Each edge is contained in at most $2|E(G)| + 3$ clauses, therefore there are at most $O(|E(G)|^2)$ clauses, and 2-SAT can be solved in the time linear in the size of the instance. ∎

Lemma 2 *If $C \subseteq B$, or $M(C) = M(C \cap \Gamma_r)$ and $C \cap \Gamma_r \subseteq B$, then the problem of finding a legal 2-factor for a graph $G \in M(C)$ can be solved in time $O(|\sqrt{|V(G)|}E(G)|)$.*

Proof Due to the reasoning similar to the one used in the previous proof, it is enough to solve the case $C \subseteq B$.

We know that if $e = (uv)$ and $e \notin O'(H_u)$ or $e \notin I'(H_v)$, the edge cannot be used in a legal 2-factor – so by throwing away all such edges from the graph, we do not change the existence of the 2-factor. In the resulting graph G' all transitions are allowed, i.e., we have transformed the problem to task of finding an oriented 2-factor in ordinary oriented graph (even simple, as we may merge parallel edges in this situation safely). By splitting the vertices into input and output parts, we may conclude that this is equivalent to finding a perfect matching in a bipartite graph, which can be solved in time $O(\sqrt{|V|}|E|)$ ([HK75], [MV80]). ∎

2 Gadgets

It remains to prove that the problem is NP-complete unless it falls into one of the above categories. We do this by reduction from problem of exact set

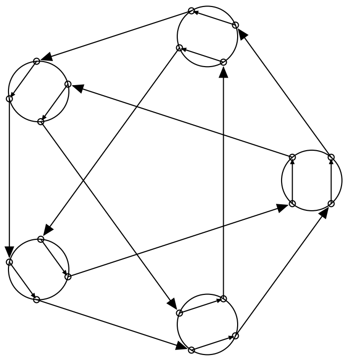


Figure 1: Gadget for simplifying graphs

cover where each element is a member of exactly three sets and each set has size exactly three, which is NP-complete due to [GJ79]. For the reduction we need some gadgets that we develop in the following lemmae.

Lemma 3 *If $C \subseteq \Gamma$ is not simple, there exists a graph $Z \in S(C)$ with a special edge f_Z such that Z has a legal 2-factor containing the edge f_Z and a legal 2-factor that does not contain containing the edge f_Z.*

Proof If $M(C) = M(C \cap \Gamma_r)$, let $C' = C \cap \Gamma_r$, otherwise $C' = C$. We may therefore assume that $C' \not\subseteq B$. It follows that there exists a graph $H \in C'$ such that it contains two disjoint allowed transitions.

The construction proceeds as follows: We construct a "section", that consists of the two sought legal 2-factors, but the edges that do not belong to it are not "wired in". We require that the number of in- and out-halfedges is the same – let it be m. We then take $m+1$ copies of the section and connect out-halfedges of copy i with some in-halfedges from mutually disjoint copies different from i (the exact matching subject to this restriction is arbitrary). The graph then clearly satisfies the conditions: It is simple, since the section is constructed as such and since no two edges leading from the same section share the endpoint. The required 2-factors are obtained by taking the two legal 2-factors in one section, choosing an arbitrary edge on the first of them as special and extending them by arbitrary legal 2-factors in the remaining sections.

The section itself may consist from two parts. The first is composed from copies of graph depicted in figure 1, where all vertices have H as a graph of allowed transitions. We may chose f as any of its edges. If H is in Γ_r, this is the whole section. If it is not, we cannot take just this one, as then the numbers of in- and out-halfedges (denote these as i and o) would not match. WLOG suppose that $i > o$. In this case we however know that $M(C) \neq M(C \cap \Gamma_r)$ and therefore there must be a graph $H' \in C'$ such that $|I(H')| < |O(H')|$ – the second part of section consists of copies of triangle with transitions H' oriented so that the triangle forms a legal part of a 2-factor – here we use the fact that the edgeless graphs are not in Γ. Each of them provides us with i' in-halfedges and o' out-halfedges, $i' < o'$. We choose the numbers of copies so that the numbers of in- and out-halfedges match, e.g. $o' - i'$ of the first one and $i - o$ of the second one. ∎

Theorem 4 *For any $C \subseteq \Gamma$, the problem of finding a legal 2-factor for a graph from $M(C)$ is NP-complete iff the problem of finding a legal 2-factor for a graph from $S(C)$ is NP-complete.*

Proof The right to left implication is trivial. For the other implication suppose that the problem for $M(C)$ is NP-complete and that C is not simple. Then for any instance G of this problem we create an equivalent instance G' belonging to $S(C)$. We do it by adding a copy Z_e of graph Z for every edge of G to it. For every edge $e = (uv)$ we take the edge $f_{Z_e} = (wz)$, remove the edges e and f_{Z_e} and add new edges (wv) and (uz) (they also replace the edges e and f_{Z_e} in the appropriate allowed transitions). The size of G' is at most $(|Z| + 1)$ times greater than of the size of G, so the reduction is polynomial. If G has a legal 2-factor, so does G':

- If e belongs to the 2-factor in G, we add the edges (wv) and (uz) to the 2-factor together with edges of a legal 2-factor of Z_e that used to pass through edge f_{Z_e}.
- Otherwise add the edges of a legal 2-factor of Z_e that does not pass through f_{Z_e}.

In the other direction, if G' has a legal 2-factor, either both (wv) and (uz) are used or both are unused in it (as they form a 2-cut). Add e to the 2-factor iff they are used.

Because Z is simple, G' is simple as well, thus this is a reduction between these two problems. ∎

Due to this theorem, we don't need to care of simplicity of the graphs we create later.

Lemma 5 *Let $C \subseteq \Gamma$ be such that $M(C) \neq M(C \cap \Gamma_r)$ and C is not simple. Let $d = GCD\{|I(H)| - |O(H)| : H \in C\}$. Then there exists a graph $A \in M(C)$ with special edges $f_A^1, \ldots, f_A^d$ such that A has a legal 2-factor, $A \setminus \{f_A^1, \ldots, f_A^d\}$ consists of two components A_1 and A_2 and all edges f_A^i lead from A_1 to A_2 (consequently they cannot be used in any 2-factor of A).*

Proof It follows from fundamental properties of GCD that there exists a finite set $\{H_1, \ldots, H_m\} \subseteq C$ and integers $t_1, \ldots, t_m$ such that

$$\sum_{i=1}^{m} t_i(|I(H_i)| - |O(H_i)|) = d.$$

If $|I(H_i)| = |O(H_i)|$, we may remove the graph H_i from the set without changing this property. Otherwise WLOG $|I(H_i)| > |O(H_i)|$, then there must exist $H_i' \in C$ with $|I(H_i')| < |O(H_i')|$, and by adding $|O(H_i')| - |I(H_i')|$ of the former and $|I(H_i)| - |O(H_i)|$ of the latter we do not change the sum. Therefore we may assume that the coefficients t_i are nonnegative. We take t_i vertices with graph of transitions H_i to the graph A – they form the part A_2. We connect all these vertices to a cycle so that it forms a legal 2-factor and connect the remaining halfedges (except for d in-halfedges) arbitrarily. Symmetrically we construct A_1 and connect the two parts by edges f_A^i. ∎

Lemma 6 *If $C \subseteq \Gamma$ is not simple then there exists a graph $N \in M(C)$ containing a special edge f_N such that*

- *N has a legal 2-factor*
- *N has no legal 2-factor containing the edge f_N.*

Proof If $M(C) \neq M(C \cap \Gamma_r)$, then we use a graph A from the previous lemma. Otherwise we may WLOG suppose that $C \subseteq \Gamma_r$.

Suppose that there is a graph $H \in C$ containing 4 mutually different vertices $i_1, i_2 \in I(H)$, $o_1, o_2 \in O(H)$ such that $(i_1 o_1) \in H$ and $(i_2 o_2) \notin H$. Number the remaining vertices of $I(H)$ and $O(H)$ from 2 to $|I(H)|$ arbitrarily. The sought graph N consists of a single vertex w with graph of allowed transitions H and $|I(H)|$ loops connecting i_k to o_k. It is clear that this graph has a legal 2-factor consisting of an edge $(i_1 o_1)$, and edge $f = (i_2 o_2)$ does not belong to any legal 2-factor.

The case when such a graph does not exist in C remains. If $|I(H)| > 2$ and it does not satisfy the condition, it must be a complete bipartite graph. If $|I(H)| \leq 2$ and $H \notin B$ then $H = 2K_2$. The only case when C is not simple then is if C contains both $2K_2$ and $K_{d,d}$ for some $d > 2$. Then we may construct the graph as follows: It consists of two vertices u and v, u having a graph of allowed transitions $2K_2$ and v having $K_{d,d}$. Let $(i_u^1 o_u^1)$ and $(i_u^2 o_u^2)$ be the allowed transitions of u, i_v^k and o_v^k an arbitrary numbering of in- and out-halfedges of v. We add edges $(o_u^1 i_v^1)$, $(o_u^2 i_v^2)$, $(o_v^1 i_u^1)$, $(o_v^2 i_u^2)$ and loops $(o_v^k i_v^k)$ for $k > 2$. This graph has two legal 2-factors and none of them uses any of the loops $(o_v^k i_v^k)$. One of them can therefore be chosen as f_N. ∎

For $C \subseteq \Gamma_r$, let us denote by C^* set of all non-edgeless induced subgraphs of members of C (then $C^* \subseteq \Gamma$, but not necessarily $C^* \subseteq \Gamma_r$).

Theorem 7 *Let $C' \subseteq C^*$ be some finite subset of C^*. Then if the problem of finding a legal 2-factor is NP-complete for $M(C')$, it is also NP-complete for $M(C)$.*

Proof If C is simple, so is C'. We may therefore suppose that C is not simple. Let the problem be NP-complete for $M(C')$. We are given an instance of $G \in M(C^*)$. We transform it to an equivalent instance $G' \in M(C)$ in the following way: For every vertex v, its corresponding graph of allowed transitions H_v^1 is a subgraph of some $H_v \in C$. We extend the graph H_v^1 to it, thus creating some free halfedges. The number of created in-halfedges and out-halfedges must be the same, as in the original graph the sum of indegrees was equal to the sum of outdegrees and graphs in C have the same in- and outdegree. We pair these free half-edges arbitrarily. For every such pair i and o we add a copy N_{io} of graph N, remove the edge $f_{N_{io}} = (uv)$ and connect i to v and o to u instead of it.

Due to properties of N, neither i nor o may be used in any legal 2-factor of G'. Thus we may restrict the legal 2-factor of G' to G. In the reverse direction, we add 2-factors inside graphs N_v to the 2-factor in G, thus obtaining a legal 2-factor in G'.

Since C' is finite, there exists $M = \max\{|H| : H^1 \in C', H \in C$ minimal such that H^1 is an induced subgraph of $H\}$. Then the size of G' is at most $|G| + M.|V(G)|.|Z|$. Both $|Z|$ and $|M|$ are constants independent on input, therefore this is a polynomial reduction. ∎

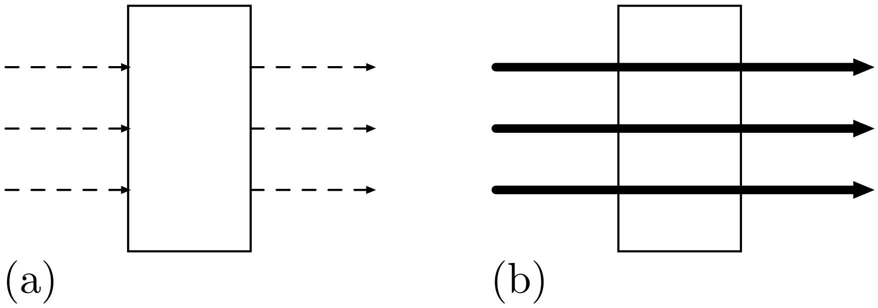


Figure 2: Copy gadget

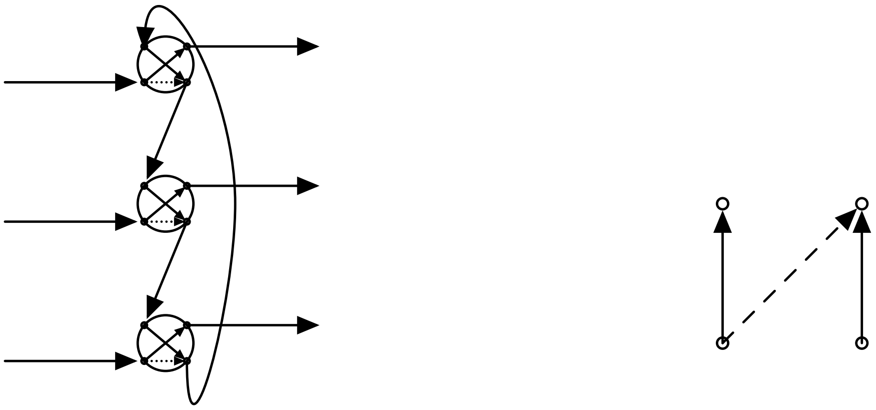
We use this theorem in the construction of the following gadgets. The subgraphs used in the constructions all have in- and outdegree at most 3, so the considered family C' is obviously finite.

The next gadget is "copy" gadget, used to ensure that after cutting its three special edges, every 2-factor either does not use them at all, or uses all three and connects the first in-halfedge to the first out-halfedge, the second one to the second one and the third one to the third one (see figure 2 (a), (b))

Lemma 8 *For $C \subseteq \Gamma_r$ such that $C \not\subseteq B$, there exists $C' \subseteq C^*$ finite and $K \in M(C')$ with three special edges f_K^1, f_K^2 and f_K^3 such that*

- *There exists a legal 2-factor of K containing all of f_K^1, f_K^2 and f_K^3.*
- *There exists a legal 2-factor of K containing none of f_K^1, f_K^2 and f_K^3.*
- *Every legal 2-factor of K is of one of these types.*
- *Let $\pi : \{1, 2, 3\} \to \{1, 2, 3\}$ be a non-identical permutation and $f_K^i = (u_i v_i)$. If we replace edges f_K^i by edges $(u_i v_{\pi(i)})$, the resulting graph has no legal 2-factor.*

Proof We use the construction from figure 3 (a). To realize it we need a graph H (figure 3 (b)) to be an induced subgraph of some member of C (then we let C' consist just of this subgraph). If H_1 does not contain H as an induced subgraph, $i \in I'(H_1)$, $o \in O'(H_1)$, then $(io) \in E(H_1)$ – otherwise non-edge (io) together with edges adjacent to i and o (that exist



(a) (b)

Figure 3: Copy gadget internals

due to definition of I' and O') form graph H. Therefore $H_1 \in B$, so unless $C \subseteq B$, we may realize this construction. ∎

The last gadget we need is "one-in-three" gadget, used to ensure that after cutting its three special edges, every 2-factor uses exactly one special in-halfedge and exactly one out-halfedge and it is possible to connect the first in-halfedge to the first out-halfedge, the second one to the second one, as well as the third one to the third one (see figure 4 (a), (b), (c)). Note that we do not require that it is not possible to match them in another way (having the requirement only for the "copy" gadget will be shown sufficient for our construction later).

Lemma 9 *For $C \subseteq \Gamma_r$, unless C is simple there exists $C' \subseteq C^*$ finite and $J \in M(C')$ with special edges f_J^1, f_J^2 and f_J^3 such that*

- *For $l = 1, 2, 3$, J has a legal 2-factor containing edge f_J^l.*
- *Let $\pi : \{1, 2, 3\} \to \{1, 2, 3\}$ be a permutation and $f_K^i = (u_i v_i)$. If we replace edges f_K^i by edges $(u_i v_{\pi(i)})$ then every legal 2-factor con-*

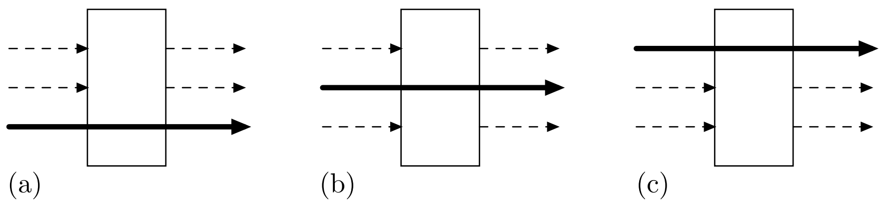


Figure 4: One-in-three gadget

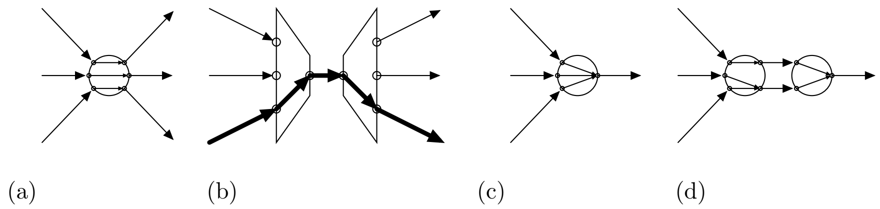


Figure 5: One-in-three gadget internals

tains exactly one of these three edges (but such a 2-factor does not necessarily have to exist if π is not the identical permutation).

Proof If $3K_1$ is a subgraph of some graph from C, we take just a single vertex with this graph of allowed transitions (figure 5 (a)).

Otherwise we compose the gadget from two parts as depicted in figure 5 (b). The incoming 2-factor is lead to the central edge and from it to one of exits. We have 2 ways how to realize the parts – one of them is in figure 5 (c), the other one is in figure 5 (d) (this is for the narrowing part, the widening one is symmetric).

Suppose now that we cannot realize the constructions, WLOG we are unable to create the narrowing part. Then we know that $C \subseteq D^+$, since if there was a graph $H \in C$ containing three edges with mutually different sources, the graph that they would form would be one of those required for construction (a), (c) or (d). ∎

3 Regular case

Now we are ready to prove the main theorem:

Theorem 10 *Let $C \subseteq \Gamma_r$ such that C is not simple. Then the problem of finding a legal 2-factor for a graph from $M(C)$ is NP-complete.*

Proof Let C' be a union of finite sets from the previous two lemmae. We prove that the problem is NP-complete for graphs from $M(C')$, this implies the result due to theorem 7.

We proceed by reduction from the problem of finding an exact set cover where each element is member of exactly three sets and each set has size exactly three ([GJ79]) as mentioned before.

Let (P, Q) $\left(Q \subseteq \binom{P}{3}\right)$ be an instance of this problem. We create an equivalent instance for our problem as follows:

For each set $q \in Q$ we add a copy K_q of graph K. For each element $p \in P$ we add a copy J_p of graph J. Now we connect them in the following manner (the idea is that each special edge of J_p corresponds to an occurrence of p in one set):

Create some matching between pairs (p, q) where $p \in q$ and mutually disjoint pairs $(f^l_{J_p}, f^m_{K_q})$ (this is always possible – just take eagerly the first still available choice). Suppose that we have such a pair, $f^l_{J_p} = (uv)$, $f^m_{K_q} = (wz)$. We remove the edges $f^l_{J_p}$, $f^m_{K_q}$ and add the edges (wv) and (uz).

If the exact set cover instance has a solution, we add edges to the 2-factor according to the following rules: If q is chosen to the exact set cover, we add edges of K_q according to the 2-factor that uses all its special edges. We also add edges that replaced the special ones. If q is not chosen, we add the edges according to the 2-factor that avoids the special edges. For each element p there exists exactly one chosen q such that $p \in q$, so exactly one incoming and the matching outgoing edge incident to J_p was added so far. We add edges according to the legal 2-factor of J that uses exactly the corresponding edge.

Reversely, if we have a solution for the corresponding 2-factor instance, we know that for each q either all edges adjacent to K_q are in the 2-factor or none is (as otherwise it could not be extended to a legal 2-factor on rest of K_q due to the properties of K). Due to the second property of J, exactly one incoming and exactly one outgoing edge adjacent to J_p must belong to the 2-factor, and due to the properties of K and the linkage in this graph they must both correspond to the same special edge in J_p. We now choose q to the exact cover iff all edges adjacent to K_q belong to the matching. As demonstrated this set covers each element of P exactly once.

Size of the created graph is $O(|P||J| + |Q||K|)$, i.e., it is polynomial in the size of (P, Q). Therefore this reduction is indeed polynomial. ∎

By combining theorems 4 and 10 we immediately obtain

Corollary 11 *Let $C \subseteq \Gamma_r$ such that C is not simple. Then the problem of finding a legal 2-factor for a graph from $S(C)$ is NP-complete.*

4 General case

Let us now consider the general case:

Theorem 12 *Let $C \subseteq \Gamma$ such that C is not simple. Then the problem of finding a legal 2-factor for a graph from $S(C)$ is NP-complete.*

Proof Due to theorem 4 it is enough to prove NP-completeness of the problem for $M(C)$.

If $M(C) = M(C \cap \Gamma_r)$, then we use the theorem 10 on $M(C \cap \Gamma_r)$, therefore we may suppose that $M(C) \neq M(C \cap \Gamma_r)$.

Let $C' \subseteq \Gamma_r$ be a set of graphs that are obtained from graphs in C by padding the smaller partition by vertices with no allowed transitions. Because C is not simple, C' is not simple, and therefore the problem of finding a legal 2-factor for a graph from $S(C')$ is NP-complete. Let G be an instance of this problem.

We create an instance G' in $S(C)$ by this procedure: remove all edges adjacent to the padding added to graphs in C'. This may leave some free halfedges from the original vertices of graphs in C (the number of in- and out-halfedges does not have to be equal). As long as there are both in- and out-halfedges, connect them arbitrarily through copies of N (thus ensuring that they cannot belong to a legal 2-factor). WLOG suppose that we have only free in-halfedges left. Let $d = GCD\{|I(H)| - |O(H)| : H \in C\}$. Clearly the number of free in-halfedges must be divisible by d. Divide them into d-tuples. For every d-tuple add a copy of graph A_1 from lemma 5 and connect the in-halfedge to the previous sources of edges f_A^i. This procedure increases the size of the instance at most $\max(|A| + 1, |N| + 1)$ times, i.e., it is polynomial.

If G' has a legal 2-factor, we obtain a legal 2-factor of G simply by restricting it to G. If G has a legal 2-factor, we know it does not use any of removed edges, so we may extend it to a legal 2-factor of G' by adding legal 2-factors to copies of parts of A and copies of N added. Therefore this indeed is the polynomial reduction between these two problems. ∎

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