TREE DEPTH, SUBGRAPH COLORING AND HOMOMORPHISM BOUNDS

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To Chantal
Shall she see the rainbows.

ABSTRACT. We define the notions tree depth and upper chromatic number of a graph and show their relevance to local - global problems for graphs partitions. Particularly we show that the upper chromatic number coincides with the maximal function which can be locally demanded in a bounded coloring of any proper minor closed class of graphs. The rich interplay of these notions is applied to a solution of bounds of minor closed classes satisfying local conditions. This solves an open problem and as an application it yields the bounded chromatic number of exact odd powers of any graph in an arbitrary proper minor closed class.

1. INTRODUCTION

This paper combines techniques related to graph colorings, local - global phenomena, graph minors and graph decompositions (tree depth). The main result (Theorem 1.2) is achieved by a combination of all these texniques and in this section we explain some of the background.

1.1. **Subgraph coloring.** How to color a graph optimally? Can we guarantee that we use *globally* as few colors and *locally* as many colors as possible? This mainstream problem of graph theory is approached here from yet another point of view using the notion of *upper chromatic number* and minimum elimination tree height (called *tree depth*) of a graph. To motivate these notions we consider the following situation:

Let to any graph H be assigned a positive integer $f(H), f(H) \leq |V(H)|$. Such a function is called a *graph function*. We want to color a graph G by N colors such that any subgraph H' of G which is isomorphic to H gets at least $f(H)$ distinct colors. Clearly $N = |V(G)|$ colors suffices and the minimal such N we denote by $\chi(f, G)$.

This definition may seem to be arbitrary but in fact it captures several of the variants of chromatic number which were recently intensively studied:

Define functions $f_1, f_2, f_3, f_4, f_5, g_k (k \geq 1)$ by:

$$f_1(H) = \begin{cases} 2, & \text{if } H \approx C_n \text{ for some } n \\ 1, & \text{otherwise} \end{cases} \qquad f_2(H) = \begin{cases} 2, & \text{if } H \approx K_2 \\ 1, & \text{otherwise} \end{cases}$$

$$f_3(H) = \begin{cases} 2, & \text{if } H \approx K_2 \\ 3, & \text{if } H \approx C_n \text{ for some } n \\ 1, & \text{otherwise} \end{cases} \qquad f_4(H) = \begin{cases} 2, & \text{if } H \approx K_2 \\ 3, & \text{if } H \approx P_4 \\ 1, & \text{otherwise} \end{cases}$$

$$f_5(H) = \begin{cases} 2, & \text{if } H \approx K_2 \\ \chi(H) + 1, & \text{if } K_{\chi(H)} \not\subseteq H \\ 1, & \text{otherwise} \end{cases} \qquad g_k(H) = \min(k, \mathrm{tw}(H)) + 1.$$

(The symbol $\approx$ denotes the homomorphism equivalence, see the next section for the undefined notions.)

It appears that the numbers $\chi(f_i, G)$ are some of the well known graph invariants:

$\chi(f_1, G)$ is equal to the point-arboricity of G (i.e. to the smallest size of a vertex partition whose parts induce forests);

$\chi(f_2, G)$ is equal to the chromatic number of G;

$\chi(f_3, G)$ is equal to the acyclic chromatic number of G (i.e. to the smallest number of colors needed for a proper coloring so that no cycle gets just 2 colors). See e.g. [5, 1])

$\chi(f_4, G)$is equal to the star-coloring number of G (i.e. the smallest number of colors needed for a proper coloring so that no path of length 3 gets 2 colors only. See e.g. [2, 14]).

The number $\chi(f_5, G)$ was studied in [13] in the context of *bounds* for graph classes in the *coloring* (homomorphism) order. This connection is explained in Section 7 of this article and in this context we solve the main problem posed in [13], Conjecture 1. The main result is discussed in this introduction and formulated as Theorem 1.2 below. This application provided a motivation for our study of numbers $\chi(f, G)$ in the context of minor closed classes.

1.2. **Minors.** Why minor closed classes? It may be seen easily (applying for example some Ramsey type argument) that all the above variants $\chi(f, G)$

of the chromatic number are unbounded for general graphs and share many properties with the chromatic number. (See [11] for local - global context of colorings of general graphs.) A bit more interestingly all the numbers $\chi(f_i, G), i = 1, \ldots, 5$, are bounded for every class of graphs which does not contain a fixed minor. While for $i = 1, 2, 3, 4$ this is well known, for f_5 this was proved only recently in [13] (see also [14]) by an involved argument. In the other words, the numbers $\chi(f_i, G), i \in \{1, \ldots, 5\}$, are all bounded for every *proper minor closed* class of graphs (i.e. a minor closed class of graphs which differs from the class of all finite graphs).

Returning to the above definition of $\chi(g_k, G)$ we have the following recent result due to Devos, Oporowski, Sanders, Reed, Seymour and Vertigan [7]:

Theorem 1.1 ([7]). *For every $k \geq 1$ and for every proper minor closed class $\mathcal{K}$ is the number $\chi(F_6, G)$ bounded for all graphs $G \in \mathcal{K}$.*

Explicitly: For every proper minor closed class $\mathcal{K}$ and integer $k \geq 1$, there are integers $i_V = i_V(\mathcal{K}, k)$ and $i_E = i_E(\mathcal{K}, k)$, such that every graph $G \in \mathcal{K}$ has a vertex partition into i_V graphs such that any $j \leq k$ parts form a graph with tree-width at most $j - 1$, and an edge partition into i_E graphs such that any $j \leq k$ parts form a graph with tree-width at most j.

We shall make use of this result in the proof of Theorem 4.7.

It seems that minor closed classes are a proper context for this type of questions (local - global phenomena). Viewing all this it is perhaps surprising that in this paper we solve the dual problem *exactly* for every minor closed class. More precisely, we determine the maximal graph function $f(H)$ such that, for any graph function g bounded by f and some fixed integer, the numbers $\chi(g, G)$ are bounded for all graphs in a fixed (but arbitrary) proper minor closed class $\mathcal{K}$ of graphs. Thus we are not interested in actual value of $\chi(g, G)$ but merely in the question whether this number is bounded for *any* minor closed class of graphs. We completely characterize (Theorem 1.2) this maximal graph function f in terms of the minimum elimination tree height, the *tree depth*. This notion is introduced in section 2 and connected to *ordered coloring* (also known as *t-ranking*). In Section 4 we give a few consequences which provided a motivation for our research. The definitions we give provide a rich spectrum of results which fit naturally to our local - global framework.

1.3. **Homomorphisms bounds.** Let $\mathcal{F}$ be a finite set of graphs. By Forb($\mathcal{F}$) we denote the class of all graphs G which satisfy

$$F \nrightarrow G$$

for every $F \in \mathcal{F}$. Here $F \nrightarrow G$ denotes the non-existence of a homomorphism $F \to G$.

Thus for $\mathcal{F} = \{K_3\}$ we get the class of all triangle-free graphs and for $\mathcal{F} = \{C_5\}$ we get the class of all graphs with *odd-girth* > 5. Now we can formulate the main result of this paper:

Theorem 1.2. *For every finite set $\mathcal{F}$ of connected graphs and for every minor closed class $\mathcal{K}$ there exists a graph $H = H(\mathcal{K}, \mathcal{F}) \in \mathrm{Forb}(\mathcal{F})$ such that $G \longrightarrow H$ for any $G \in \mathcal{K} \cap \mathrm{Forb}(\mathcal{F})$.*

In the other words the class $\mathcal{K} \cap \mathrm{Forb}(\mathcal{F})$ is bounded in the class $\mathrm{Forb}(\mathcal{F})$. See [15, 13] where this is stated as a problem and the boundedness is related e.g. to the Hadwiger conjecture. One can interpret Theorem 1.2 as a finite approximation to Hadwiger conjecture. See Section 6 of this paper.

1.4. **An application - exact powers.** We now explain a consequence of our main result in a greater detail. Let G be a graph, p a positive integer. Denote by $G^{(p)}$ the graph (V, E') where $\{x, y\}$ is an edge of E' iff the distance $d_G(x, y) = p$. The graph $G^{(p)}$ could be called *exact p-power* of G. Clearly graphs $G^{(2)}$ and $G^{(p)}, p$ even, have unbounded chromatic number even for the case of trees (consider possibly subdivided stars), and the only (obvious) bound is $\chi(G^{(p)}) \leq \Delta(G)^p + 1$. Similarly, for any odd p there are graphs G for which is the chromatic number $\chi^{(p)}$ arbitrarily large. However for p odd and arbitrary proper minor closed class we have the following (perhaps surprising):

Theorem 1.3. *For every p odd and every minor closed class $\mathcal{K}$ there exists an integer $N = N(p, \mathcal{K})$ such that all the graphs $G^{(p)}, G \in \mathcal{K}$ and $\mathrm{odd-girth}(G) > p$ have chromatic number $\leq N$.*

This is a generalization of [14] where this is proved for $p = 3$ by a different method. This result is non-trivial even for planar graphs and we obtained only a very rough bounds even for this particular case. See [**?**] for better bounds for planar graphs.

It is easy to see that Theorem 1.2 implies Theorem 1.3:

Proof. Let p be odd > 1. Put $\mathcal{F} = \{C_p\}$. Let H satisfies Theorem 1.2. Put explicitly $H = (V, E), |V| = N$. Then any homomorphism $f : G \longrightarrow H$ satisfies $f(x) \neq f(y)$ whenever $d_G(x, y) = p$ (as in this case the image of G under f contains an odd cycle of length $\leq p$ which is contradiction with $H \in \mathrm{Forb}(\mathcal{F})$). Thus the homomorphism f may be thought of as an N-coloring of the graph $G^{(p)}$. $\square$

Theorems 1.2 and 1.3 motivated the present paper.

The paper is organized as follows: In Section 2 we introduce tree depth of a graph and derive some properties of this concept relevant in our context. In Section 3 we prove some finiteness and reduction theorems which will unveil the efficiency of tree-depth. In Section 4 we introduce centered colorings, relate them to vertex rankings and use Theorem 1.1 to prove that local centered colorings (*p-centered colorings*) may be obtained in proper minor closed classes with a bounded number of colors. In Section 5 we deal with subgraph colorings. It is here where we define the notion of *upper chromatic number* and prove that this coincides with the tree depth. In Section 6 we deal with homomorphisms and prove Theorem 1.2. In Section 7 we state some remarks and open problems. All the graphs considered in this paper are simple and finite.

Let us recall right at this place that a *coloring* of a graph G is an assignment of colors to the vertices of the graph, one color to each vertex. A coloring is *proper* if each the vertices of any K_2 in G gets at least 2 colors. The minimum number of colors required for a proper coloring of G is the *chromatic number* of G, denoted $\chi(G)$.

2. Tree-depth

Advancing proof of Theorem 1.2 we develop fragments of a theory of tree-depth. It will appear that this parameter can be defined equivalently in several seemingly different ways and we shall make use of this in our proofs.

2.1. **Tree-depth and Elimination Trees.** A *rooted forest* is a disjoint union of rooted trees. The *height* of a rooted forest F is the maximal number of vertices of a path from the root of a tree of F to one of its leaf and is noted $\mathrm{height}(F)$. The height of a vertex x in a rooted forest F is the number of vertices of a path from the root (of the tree to which x belongs to) to x and is noted $\mathrm{height}(x, F)$. The *closure* $\mathrm{clos}(F)$ of a rooted forest F is the graph with vertex set $V(F)$ and edge set $\{\{x, y\} : x \text{ is an ancestor of } y \text{ in } F, x \neq y\}$. A rooted forest F defines a partial order on its set of vertices, which comparability graph is $\mathrm{clos}(F)$: $x \leq_F y$ if x is an ancestor of y in F.

Definition 2.1. The *tree-depth* $\mathrm{td}(G)$ of a graph G is the minimum height of a rooted forest F such that $G \subseteq \mathrm{clos}(F)$.

This definition is analogous to the definition of *rank function* of a graph which has been recently used for analysis of countable graphs, see e.g. [16].

Let G be a connected graph. An *elimination tree* for G is a rooted tree Y with vertex set $V(G)$ defined recursively as follows. If $V(G) = \{x\}$ then Y is reduced to $\{x\}$. Otherwise choose a vertex $r \in V(G)$ as the root of Y. Let $G_1, \dots, G_p$ be the connected components of $G - r$. For each component G_i let Y_i be an elimination tree. Y is defined by making each root r_i of Y_i adjacent to r.

Lemma 2.1. *Let G be a connected graph. A rooted tree Y is an elimination tree for G if and only if $G \subseteq \operatorname{clos}(Y)$. Hence $\operatorname{td}(G)$ is the minimum height of an elimination tree for G.*

Proof. We prove the lemma by induction over the order of G. This is true if $V(G) = \{x\}$. Otherwise let r be the root of Y and let $G_1, \dots, G_p$ be the connected components of $G - r$. Then Y is an elimination tree for G if and only if the connected components of $Y - r$ may be labeled $Y_1, \dots, Y_p$ in such a way that, for any $1 \leq i \leq p$, Y_i is an elimination tree for G_i. By induction, this is equivalent to the existence of a labeling $Y_1, \dots, Y_p$ of the connected components of $G - r$ such that $G_i \subseteq \operatorname{clos}(Y_i)$, for any $1 \leq i \leq p$ and this is obviously equivalent to $G \subseteq \operatorname{clos}(Y)$. $\square$

2.2. **Basic Properties.** From Lemma 2.1 we deduce the following inductive form of the tree-depth:

Lemma 2.2. *Let G be a graph and let $G_1, \dots, G_p$ be its connected components. Then:*

$$\operatorname{td}(G) = \begin{cases} 1, & \textit{if } |V(G)| = 1; \\ 1 + \min_{v \in V(G)} \operatorname{td}(G - v), & \textit{if } p = 1 \textit{ and } |V(G)| > 1; \\ \max_{i=1}^{p} \operatorname{td}(G_i), & \textit{otherwise.} \end{cases}$$

Proof. We shall prove the lemma by induction on $\operatorname{td}(G)$. It is straightforward for graphs having tree-depth 1. Assume the lemma has been proved for graphs with tree-depth at most $t - 1$ and assume $\operatorname{td}(G) = t$.

If G is connected and has tree-depth t, there exists a rooted tree Y of height t such that $G \subseteq \operatorname{clos}(Y)$. Let r be the root of Y and let $Y - r$ denote the rooted forest of height $t - 1$ obtained from Y by removing r and considering the sons of r in Y as roots. Then $\operatorname{td}(G - r) \leq t - 1$ as $G - r \subseteq \operatorname{clos}(Y - r)$ and $\operatorname{td}(G) \geq \operatorname{td}(G - r) + 1$. Conversely, let v be any vertex of G and let F be a rooted forest of height $\operatorname{td}(G - r)$ such that $G - v \subseteq \operatorname{clos}(F)$. Let Y be the tree obtained from F by adding r adjacent to the roots of the components of F. Rooting Y at r we get $G \subseteq \operatorname{clos}(Y)$ thus $\operatorname{td}(G) \leq \operatorname{td}(G - r) + 1$. Hence $\operatorname{td}(G) = \min_{v \in V(G)} \operatorname{td}(G - v)$.

If G is disconnected, it is obvious that the minimum height rooted forest F such that $G \subseteq \operatorname{clos}(F)$ is the union of those computed for each connected components of G. Hence $\operatorname{td}(G) = \max_i \operatorname{td}(G_i)$. $\square$

We shall also stress a fundamental property of tree-depth - its monotony according to minor ordering:

Lemma 2.3. *If H is a minor of G then* $\operatorname{td}(H) \leq \operatorname{td}(G)$.

Proof. Let F be a rooted forest of height $\operatorname{td}(G)$ such that $G \subseteq \operatorname{clos}(F)$ and let $e = \{x, y\}$ be an edge of G, where x is an ancestor of y in F. Let $\pi(y)$ be the father of y in F. Then $G - e \subseteq \operatorname{clos}(F)$ and $G/e \subseteq \operatorname{clos}(F/\{y, \pi(y)\})$. $\square$

Although there is an (easy) polynomial algorithm to decide whether $td(G) \leq k$ for any fixed k, if P$\neq$NP then no polynomial time approximation algorithm for the tree-depth can guarantee an error bounded by n^ϵ, where ϵ is a constant with $0 < \epsilon < 1$ and n is the order of the graph [4].

2.3. **Tree-depth and Vertex Separators.** Let G be a graph of order n. An *α-vertex separator* of G is a subset S of vertices such that every connected component of $G - S$ contains at most αn vertices.

Lemma 2.4. *Let G be a graph of order n and let $s_G : \{1, \ldots, n\} \to \mathbb{N}$ be defined by*

$$s_G(i) = \max_{\substack{|A| \leq i,\\ A \subseteq V(G)}} \min\{|S| : S \text{ is a } \tfrac{1}{2}\text{-vertex separator of } G[A]\}$$

Then:

$$\operatorname{td}(G) \leq \sum_{i=1}^{\log_2 n} s_G\big(\frac{n}{2^i}\big)$$

Proof. We prove the lemma by induction on n. The lemma is straightforward if $n = 1$. Assume the lemma has been proved for graphs of order at most $n - 1$.

By definition of s_G, G has a $\frac{1}{2}$-vertex separator S of size at most $s_G(n)$. Let $G_1, \ldots, G_p$ be the connected components of $G - S$. Then, according to Lemma 2.2 and the fact that the function s_{G_i} corresponding to G_i is obviously bounded by s_G:

$$\operatorname{td}(G) \leq |S| + \max_i \operatorname{td}(G_i) \leq s_G(n) + \sum_{i=1}^{\log_2(n/2)} s_G\left(\frac{n/2}{2^i}\right) \leq \sum_{i=1}^{\log_2 n} s_G\left(\frac{n}{2^i}\right)$$

$\square$

Corollary 2.5 (see also [4]). *For any connected graph G of order n,* $\mathrm{td}(G) \leq (\mathrm{tw}(G)+1)\log_2 n$.

Proof. It is proved in [18] that any graph of tree-width at most k has a $\frac{1}{2}$-vertex separator of size at most $k+1$. Hence $s_G(i) \leq \mathrm{tw}(G)+1$ for all i. The result follows. □

Notice that this result is somehow optimal for tree-depth, as shown by the example of paths of length n: $\mathrm{tw}(P_n) = 1$, but $\mathrm{td}(P_n) = \lceil \log_2(n+1) \rceil$.

Corollary 2.6. *Every graph G of order n with no minor isomorphic to K_h has tree-depth at most $(2+\sqrt{2})\sqrt{h^3 n}$.*

Proof. It is proved in [3] that a graph of order i with no K_h minor has a separator of size at most $\sqrt{h^3 i}$. Hence $s_G(i) \leq \sqrt{h^3 i}$ and:

$$\mathrm{td}(G) \leq \sum_{i=1}^{\log_2 n} s_G\left(\frac{n}{2^i}\right) \leq \sqrt{h^3 n} \sum_{i=1}^{\log_2 n} \left(\frac{1}{\sqrt{2}}\right)^i \leq (2+\sqrt{2})\sqrt{h^3 n}$$

□

3. Reductions and Finiteness

In this section we shall prove two powerful reduction theorems (and finitness results) related to tree-depth.

Let G be a graph. Note $\mathcal{Y}(G)$ the set of the couples (Y, f) such that Y is a rooted forest and f an injective homomorphism from G to $\mathrm{clos}(Y)$. As shown before, $\mathrm{td}(G) = \min_{(Y,f)\in\mathcal{Y}(G)} \mathrm{height}(Y)$. An element $(Y, f) \in \mathcal{Y}(G)$ defines a coloration $\lambda_{(Y,f)}$ of vertices of G as follows:

$$\lambda_{(Y,f)}(x) = \{\mathrm{height}(x, Y)\}$$
$$\times \{(\mathrm{height}(u, Y), \mathrm{height}(v, Y)) : f(u) \leq_Y f(v) \leq_Y f(x) \text{ and } \{u, v\} \in E(G)\}$$

Definition 3.1. Let G be a graph. An automorphism $f : G \to G$ has the *fixed point property* if, for every connected subgraph H of G, $f(H) \cap H$ is either empty or contains a fixed point of f.

Theorem 3.1. *There exists a function $\digamma : \mathbb{N} \times \mathbb{N} \to \mathbb{N}$ with the following property: For any integer N, any graph G of order $n > \digamma(N, \mathrm{td}(G))$ and any mapping $g : V(G) \to \{1, \ldots, N\}$, there exists a non trivial involutive g-preserving automorphism $\mu : G \to G$ with the fixed point property (a homomorphism μ is g-preserving if $\mu \circ g = g$).*

Proof. We prove the lemma by induction over $\mathrm{td}(G)$. If $\mathrm{td}(G) = 1$, the lemma is straightforward as G has only isolated vertices. Hence if $n > N$ the graph G has two vertices with the same g-value and any automorphism has the fixed point property. Thus $\digamma(N, 1) = N$ will do. Assume the lemma has been proved for graphs of tree-depth at most $t \geq 1$ and let G be a graph of tree-depth $t + 1$.

If G is connected, there exists a vertex r such that $\mathrm{td}(G - r) = t$. Let $G_1, \ldots, G_p$ be the connected components of $G - r$. Define the mapping g' on $V(G - r)$ by

$$g'(x) = \begin{cases} (g(x), 1), & \text{if } \{x, r\} \in E(G) \\ (g(x), 0), & \text{otherwise} \end{cases}$$

If $n > \digamma(2N, t) + 1$ then, by induction, $G - r$ has a non-trivial involutive g'-preserving automorphism μ' having the fixed point property. Define μ as the extension of μ' to $V(G)$ such that $\mu(r) = r$. By construction, μ is a non-trivial involutive g-preserving automorphism of G, and it has the fixed-point property: if H is a connected subgraph of G and $\mu(H) \cap H \neq \emptyset$ then either $r \in H$ or $H \subseteq G - r$ and hence μ has a fixed point in H.

Assume G is not connected and let $G_1, \ldots, G_p$ be the connected components of G. If one of the connected components has order greater than $\digamma(2N, t) + 1$ then, according to the previous case, it has a non-trivial involutive g-preserving automorphism μ' with the fixed-point property. Extending μ to $V(G)$ by the identity, we still have a non-trivial g-preserving automorphism with the fixed-point property. Otherwise, as there exists at most $N^k 2^{\binom{k+1}{2}}$ N-colored graphs of order at most k. Hence, if $p > N^{\digamma(2N,t)+1} 2^{\binom{\digamma(2N,t)+2}{2}}$ there exists $1 \leq i < j \leq p$ and a g-preserving isomorphism from G_i to G_j. This isomorphism obviously defines a non-trivial involutive g-preserving automorphism μ of G with the fixed-point property: assume G has a connected subgraph H such that $\mu(H) \cap H \neq \emptyset$ and H contains no fixed point of μ. Then H meets G_i and G_j but include no vertices outside $G_i \cup G_j$. As H is connected and as there are no edges between vertices in G_i and vertices in G_j, we are led to a contradiction.

Thus defining $\digamma(N, t + 1) = (\digamma(2N, t) + 1) N^{(\digamma(2N,t)+1)} 2^{\binom{\digamma(2N,t)+2}{2}}$ will do. $\square$

The following two consequences indicate that tree-depth is a good "scale" for rigid graphs and even cores: For each given tree-depth we get only finitely many cores. (Note that this does not hold for tree width in a very strong sense: According to [10] the class of series parallel graphs is (countably) universal.)

Corollary 3.2. *Any rigid graph of tree-depth t has order at most $\digamma(1,t)$.*

Corollary 3.3. *For any graph G and any mapping g from $V(G)$ to a set of cardinality N, there exists a subset A of $V(G)$ of cardinality at most $\digamma(N,t)$, such that G has a g-preserving homomorphism to $G[A]$.*

In particular, any graph G is hom-equivalent to one of its induced subgraph of order at most $\digamma(1,\mathrm{td}(G))$.

Proof. If G has order $n > \digamma(N,t)$ then there exists, according to Theorem 3.1, a non-trivial g-preserving automorphism μ of G with the fixed-point property. Let F be the set of the fixed points of μ. As μ has the fixed point property, $V(G) \setminus F$ may be partitioned into complementary subsets A and B such that $B = \mu(A)$ and such that no edge exists between a vertex in A and a vertex in B.

Define $f : V(G) \to V(G) \setminus A$ by

$$f(x) = \begin{cases} \mu(x), & \text{if } x \in A \\ \mu(x) = x, & \text{if } x \in F \\ x, & \text{if } x \in B \end{cases}$$

As there is no edge between vertices of A and $\mu(A)$, $\{x,y\} \in E(G)$ imply that either $\{x,y\} \subseteq A \cup F$, in which case $f(x) = x$ and $f(y) = y$ thus $\{f(x), f(y)\} \in E(G)$, or $\{x,y\} \subseteq F \cup B$, in which case we get $f(x) = \mu(x)$ and $f(y) = \mu(y)$ thus $\{f(x), f(y)\} \in E(G)$. Altogether, f is a homomorphism from G to one of its proper induced subgraph (as μ is non-trivial, A is not empty). Iterating this construction, we eventually get a sequence of homomorphisms whose composition is a homomorphism from G to one of its induced subgraph of order at most $\digamma(N,t)$. $\square$

Corollary 3.4. *Let $k \geq 1$ be an integer. Then, the class $\mathcal{D}_k$ of all graphs G with $td(G) \leq k$ includes a finite subset $\hat{\mathcal{D}}_k$ such that, for every graph $G \in \mathcal{D}_k$, there exists $\hat{G} \in \hat{\mathcal{D}}_k$ which is hom-equivalent to G and isomorphic to an induced subgraph of G.*

Advancing yet another finitness result (Theorem 3.6) we take time out for a lemma:

Lemma 3.5. *Let G be a tree of size m having p leaves and tree-depth k. Then, $m \leq (2^{k-1} - 1)p$.*

Proof. We prove the inequality by induction over k. The inequality is obviously true for $k = 1$ and we now assume it is true for $k - 1$. Let Y be a rooted tree of height k such that $G \subseteq \mathrm{clos}(Y)$ and let v be the root of Y. The graph $G - v$ has connected components $G_1, \dots, G_{d(v)}$ which are trees of order $m_1, \dots, m_{d(v)}$ having $p_1, \dots, p_{d(v)}$ leaves, where $m = d(v) + \sum_i m_i$ and $p \leq \sum_i (p_i - 1)$. By induction, $m_i \leq (2^{k-2} - 1)p_i$. Hence, $m \leq d(v) + (p + d(v))(2^{k-2} - 1) = (p + d(v))2^{k-2} - p$. Moreover, $p \geq d(v)$ has each G_i includes at least one leaf of G. Thus, $m \leq 2^{k-1}p - p = (2^{k-1} - 1)p$. □

Theorem 3.6. *There exists a function $\mu : \mathbb{N} \to \mathbb{N}$, such that any graph G has a connected subgraph $H \subseteq G$, so that* $\mathrm{td}(H) = \mathrm{td}(G)$ *and* $|E(H)| \leq \mu(\mathrm{td}(G))$.

Proof. $\mathrm{td}(G) = 1$ means that G is isomorphic to $\overline{K_n}$, thus we can choose any vertex subgraph for H and put $\mu(1) = 0$. Assume $\mathrm{td}(G) \geq 2$ and let $k = \mathrm{td}(G)$. According to Lemma 2.3, the class $\mathcal{D}_{k-1} = \{G : \mathrm{td}(G) \leq k - 1\}$ is a proper minor closed class of graphs. Thus, (using Robertson - Seymour minor graph theorem) there exists a finite set $\mathcal{F}_{k-1}$ of forbidden minors for the class $\mathcal{D}_{k-1}$. As $G \notin \mathcal{D}_{k-1}$, there exists $K \in \mathcal{F}_{k-1}$, so that K is a minor of G. Moreover, we may assume that G is minimal in the sense that any edge deletion decreases the tree depth of G. Thus G is connected and, for any edge e, K is not a minor of $G - e$. Hence, K is obtained from G by contracting some connected trees into single vertices, and deleting at most one edge for any connected components of K but one. By minimality of G, each vertex v of K is obtained by contracting a tree G_v of G of tree-depth at most $k - 1$ having at most $d(v)$ extremal vertices. According to Lemma 3.5, G_v has size at most $(2^{k-2} - 1)d(v)$. Altogether, G has at most $2^{k-2}|E(K)| + c_0(K) - 1$ edges, where $c_0(K)$ is the number of connected components of K. Put $\mu(k) = \max_{K \in \mathcal{F}_k} 2^{k-2}|E(K)| + c_0(K) - 1$. □

4. Centered Colorings

4.1. Centered Colorings and Vertex Rankings.

Definition 4.1. A *centered coloring* of a graph G is a vertex coloring such that, for any (induced) connected subgraph H, some color $c(H)$ appears exactly once in H.

Note that a centered coloring is necessarily proper. We can relate the minimum number of colors in a centered coloring to the notion of vertex ranking number which has been investigated in [6],[19]: The *vertex ranking* (or *ordered coloring*) of a graph is a vertex coloring by a linear ordered set of colors such that for every path in the graph with end vertices of the same color there is a vertex on this path with a higher color. A vertex-coloring $c : V(G) \to \{1, \dots, t\}$ with this property is a *vertex t-ranking* of G. The minimum t such that G has a vertex t-ranking is the *vertex ranking number* of G and is noted $\chi_r(G)$ (see [6],[19]).

Lemma 4.1. *Any vertex ranking is a centered coloring and conversely any centered coloring defines a vertex ranking with the same number of colors. Thus $\chi_r(G)$ is the minimum number of colors in a centered coloring of G.*

Proof. Assume c is a vertex ranking of a graph G and let H be a connected subgraph of G. Let $i = \max_{v \in V(H)} c(v)$. Then H has at most one vertex colored i as for otherwise the path linking them would include a vertex with color $j > i$.

Conversely, assume f is a centered coloration of G using t colors. We shall prove by induction over t that f defines a vertex t-ranking of G. As we may consider independently each connected component of G, we may assume G is connected. As f is a centered coloring there exists a color α which appears exactly once in G, at a vertex v. As the restriction of f to $G - v$ is a centered coloring using $t - 1$ colors, it defines (by induction) a vertex $(t-1)$-ranking c of $G - v$. We extend c to G by defining $c(v) = t$. Now any path linking two vertices with the same c-color i is either a path of $G - v$ (so includes a vertex of c-color $j > i$) or includes v which has c-color t. □

Lemma 4.2. *Let G be a graph. Then,* $\mathrm{td}(G)$ *is the minimum number of colors in a centered coloring of G.*

Proof. Notice that the minimum number of colors in a centered coloring of G is the maximum of the minima computed on the connected components of G. As $\mathrm{td}(G)$ is the maximum tree-depth of the connected components of G, we may restrict our proof to the case where G is connected.

We first prove that $\mathrm{td}(G)$ is at most equal to the number of colors in any centered coloring of G, by induction on the number k of colors in the centered coloring. If $k = 1$, $G = K_1$ and thus $\mathrm{td}(G) = 1$. Assume we have proved $\mathrm{td}(G) \leq k$ if $k \leq k_0$, and assume $k = k_0 + 1$. There exists a color c_0 which appears only once in G, at a vertex v_0. Each of the connected components $G_1, \dots, G_p$ of $G - v_0$ has a centered coloring using $k - 1$ colors, and thus has depth at most $k-1$. Let $Y_1, \dots, Y_p$ be trees rooted at $r_1, \dots, r_p$, such that $G_i \subseteq \mathrm{clos}(Y)$ and $\mathrm{height}(Y_i) = \mathrm{td}(G_i)$. Then the tree Y with root

v_0 and subtrees $Y_1, \ldots, Y_p$ is such that $G \subseteq \operatorname{clos}(Y)$ and $\operatorname{height}(Y) \leq k+1$. Thus, $\operatorname{td}(G) \leq k+1$.

Now, we prove the opposite inequality, that is that $\operatorname{td}(G)$ is at least equal to the number of colors in some centered coloring of G: Let Y be a rooted tree of height $\operatorname{td}(G)$, such that $G \subseteq \operatorname{clos}(Y)$. Color each vertex by its height in Y, thus using $\operatorname{td}(G)$ colors. According to the structure of $\operatorname{clos}(Y)$, any connected subgraph H of $\operatorname{clos}(Y)$ (and thus any connected subgraph of G) has a vertex which is minimum in Y. The color assigned to this vertex hence appears exactly once in H, and the constructed coloring is thus a centered coloring of G. □

Remark 4.3. According to the construction used above, if G has a centered coloring and $e = \{x, y\}$ is an edge of G, then the graph G/e has a centered coloring. This can be deduced by modifying a centered coloring of G: the vertex corresponding to x and y has either the color of x or the color of y and all the other vertices of G/e have the same color they have in G.

In the case G is connected, we obtain:

Corollary 4.4. *Let G be a connected graph. Then, $\operatorname{td}(G), \chi_r(G)$, the minimum height of an elimination tree for G and the minimum number of colors in a centered coloring of G are equal numbers.*

Remark that the equality of $\chi_r(G)$ and of the minimum height of an elimination tree [1] already appears in [6], but we reproved it here for completeness.

4.2. **p-centered colorings of minor closed classes.** We introduce p-centered colorings, as a local approximation of centered-colorings:

Definition 4.2. A *p-centered coloring* of a graph G is a vertex coloring such that, for any (induced) connected subgraph H, either some color $c(H)$ appears exactly once in H, or H gets at least p colors.

We are aiming for Theorem 4.7. The proof will be an easy combination of finiteness Theorem 3.6 and of the following lemmas.

Lemma 4.5. *Let G, G_0 be a graph, let $p = \operatorname{td}(G_0)$, let c be a q-centered coloring of G where $q \geq p$. Then any subgraph H of G isomorphic to G_0 gets at least p colors in the coloring of G.*

[1] In [6], the "height" is the maximum length of a path from the root to a leaf, that is 1 less than given by our definition.

Proof. We prove the lemma by induction on the order of G_0. If $G_0 \approx K_1$, the lemma is straightforward. Assume the lemma has been proved for graphs G_0 of order at most $n-1$ and let G_0 be a graph of order $n > 1$.

If G_0 is not connected, the tree-depth of G_0 equals the tree-depth of one of its connected components whose copies, by induction, get at least p colors. So we are done.

Otherwise, let H be a subgraph of G isomorphic to G_0. According to the definition of a q-centered coloring, either H gets at least $q \geq p$ colors, or there exists a color which appears only once on $V(H)$, at a vertex r. According to Lemma 2.2, $\mathrm{td}(H) \leq 1 + \mathrm{td}(H-r)$ thus $\mathrm{td}(H-r) \geq p-1$. By induction, $H-r$ gets at least $p-1$ colors in the coloring of G. Hence H gets at least p colors. □

Lemma 4.6. *Let p, k be integers. Then, there exists an integer $N(p,k)$, such that any graph G with tree width at most k has a p-centered coloring using $N(p,k)$ colors.*

Proof. It is sufficient to prove the lemma in the case where G is a k-tree.

As G is a k-tree, it has an *acyclic fraternal orientation*, that is an acyclic orientation such that $(x,z) \in E(G)$ and $(y,z) \in E(G)$ implies $(x,y) \in E(G)$ or $(y,x) \in E(G)$. In such an orientation, vertices have indegree at most $k-1$.

Let G^+ be the directed graph obtained from G by adding an arc (x,y) if x and y are not adjacent and if there exists in G a directed path of length at most p from x to y. This way, the in-degrees increase by at most k^p, and thus the chromatic number of G^+ is bounded by a function $N(p,k)$. In the following we consider a coloration of the vertices of G induced by a proper coloration of G^+ using at most $N(p,k)$ colors.

Consider a connected induced subgraph H of G and the partial order induced by the orientation of G. As all the in-neighbors in H of a vertex of H form a clique, they are comparable and thus each vertex has a unique predecessor in the partial order. The arc joining a vertex to its predecessor defines a tree $Y \subseteq H$ (by connectivity of H) and the arcs entering a vertex x only comes from ancestors of x.

On the one hand, if a vertex x of H has at least $(p-1)$ ancestors in Y, it is the endpoint of a directed chain of G of length p (as $Y \subseteq G$). In this case $G^+[V(H)]$ includes a clique of size p, and hence $V(H)$ receives at least p colors in a proper coloring of G^+.

On the other hand, if no vertex of H has at least $(p-1)$ ancestors in Y then $G^+[V(H)] = \mathrm{clos}(Y)$ and the root of Y has a color which appears exactly once in H.

Altogether, the coloration of G deduced from a proper coloration of G^+ (using at most $N(p,k)$ colors) is a p-centered coloration. $\square$

After all these steps we arrive to the following:

Theorem 4.7. *For every graph K and integer $p \geq 1$, there exists integer $p_V = p_V(K,p)$, such that every graph with no K-minor has a p-centered coloring using p_V colors.*

Proof. Let G be a graph with no K-minor. According to Theorem 1.1, there exists a vertex partition into $i_V = i_V(K, p+1)$ parts, such that any p parts form a graph of tree width at most $p-1$. Let G_i be the graph induced by all the parts but the ith (for $1 \leq i \leq i_V$). According to Lemma 4.6, each of the G_i has p-centered coloring using $N(p, p-1)$ colors. Take the product of the coloring of G by i_V colors and of the colorings of the G_i as a new coloring of G (with $p_V = i_V N(p, p-1)^{i_V}$ colors). Let H be a connected subgraph of G. Then, either G gets at least p colors, or $V(H)$ is included in some subgraph G_i of G induced by $p-1$ parts. In the later case, some color appears exactly once in H. $\square$

We shall prove that Theorem 4.7 is optimal in the following sense:

Proposition 4.8. *For any integers p, k, N, for any graph H with tree depth p, there exists a graph G with no K_{p+1} minor, such that for any N-coloring of the vertices of G, there exists a subgraph of G isomorphic to H that receives at most p colors.*

Proof. The class $\mathcal{D}_k = \{G : \mathrm{td}(G) \leq p\}$ has K_{p+1} as a forbidden minor. According to Ramsey theorem, any sufficiently "large" N-colored rooted forest Y of height p contains a "large" sub-forest Y' of height p (such that $H \subseteq \mathrm{clos}(Y')$) whose levels are monochromatic. Thus, this subgraph receives at most p colors. $\square$

Remark 4.9. Notice that this coloring is actually p-centered.

5. Subgraph coloring

Our way of generalizing proper colorings is to consider, (for a given graph function f) the minimum number of colors required, so that any subgraph H of G gets at least $f(H)$ colors. (Recall for instance that the star coloring corresponds to the graph function where any P_4 gets at least 3 colors.)

As explained in the introduction, in this context "natural" families of graphs are proper minor closed classes of graphs. These families have bounded density, bounded chromatic number, bounded star coloring number, etc. This boundedness property is captured by the following definition:

Definition 5.1. The *upper chromatic number* of a graph H is the greatest integer $\overline{\chi}(H)$, such that, for *any* proper minor closed class of graph $\mathcal{K}$, there exists a constant $k(\mathcal{K}, H)$, such that any graph $G \in \mathcal{K}$ has a coloring using at most $k(\mathcal{K}, H)$ colors so that any subgraph of G isomorphic to H gets at least $\overline{\chi}(H)$ colors.

A bit surprisingly the upper chromatic number is not a new parameter and it can be determined by means of results of previous section:

Theorem 5.1. *For any graph G, $\overline{\chi}(G) = \mathrm{td}(G)$.*

Proof. According to Proposition 4.8, there exists a proper minor closed class of graph such that, for any integer N, there exists a graph in the class which will include a copy of G with at most $\mathrm{td}(G)$ colors, whatever N-coloration we choose on the graph. Thus $\overline{\chi}(G) \leq \mathrm{td}(G)$.

Let $p = \mathrm{td}(G)$. According to Theorem 4.7, for any proper minor closed class $\mathcal{K}$ of graphs, there will exist a integer p_V such that any graph $X \in \mathcal{K}$ has a p-centered coloring using p_V colors. According to Lemma 4.5, any copy of G will get at least p colors. Thus $\overline{\chi}(G) \geq \mathrm{td}(G)$. □

¿From the elementary properties of $\mathrm{td}(G)$, we get:

Corollary 5.2. *If H is a minor of G, then $\overline{\chi}(H) \leq \overline{\chi}(G)$.*

It follows that $\overline{\chi}$ is a minor monotone invariant (as opposed to the chromatic number). But there seems to be even more structure here as indicated by the following:

Given a graph G, we may also be concerned by the minimum number of colors ensuring that any subgraph has many colors. More formally, we shall introduce the following family of chromatic numbers:

Definition 5.2. Let G be a graph of order n and let k be an integer. The kth chromatic number $\chi_k(G)$ is the smallest integer N, such that G may be N-colored in such a way that, for any $H \subseteq G$, H gets at least $\min(k, \overline{\chi}(H))$ colors.

Thus:

$$1 = \chi_1(G) \leq \chi(G) = \chi_2(G) \leq \chi_3(G) \leq \cdots \leq \chi_n(G) = \cdots = \chi_\infty(G) \tag{1}$$

It follows that Theorem 4.7 may be reformulated as follows:

Corollary 5.3. *For any proper minor closed class of graphs $\mathcal{K}$ and for any fixed integer $p \geq 1$, $\chi_p(G)$ is bounded on $\mathcal{K}$.*

The following theorem justifies the term of "upper chromatic number" and the restriction that p shall be bounded in the previous corollary:

Theorem 5.4. *For any graph G, $\chi_\infty(G) = \overline{\chi}(G)$.*

Proof. On one hand, considering $H = G$ in the definition, we get $\chi_\infty(G) \geq \overline{\chi}(G)$. On the other hand, let Y be a rooted forest of height $\mathrm{td}(G) = \overline{\chi}(G)$ (according to Theorem 5.1). Then, $G \subseteq \mathrm{clos}(Y)$. Color the vertices of Y according their height in Y (thus, with $\overline{\chi}(G)$ colors). Then, for any connected subgraph $H \subseteq G$, let Y' be the subgraph of Y with vertices having the same color than at least one vertex in $V(H)$. Then, $H \subseteq \mathrm{clos}(Y')$ and thus, choosing the connected components of Y' whose closure include H, we may get a sub-forest Y'' of Y, such that $H \subseteq \mathrm{clos}(Y'')$, and thus $\mathrm{td}(H) \leq \mathrm{height}(Y'')$. As $\mathrm{height}(Y'')$ is the number of colors in H, H gets at least $\mathrm{td}(H)$ colors. $\square$

6. Homomorphisms

We found useful to define the following "truncated" products. This construction is similar to those given in [13, 14].

Definition 6.1. Let K be a finite graph and let $p \geq 2$ be an integer. The *p-extension* $K^{\Uparrow p}$ is the graph with vertex set $W_1 \cup \cdots \cup W_p$, where

$$\begin{aligned} W_i &= \overbrace{V(K) \times \cdots \times V(K)}^{i-1} \times \{\omega\} \times \overbrace{V(K) \times \cdots \times V(K)}^{p-i} \\ &= \{(x_1, \ldots, x_{i-1}, \omega, x_{i+1}, \ldots, x_p) : x_k \in V(K) \text{ for } k \neq i\} \end{aligned}$$

and

$$\begin{aligned} E(K^{\Uparrow p}) = \{\{&(x_1, \ldots, x_{i-1}, \omega, x_{i+1}, \ldots, x_p), (y_1, \ldots, y_{j-1}, \omega, y_{j+1}, \ldots, y_p)\} : \\ &i \neq j \text{ and } \forall k \notin \{i, j\}, \{x_k, y_k\} \in E(K)\} \end{aligned}$$

Lemma 6.1. *Let $\mathcal{F}$ be a finite set of finite graphs, let G be a graph and let p be an integer strictly greater than* $\max\{|V(K)| : K \in \mathcal{F}\}$.

If $G \in \mathrm{Forb_h}(\mathcal{F})$, then $G^{\Uparrow p} \in \mathrm{Forb_h}(\mathcal{F})$.

Proof. Assume there exists an homomorphism $\phi : K \to G^{\Uparrow p}$, where $K \in \mathcal{F}$. As $|V(K)| < p$, there exists k, such that $\phi(x) \notin W_k$ for all $x \in V(K)$ (recall W_k is the set of the vertices of $G^{\Uparrow p}$ having ω at position k). Hence, denoting $\phi_k(x)$ the kth coordinate of $\phi(x)$, we have $\{\phi_k(x), \phi_k(y)\} \in E(G)$

for any edge $\{x,y\}$ of K. Thus, $\phi_k \circ f_k$ is a homomorphism from K to G, a contradiction. Hence, $G^{\Uparrow p} \in \mathrm{Forb_h}(\mathcal{F})$. $\square$

Lemma 6.2. *Let U be a graph and let $p > 1$ be an integer.*

Assume a graph G has a p-proper coloring inducing a vertex partition $V(G) = V_1 \cup \cdots \cup V_p$ such that $G[V(G) \setminus V_i] \to U$, for any $1 \leq i \leq p$.

Then, $G \to U^{\Uparrow p}$.

Proof. Let f_i $(1 \leq i \leq p)$ be an homomorphism from $G[V(G) \setminus V_i]$ to U.

Let $f : V(G) \to V(U^{\Uparrow p})$ be defined on V_i by

$$f(x) = (f_1(x), \ldots, f_{i-1}(x), \omega, f_{i+1}(x), \ldots, f_p(x)\}$$

Let $\{x,y\}$ be an edge of $E(G)$, $x \in V_i$, $y \in V_j$ ($i \neq j$ as the coloring is proper). Then, for any $k \notin \{i,j\}$, as f_k is an homomorphism from $G[V(G) \setminus V_k]$ to U, $\{f_k(x), f_k(y)\} \in E(U)\}$. Hence, $\{f(x), f(y)\} \in E(U^{\Uparrow p})$, what proves $G \to U^{\Uparrow p}$. $\square$

Lemma 6.3. *Let $\mathcal{F}$ be a finite set of finite graphs, let U be a graph in $\mathrm{Forb_h}(F)$, let p, q be integers such that $q > p \geq \max\{|V(K)| : K \in \mathcal{F}\}$.*

Assume a graph G has a q-proper coloring inducing a vertex partition $V(G) = V_1 \cup \cdots \cup V_q$ such that any subgraph of G induced by p colors has a homomorphism to U.

Then, $U' = U^{\Uparrow(p+1)\cdots\Uparrow q}$ is such that $U' \in \mathrm{Forb_h}(\mathcal{F})$ and $G \to U'$

Proof. We prove this lemma inductively on $q - p$. If $q = p + 1$, this is Lemma 6.2 and Lemma 6.1.

Assume the lemma has been proved for $q-p = a$ and assume $q = p+a+1$. According to Lemma 6.2 (considering the subgraphs induced by p colors of a subgraph of G induced by $(p+1)$ colors), any subgraph H of G induced by $p+1$ colors has a homomorphism to $U^{\Uparrow(p+1)}$ and, according to Lemma 6.1, $U^{\Uparrow(p+1)} \in \mathrm{Forb_h}(\mathcal{F})$. Thus, the result follows from the induction hypothesis applied on $\mathcal{F}, U^{\Uparrow(p+1)}, (p+1)$ and q. $\square$

Actually, we are now able to prove that the problem of finding a universal graph for $\mathcal{K} \cap \mathrm{Forb_h}(\mathcal{F})$ only needs a resolution for the cases where $\mathcal{K}$ is a class of graph with bounded tree depth:

Theorem 6.4. *Let $\mathcal{F}$ be a finite set of finite connected graphs. Then, for any proper minor closed class of graph $\mathcal{K}$ there exists a finite graph $U(\mathcal{K}, \mathcal{F}) \in \mathrm{Forb_h}(\mathcal{F})$ such that any graph of $\mathcal{K} \cap \mathrm{Forb_h}(F)$ has a homomorphism to $U(\mathcal{K}, \mathcal{F})$.*

Proof. Let $p = \max_{K\in\mathcal{F}} |V(K)| + 1$. There exists an integer N, such that any graph $G \in \mathcal{K}$ has a proper N-coloring in which any p colors induce a graph of tree depth at most p. According to Corollary 3.4, there exists a finite set $\hat{\mathcal{D}}_k$ of graphs with tree depth at most k, so that any graph with tree-depth at most k is hom-equivalent to one graph in the set. Let $U(\mathcal{D}_k, F)$ be the disjoint union of the graphs in $\hat{\mathcal{D}}_k \cap \mathrm{Forb_h}(F)$. According to Lemma 6.3, $U(\mathcal{K},\mathcal{F}) = U(\mathcal{D}_k, F)^{\Uparrow(p+1)\cdots\Uparrow N}$ will work. □

Corollary 6.5. *For any proper minor closed class $\mathcal{K}$ and any odd integer $p \geq 1$, there exists an integer $N(\mathcal{K}, p)$ so that, for any $G \in \mathcal{K}$:*

$$\mathrm{odd-girth}(G) > p \Longrightarrow \chi(G^{(p)}) \leq N(\mathcal{K}, p)$$

Proof. Let U be a graph in $\mathrm{Forb_h}(C_p)$ which is universal for $\mathcal{K}\cap\mathrm{Forb_h}(C_p)$. Then, for any $G \in \mathcal{K}$ with odd girth strictly greater than p, $G \to U$. Thus, $G^{(p)} \to U^{(p)}$ and hence $\chi(G^{(p)}) \leq \chi(U^{(p)})$. □

7. Remarks and Open Problems

Some of the results of this paper may be formulated (and in fact are motivated) by the quasiorder (and partial order) induced by the existence of a homomorphism:

Given graphs G, H we denote by $G \leq H$ the existence of a homomorphism $G \to H$. Clearly $\leq$ is a quasiorder. If we consider isomorphism types of cores then we obtain a partial order. This quasiorder (and partial order) is called *colouring order* (or *homomorphism order*) and it is denoted by $\mathcal{C}$. We also denote by $<$ the strict version of $\leq$. For a graph H we denote by $\mathcal{C}_H$ the principal ideal determined by H: $\mathcal{C}_H = \{G; G \leq H\}$. $\mathcal{C}_H$ is also called a *colour class*. This name is justified by interpreting homomorphisms as generalized colourings: Indeed, a homomorphism $G \to K_k$ is a just a (proper) k-colouring of graph G and, more generally, a homomorphism $G \to H$ is called a H–colouring. Thus $\mathcal{C}_H$ is the class of all H-colourable graphs; hence the name colour class. It follows that the question whether $G \leq H$ is difficult to decide (and it is NP-complete in a very strong sense).

It is perhaps surprising how many fine combinatorial questions are captured by *order – theoretic* properties of the colouring order $\mathcal{C}$. Our paper is related to extremal elements of this order: greatest and maximal elements, suprema and (upper) bounds in general. It appears that these extremal graphs capture various problems which are as remote as *duality theorem* ([17]) and Hadwiger conjecture (as shown in [15]). These interpretations also lead to some, hopefully interesting, problems.

Given a class $\mathcal{K}$ of graphs it is usually a difficult question to find a graph H which is maximal (or greatest, or supremum) of $\mathcal{K}$ in $\mathcal{C}$ as such a result yields maximal chromatic number of a graph in $\mathcal{K}$. We review these familiar concepts in the setting of colouring order $\mathcal{C}$:

A graph H is said to be an *(upper)bound* of $\mathcal{K}$ if every graph $G \in \mathcal{K}$ satisfies $H \leq G$. If in addition $H \in \mathcal{K}$ then H is said to be *greatest* graph in $\mathcal{K}$.

A graph H is said to be *maximal* of $\mathcal{K}$ if $H \in \mathcal{K}$ and no graph $G \in \mathcal{K}$ satisfies $G < H$.

A graph H is said to be *supremum* of $\mathcal{K}$ if $G \leq H$ for every $G \in \mathcal{K}$ and if for every graph $H' < H$ there exists a graph $G \in \mathcal{K}$ such that $G \not\leq H'$.

For example, using this terminilogy, the 4-colour theorem says that K_4 is the greatest graph in the class of all planar graphs. This obviously cannot be improved. On the other hand, Grötzsch's theorem says that K_3 is an upper bound of the class of all planar K_3–free graphs. However this may be improved as K_3 fails to be a supremum of this class. Indeed, by [14] and also by Theorem 1.2, there exists a graph H which is triangle free and which is an upper bound for the class of all triangle free planar graphs. Then the graph $H' = H \times K_3$ is also a bound which moreover satisfies $H' < K_3$.

Note, that in this case we do not know whether a supremum exits.

Using this terminology Theorem 1.2 may be formulated as follows:

Theorem 7.1. *For every finite set $\mathcal{F}$ of connected graphs and for every minor closed class $\mathcal{K}$ the class $\mathcal{K} \cap \mathrm{Forb}(\mathcal{F})$ is bounded in the class* $\mathrm{Forb}(\mathcal{F})$*.*

This results also nicely complements a similar result obtained for classes of bounded degree graphs (instead of proper minor closed classes), see [8, 9].Perhaps more interestingly, each of the classes $\mathrm{Forb}(\mathcal{F}) \cap \mathcal{C}_H$ has supremum H.

As pointed above the Hadwiger conjecture amounts to the fact that any minor closed class $\mathcal{K}$ of graphs has greatest element (in the homoorphism order) and this element is the complete graph. If true then it follows that K_h is a bound for a minor closed class $\mathcal{K}$ of graphs where $h = h(\mathcal{K})$ is the Hadwiger number of $\mathcal{K}$. While this is an open problem we at least found a bound not containing K_{h+1}.

Note that for undirected graphs (without structural restrictions) the class $\mathrm{Forb}(\mathcal{F})$ is bounded only in trivial cases (bipartite graphs). Perhaps more interestingly, each of the classes $\mathrm{Forb}(\mathcal{F}) \cap \mathcal{C}_H$ has supremum H, see [14].

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