An Improved Approximation Algorithm for the Asymmetric TSP with Strengthened Triangle Inequality

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Abstract

We consider the asymmetric traveling salesperson problem with γ-parameterized triangle inequality for $\gamma \in [\frac{1}{2}, 1)$. That means, the edge weights fulfill $w(u,v) \leq \gamma \cdot (w(u,x) + w(x,v))$ for all nodes u, v, x. Chandran and Ram [8] recently gave the first constant factor approximation algorithm with polynomial running time for this problem. They achieve performance ratio $\frac{\gamma}{1-\gamma}$. We devise an approximation algorithm with performance ratio $\frac{1+\gamma}{2-\gamma-\gamma^3}$, which is better than the one of Chandran and Ram for $\gamma \in [0.5437, 1)$, that is, for the particularly interesting large values of γ.

1 Introduction

The traveling salesperson problem is a well-known NP optimization problem. Given a complete loopless graph G and a weight function w that assigns to each edge a nonnegative weight, our goal is to find a tour of minimum weight

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that visits each node exactly once. In general, the graph G may be directed. In this case, one also speaks of the asymmetric traveling salesperson problem (ATSP). An important and well-studied special case is the case where w is symmetric (TSP), that is, $w(u,v) = w(v,u)$ for all $u, v \in V$. In other words, the underlying graph can be considered undirected.

TSP and henceforth ATSP are both NPO-complete. Thus there is no good approximation algorithm for these two problems, unless $\mathsf{NP} = \mathsf{P}$. A natural restriction is that the weight function w should fulfill the triangle inequality

$$w(u,v) \leq w(u,x) + w(x,v) \quad \text{for all } u, x, v \in V. \tag{1}$$

We call the corresponding problems Δ-ATSP and Δ-TSP in the asymmetric and symmetric case, respectively. For Δ-TSP, Christofides [9] devised a $\frac{3}{2}$ approximation algorithm with polynomial running time, whereas the best approximation algorithm for Δ-ATSP has only performance ratio $\log n$. This was shown by Frieze, Galbiati, and Maffioli [10]. See also [4] and [11] for some improvement ($0.999 \log n$ and $0.841 \log n$). Many researchers conjecture that there is a constant factor approximation algorithm also for Δ-ATSP, but this question is still open after more than two decades.

Here we consider a strengthening of the triangle inequality (1), which allows a constant factor approximation: Let γ be some constant with $\frac{1}{2} \leq \gamma < 1$. An instance of the problem $\Delta(\gamma)$-ATSP is a complete loopless directed graph G with node set V and a weight function w assigning to each edge of G a nonnegative weight. The weight function fulfills the γ-*parameterized triangle inequality*, i.e.,

$$w(u,v) \leq \gamma \cdot (w(u,x) + w(x,v)) \quad \text{for all } u, x, v \in V. \tag{2}$$

The goal is to compute a TSP tour of minimum weight.

One can also view the γ-parameterized triangle inequality as a data dependent bound. Given an instance of Δ-(A)TSP, we can compute $\tilde{\gamma} = \max\{\frac{w(u,v)}{w(u,x)+w(x,v)}\}$ and use an algorithm for $\Delta(\tilde{\gamma})$-(A)TSP to obtain better performance guarantees on instances where $\tilde{\gamma}$ is small enough.

1.1 Previous and new results

As mentioned above, for Δ-ATSP and Δ-TSP, there are approximation algorithms with polynomial running time achieving performance ratios $\log n$ and $\frac{3}{2}$, respectively.

Böckenhauer et al. [6] studied the symmetric traveling salesperson problem with γ-parameterized triangle inequality for $\gamma \in [\frac{1}{2}, 1)$. They achieve approximation performance $\min\{1+\frac{2\gamma-1}{3\gamma^2-2\gamma+1}, \frac{2}{3}+\frac{\gamma}{3(1-\gamma)}\}$. Then Böckenhauer et al. [7] (improving a result by Andreae and Bandelt [2]) as well as Bender and Chekuri [3] considered the symmetric case with γ-parameterized triangle inequality for $\gamma \geq 1$. Combining their algorithms, we get an approximation algorithm with performance guarantee $\min\{\frac{3}{2}\gamma^2, 4\gamma\}$.

Recently, Chandran and Ram [8] studied the asymmetric traveling salesperson problem with γ-parameterized triangle inequality for $\gamma \in [\frac{1}{2}, 1)$. They designed a constant factor approximation algorithm with performance ratio (asymptotically) $\frac{\gamma}{1-\gamma}$, in contrast to the $\log n$ upper bound for Δ-ATSP. Since in the asymmetric case we even do not know whether for $\gamma = 1$ an approximation algorithm with constant performance ratio exists, studying the case $\gamma \geq 1$ does not look very promising at the moment.

As our main result, we present an approximation algorithm SHORTCUT with performance ratio $\frac{1+\gamma}{2-\gamma-\gamma^3}$. This improves the result by Chandran and Ram for $\gamma \in [0.5437, 1)$, that is, for the particularly interesting large values of γ. The running time of our algorithm is $O(n^3)$, which matches the running time of the algorithm by Chandran and Ram. Our new algorithm also improves upon the conference version of this paper [5] where an approximation ratio of $\frac{2}{2-\gamma-\gamma^3}$ is obtained.

The approximation performance of SHORTCUT is better than the one obtained by Chandran and Ram [8], if

$$\frac{\gamma}{1-\gamma} \geq \frac{1+\gamma}{2-\gamma-\gamma^3} = \frac{1+\gamma}{(1-\gamma)(2+\gamma+\gamma^2)}.$$

For $\gamma < 1$ this is equivalent to $\gamma + \gamma^2 + \gamma^3 \geq 1$. The only real valued root of the corresponding polynomial can be computed exactly, it is $\frac{X}{3} - \frac{2}{3X} - \frac{1}{3}$, where $X = (17 + 3\sqrt{33})^{\frac{1}{3}}$; the numerical value of the root is approximately 0.5437. In particular, SHORTCUT is better for $\gamma \in [0.5437, 1)$. For γ close to 1, the improvement is by a factor close to 2. Figure 1 compares the performances in dependence on γ.

1.2 Notations and conventions

For a set of nodes V, let $K(V)$ denote the set of edges $(V \times V) \setminus \{(v, v) \mid v \in V\}$. Throughout this work, we are considering directed graphs $G = (V, K(V))$ together with a weight function $w : K(V) \to \mathbb{Q}_{\geq 0}$ and a

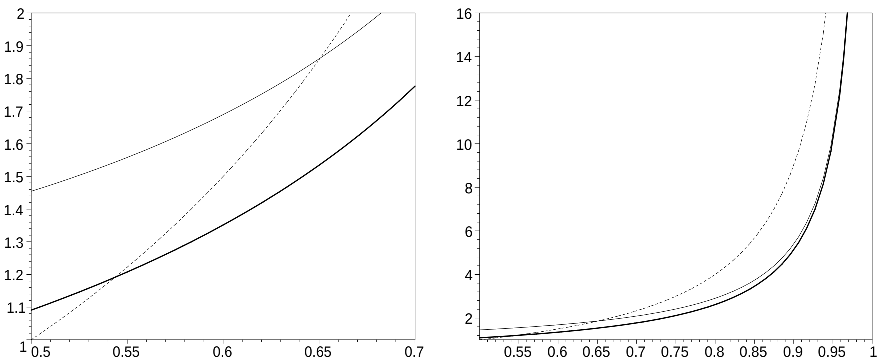

Figure 1: The approximation performance of the algorithm SHORTCUT (drawn thick) compared to the one by Chandran and Ram [8] (drawn dashed) and Bläser [5] (drawn thin).

parameter $\gamma \in [\frac{1}{2}, 1)$. We always require that w fulfills the γ-parameterized triangle inequality (2). (Note that if w fulfills the γ-parameterized triangle inequality for some γ, then necessarily $\gamma \geq \frac{1}{2}$. Thus the lower bound is no restriction.)

A cycle cover of a directed graph G is a spanning subgraph that consists solely of node disjoint directed cycles. For any subgraph $S = (V, E)$ of G, the weight $w(S)$ of S is defined as the sum of the weights of the edges in E, that is, $w(S) = \sum_{e \in E} w(e)$. In particular, this defines the weight of cycle covers and TSP tours.

For a given directed graph G with weight function w, let $\mathrm{AB}(G)$ denote the weight of a minimum weight cycle cover. (This is also called the *assignment bound.*) Furthermore, let $\mathrm{TSP}(G)$ denote the weight of a minimum weight TSP tour of G. Obviously, we have $\mathrm{AB}(G) \leq \mathrm{TSP}(G)$. $\mathrm{AB}(G)$ and a corresponding minimum weight cycle cover can be computed in polynomial time. This can be done in time $O(n^3)$; there are various algorithms with this time bound—based on bipartite matching, for example—see [1] for an overview.

2 The approximation algorithm

Our new approximation algorithm SHORTCUT generalizes the repeated cycle cover approach by Frieze, Galbiati, and Maffioli [10]. The algorithm of Frieze, Galbiati, and Maffioli without any modifications already yields a $\frac{1}{1-\gamma}$ approximation for $\Delta(\gamma)$-ATSP, as noticed in [5].

We sketch the improved algorithm now. First a minimum weight cycle cover C is computed. We split the nodes into two sets V_1 and V_2 so that they alternate in each cycle of C. Then we recursively compute two TSP tours T_1 and T_2, one in the graph G_1 induced by V_1 (i.e., $G_1 = (V_1, K(V_1))$ with weight function w_1 where w_1 is the restriction of w to $K(V_1)$) and the other one in the graph G_2 induced by V_2. Finally, we combine C and one of T_p into an Eulerian tour and a tour T by taking shortcuts. We output the shortest T of a few candidates.

Taking shortcuts is the place where we use the power of the strengthened triangle inequality. Suppose that walk S' is obtained by shortcutting from another walk S. Even with regular triangle inequality $w(S') \leq w(S)$. The additional power of the γ-parameterized triangle inequality is used when we bound $w(S')$ by the sum of the contributions of (the occurrences of) the edges $e \in S$. If the edge e appears in S', its contribution is $w(e)$. If the edge e is contracted at least once, i.e., one of its endpoints is omitted, its contribution is bounded by $\gamma \cdot w(e)$. If the edge e is contracted at least twice—for example, if both of its endpoints are omitted—its contribution further decreases and is bounded by $\gamma^2 w(e)$.

To get the best approximation factors we need to arrange the shortcuts quite carefully.

Algorithm SHORTCUT
Input: directed graph $G = (V, K(V))$ with weight function w where w fulfills the γ-parameterized triangle inequality (2) for some $\frac{1}{2} \leq \gamma < 1$
Output: TSP tour T

1. Compute a minimum weight cycle cover C of G. If C has a single cycle, let $T = C$ and stop.
2. Let $C_1, \ldots, C_t$ be the cycles of C. Denote the nodes in C_i by $v_{i1}, v_{i2}, \ldots, v_{ik_i} = v_{i0}$, in the order along the cycle starting so that the edge (v_{i0}, v_{i1}) has the minimal weight of all the edges of C_i.

Let $V_1 = \{v_{ij} \mid i = 1, \ldots, t,\ j \text{ odd}\}$ and $V_2 = \{v_{ij} \mid i = 1, \ldots, t,\ j \text{ even}\}$. Furthermore, let $\bar{V}_1 = \{v_{i1} \mid i = 1, \ldots, t\}$ and $\bar{V}_2 = \{v_{i2} \mid i = 1, \ldots, t\}$. Note that V is a disjoint union of V_1 and V_2; furthermore, each $\bar{V}_p$ has exactly one node in each cycle of C.

3. Recursively compute two TSP tours T_1 and T_2 of the graphs G_1 and G_2 that are induced by V_1 and V_2.
4. For each $p \in \{1, 2\}$, construct an Eulerian tour E_p of $(V, C \cup T_p)$ as follows. Visit the nodes of V_p in the order given by T_p. For each node $v \in \bar{V}_p$, the tour runs through the (unique) cycle in C that v belongs to; after that, it continues with the next node of T_p. For $v \in V_p \setminus \bar{V}_p$, the tour continues immediately with the next node of T_p.

   Note that each node of V_p occurs exactly twice on E_p and each node of $V - V_p$ occurs exactly once.
5. For each $p \in \{1, 2\}$, construct two Hamiltonian tours H_{p1} and H_{p2} from E_p by taking shortcuts. To do this, we need to determine for each $u \in V_p$ which of its two occurrences in E_p is removed. There are two possibilities for each node, we make sure that each of H_{pq} uses a different one. In addition, we choose a special way to do it:
   (a) First, we choose a starting node u_1 as follows: If $V_p = \bar{V}_p$, let u_1 be an arbitrary node from V_p. Otherwise let u be an arbitrary node from $V_p \setminus \bar{V}_p$; choose (u_0, u_1) to be the lighter edge of the two edges of T_p incident with u (i.e., $u = u_0$ or $u = u_1$). Denote the nodes of V_p by u_1, u_2, $\ldots$, $u_k = u_0$ in the order along the tour T_p.
   (b) For u_1, do the following: In H_{p1}, omit (i.e., take a shortcut at) the occurrence of u_1 where E_p continues by an edge from T_p (i.e., the next node is u_2); in H_{p2} omit the occurrence of u_1 where E_p continues along C.
   (c) Process the nodes u_i, for $i = 2, \ldots, k$ as follows. Consider the consecutive occurrences of u_{i-1} and u_i in E_p (i.e., the place when E_p traverses the edge (u_{i-1}, u_i). Let H_{pq} be the tour where this occurrence of u_{i-1} was not skipped. Skip (and shortcut) the occurrence of u_i following u_{i-1} in H_{pq} and skip the other occurrence of u_i in $H_{p,3-q}$.

6. Output as T the shortest tour of the four tours H_{pq}.

The next lemma uses the analysis of shortcuts to bound the weight of a minimum weight TSP tour of G_1 and G_2 in terms of the weight of a minimum weight TSP tour of G.

Lemma 2.1 *Let $V_1, V_2 \subseteq V$ be two disjoint sets of nodes. Let G_1 and G_2 be the graphs induced by V_1 and V_2, respectively. Then*

$$\mathrm{TSP}(G_1) + \mathrm{TSP}(G_2) \leq (1 + \gamma^2)\,\mathrm{TSP}(G).$$

Proof. Let T be a minimum weight TSP tour of G. Thus $w(T) = \mathrm{TSP}(G)$. We construct two TSP tours T_1 and T_2 of G_1 and G_2, respectively, such that $w(T_1)+w(T_2) \leq (1+\gamma^2)w(T)$. This proves the claim of the lemma.

Given T, we construct T_1 and T_2 by taking shortcuts. We move along the tour T starting with an arbitrary node in V_1 or V_2, respectively. Whenever we would visit a node not in V_1 or V_2, respectively, we directly go to the next node of T that is in V_1 or V_2.

Let $e = (u, v)$ be an edge of T. If both u and v belong to V_1, then the edge e appears in T_1 but is contracted (at least) twice when constructing T_2. Thus e contributes weight $w(e)$ to T_1 and $\gamma^2 w(e)$ to T_2 yielding a total contribution of $(1 + \gamma^2)w(e)$. If both u and v belong to V_2, the same analysis works. If u belongs to V_1 and v belongs to V_2 or vice versa, then e is contracted at least once to obtain T_1 and at least once to obtain T_2. Thus the total contribution is at most $2\gamma \cdot w(e) \leq (1 + \gamma^2)w(e)$. Summing over all edges e of T yields the result. $\square$

Next we use a similar analysis to show that the algorithm guarantees that the total weight of the four tours H_{pq} is bounded by $(2 + 2\gamma)w(C) + 2\gamma(w(T_1)+w(T_2))$. If the cycle cover C consists solely of cycles with exactly two nodes, it is easy to see that each edge of T_p is contracted in both H_{p1} and H_{p2} and each edge of C is contracted in exactly one of H_{p1} and H_{p2}; the bound then follows. In the general case, it may happen that some cycles in C are odd and then one of its edges may be contracted twice in one of the four tours instead of being contracted in two distinct tours. Similar problem may also happen for an edge in T_p. This would multiply the weight of the edge by $1 + \gamma^2$ instead of 2γ and thus increase its contribution. The next lemma, together with a careful choice of H_{pq}, allows a tighter bound in these cases.

Lemma 2.2 *Let $v, x, y, z \in V$ be such that $w(x, y) \le \max\{w(v, x), w(y, z)\}$. Then*

$$w(v, z) \le \gamma w(v, x) + (2\gamma - 1)w(x, y) + \gamma w(y, z).$$

Proof. Suppose that $w(x, y) \le w(v, x)$; the other case is symmetric. Then, by contracting first the node x and then y we have

$$w(v, z) \le \gamma^2 w(v, x) + \gamma^2 w(x, y) + \gamma w(y, z).$$

For any $\gamma \in [\frac{1}{2}, 1]$, we have

$$0 \le \gamma^2 - 2\gamma + 1 \le \gamma - \gamma^2,$$

as the corresponding equation has roots $\frac{1}{2}$ and 1. Thus

$$(\gamma^2 - 2\gamma + 1)w(x, y) \le (\gamma - \gamma^2)w(v, x)$$

and

$$\begin{aligned} w(v, z) &\le \gamma^2 w(v, x) + \gamma^2 w(x, y) + \gamma w(y, z) \\ &\le \gamma w(v, x) + (2\gamma - 1)w(x, y) + \gamma w(y, z). \end{aligned}$$

□

Lemma 2.3 *The algorithm SHORTCUT guarantees for any input that*

$$w(H_{11}) + w(H_{12}) + w(H_{21}) + w(H_{22}) \le (2 + 2\gamma)w(C) + 2\gamma(w(T_1) + w(T_2)).$$

Proof. We estimate the contribution of each edge $e \in E_p$, $p \in \{1, 2\}$, to the four tours H_{pq} produced by shortcuts.

First consider $e = (x, y) \in C$; such e contributes to all four tours H_{pq} and we shall bound its contribution by $(2 + 2\gamma)w(e)$. If $x \in V_p$ then e is contracted because of removing this occurrence of x in exactly one of the tours H_{p1} and H_{p2}; similarly for y. If these two contractions happen in different tours, the contribution of e is at most $(2+2\gamma)w(e)$ as claimed. If the two contractions happen in the same tour, it has to be the case that $x, y \in V_p$ for some p. By the choice of V_p (step 2 of the algorithm), for some cycle C_i, $(x, y) = (v_{i0}, v_{i1})$, and thus (x, y) is the lightest edge of C_i; in addition $p = 1$. This means that in the tour E_1, the edge (x, y) is preceded by another edge of C_i which is larger. Therefore we can apply Lemma 2.2 and bound the contribution of (x, y) to the tour with two contractions by $(2\gamma - 1)w(x, y)$

(instead of γ^2); its total contribution is then at most $(3+2\gamma-1)w(x,y)$ as claimed. Lemma 2.2 increases the bound of the preceding edge in C_i from γ^2 to γ times its weight; however, the coefficient used in the previous analysis was γ, so the analysis for the other edges remains valid.

Next consider $e=(u_i,u_{i+1})\in T_p$ for some p and numbering of nodes in T_p as in the step 5(a) of the algorithm; such e contributes to two tours H_{p1} and H_{p2}, and we shall bound its contribution by $2\gamma\cdot w(e)$. Each of the two occurrences of a node u_i in E_p is contracted in exactly one tour of H_{p1} and H_{p2}. If these contractions for the occurrences of u_i and u_{i+1} as the endpoints of e happen in two different tours, the contribution of e is at most $2\gamma\cdot w(e)$ as claimed. If the two contractions happen in a single tour, it has to be the case that $e=(u_0,u_1)$, by the choices in the step 5(c) of the algorithm. Furthermore, it has to be the case that $\bar{V}_p\neq V_p$: otherwise at every node in V_p the tour E_p follows the cycle C_i before continuing on T_p and in H_{p1} all occurrences of u_i continuing along E_p are contracted—thus the endpoints of e cannot be both contracted in the same tour. This means that one of the endpoints of e was chosen as $u\in V_p\setminus\bar{V}_p$ in step 5(a), in the tour E_p the two edges of T_p incident with u follow each other, and e is chosen as the lighter one of these two edges. Thus we can apply Lemma 2.2 (with e playing the role of (x,y) in the lemma) and bound the contribution of e to the tour with two contractions by $(2\gamma-1)w(e)$ (instead of γ^2); its total contribution is then at most $(1+2\gamma-1)w(e)$ as claimed. As before, Lemma 2.2 increases the bound of one other edge in E_p from γ^2 to γ times its weight, but this does not change the previous analysis.

The lemma now follows by summing the contributions of all the edges in C and both T_1 and T_2. The only remaining subtle point is that an edge may appear in both C and some T_p; in that case we need to sum over both of its occurrences, according to both cases above. $\square$

Now we can estimate the approximation performance of our algorithm.

Theorem 2.4 *The approximation ratio of the algorithm SHORTCUT is bounded by*

$$R=\frac{1+\gamma}{2-\gamma-\gamma^3}.$$

The running time of the algorithm is cubic in the number of nodes.

Proof. The bound on the approximation performance is shown by induction on the number of nodes. If C in the algorithm has a single cycle (which covers all graphs with at most three nodes), the output is an optimal tour.

Suppose that C has more cycles. By the induction hypothesis and Lemma 2.1,

$$w(T_1) + w(T_2) \le R(\mathrm{TSP}(G_1) + \mathrm{TSP}(G_2)) \le (1+\gamma^2)R \cdot \mathrm{TSP}(G). \qquad (3)$$

The TSP tour T computed in step 6 has weight at most

$$\begin{aligned} w(T) &\le \frac{w(H_{11}) + w(H_{12}) + w(H_{21}) + w(H_{22})}{4} \\ &\le \frac{1+\gamma}{2} w(C) + \frac{\gamma}{2}(w(T_1) + w(T_2)) \\ &\le \frac{1+\gamma}{2} \mathrm{TSP}(G) + \frac{\gamma+\gamma^3}{2} R \cdot \mathrm{TSP}(G) \\ &= R \cdot \mathrm{TSP}(G). \end{aligned}$$

by Lemma 2.3, equation (3), and the choice of R. This proves the claim about the approximation performance.

Let $S(n)$ denote the worst case running time of the algorithm on instances with n nodes. We have $S(1) = 1$ and $S(n) \le 2 \cdot S(\frac{2n}{3}) + O(n^3)$, for all $n > 1$, because each instance is divided into two subproblems of size at most $\frac{2n}{3}$. The time for computing the two subinstances is dominated by the time $O(n^3)$ used to construct the cycle cover C. Solving the recurrence, we obtain $S(n) = O(n^3)$. □

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