The Subchromatic Index of Graphs

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Abstract

In an edge coloring of a graph, each color class forms a subgraph without path of length two (a matching). An edge subcoloring generalizes this concept: Each color class in an edge subcoloring forms a subgraph without path of length three. While every graph with maximum degree at most two is edge 2-subcolorable, we point out in this paper that recognizing edge 2-subcolorable graphs with maximum degree three is NP-complete, even when restricted to triangle-free graphs. As by-products, we obtain NP-completeness results for the star index and the subchromatic number for several classes of graphs. It is also proved that recognizing edge 3-subcolorable graphs is NP-complete.

Moreover, edge subcolorings and subchromatic index of various basic graph classes are studied. In particular, we show that every unicyclic graph is edge 3-subcolorable and edge 2-subcolorable unicyclic graphs have a simple structure, allowing an easy linear time recognition. We also present an algorithm for testing edge k-subcolorability for graphs of bounded treewidth.

1 Introduction

Let $G = (V, E)$ be a graph. An *independent set* (a *clique*) is a set of pairwise nonadjacent (adjacent) vertices. For $W \subseteq V$, the subgraph of G induced by

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W is denoted by $G[W]$. For $F \subseteq E$, $V(F)$ denotes the set of endvertices of edges from F, and $G(F) = (V(F), F)$ is the subgraph of G induced by the edge set F.

A (proper) *vertex r-coloring* of G is a partition $V_1, \dots, V_r$ into disjoint independent sets, called *color classes* of the coloring. The *chromatic number* $\chi(G)$ is the smallest number r for which G admits a vertex r-coloring. One of the most interesting generalizations of the classical vertex coloring is the notion of vertex subcoloring; see [2, 9, 12, 15]. A *vertex r-subcoloring* is a partition $V_1, \dots, V_r$ of V where each color class V_i consists of disjoint cliques (of various sizes). The smallest number r for which G has a vertex r-subcoloring is called the *subchromatic number* $\chi_s(G)$ of G.

Note that a partition $V_1, \dots, V_r$ of V is a vertex r-coloring of $G = (V, E)$ if and only if, for each i, $G[V_i]$ does not contain a P_2 as an (induced) subgraph, and the partition is a vertex r-subcoloring if and only if, for each i, $G[V_i]$ does not contain a P_3 as an induced subgraph (P_k denotes the path on k vertices).

A (proper) *edge r-coloring* is a partition $E_1, \dots, E_r$ of E into color classes E_i in which every two distinct edges do not have an endvertex in common, i.e., each E_i forms a matching. The *chromatic index* $\chi'(G)$ is the smallest number r for which G admits an edge r-coloring. Clearly, a partition $E_1, \dots, E_r$ of E is an edge r-coloring of $G = (V, E)$ if and only if, for each i, $G(E_i)$ does not contain a P_3 as a (not necessarily induced) subgraph. This observation leads to the following natural generalization of the classical edge coloring.

Definition. *An* edge r-subcoloring *of the edges of a graph $G = (V, E)$ is a partition $E_1, \dots, E_r$ of E into disjoint color classes E_i such that each $G(E_i)$ does not contain a P_4 as a (not necessarily induced) subgraph. The* subchromatic index $\chi'_s(G)$ *is the smallest number r for which G admits an edge r-subcoloring.*

Remark. Obviously, a partition $E_1, \dots, E_r$ of $E(G)$ is an edge r-coloring of G if and only if, for each i, the connected components of $G(E_i)$ are stars or triangles, where a star is a complete bipartite graph $K_{1,s}$ for some $s \geq 1$.

A related notion that has been studied in the literature is as follows. A partition $E_1, \dots, E_r$ of $E(G)$ is a *star partition* of G if, for each i, the connected components of $G(E_i)$ are stars. The *star index* $\chi^*(G)$ of G is the smallest number r for which G has a star partition into r subsets E_i; cf. [1, 3, 4, 5, 11, 16, 19].

Clearly, for all graphs G, $\chi'_s(G) \leq \chi^*(G)$, and $\chi'_s(G) = \chi^*(G)$ whenever G is triangle-free.

Recall that the *line graph* $L(G)$ of a graph G has the edges of G as vertices and two distinct edges e, e' are adjacent in $L(G)$ whenever they have an endvertex in common. It is well-known that proper edge colorings of G correspond to proper vertex colorings of $L(G)$ and vice versa. In particular, $\chi'(G) = \chi(L(G))$. Likewise, we have the following easy to see fact.

Fact 1 *Edge subcolorings of a graph G correspond to vertex subcolorings of the line graph $L(G)$ of G and vice versa. In particular, $\chi'_s(G) = \chi_s(L(G))$.*

Our terminology of edge subcoloring is intended to recall this fact.

2 Basic properties and examples

First, since the subchromatic index of a graph is the maximum subchromatic index among those of its connected components, so we assume throughout this paper that all graphs are connected.

The next proposition means that subcolorings provide monotone property.

Proposition 1 *For any graph G and any vertex v of G,*

$$\chi'_s(G) \leq \chi'_s(G \setminus v) + 1.$$

Proof: Since any subcoloring of $G \setminus v$ can be extended to a subcoloring of G by using an extra new color on all edges incident with v. $\Box$

General lower and upper bounds for the subchromatic index are given below. Let $\Delta(G)$ be the maximum degree of a vertex in the graph G.

Proposition 2 *For any graph G with m edges on n vertices,*

$$\frac{m}{n} \leq \chi'_s(G) \leq \Delta(G).$$

Moreover, if G is triangle-free, $\chi'_s(G) \geq \frac{m}{n} + 1$.

Proof: Since every color class consists of stars and triangles, it may contain at most n edges. In the subcoloring each edge has to be colored and the lower bound follows.

Note that a color class in a graph on n vertices can have n edges, iff n is a multiple of 3 and the class itself is a covering of the vertices by $\frac{n}{3}$ disjoint triangles. Hence for triangle-free graphs the lower bound can be shifted by at least 1.

To obtain the upper bound, we have from Fact 1: $\chi'_s(G) = \chi_s(L(G)) \leq \lceil\frac{\Delta(L(G))}{2}\rceil + 1 = \lceil\frac{2\Delta(G)-2}{2}\rceil + 1 = \Delta(G)$, where the inequality for the subchromatic number was shown in [2].

To find a valid subcoloring using at most $\Delta(G)$ edge colors efficiently we may proceed greedily on the vertex set: With each new vertex u assign colors to its adjacent edges as follows: for an edge (u, v) pick a color that is not used on no already colored edge incident with v. Such a subcoloring is triangle-free and all stars have the property that the center of the star is the latest vertex of the star in the order. □

Observe that the upper bound is attained e.g. for the 5-cycle $\chi'_s(C_5) = 2 = \Delta(C_5)$ or the Petersen graph $\chi'_s(P) = 3 = \Delta(P)$. The last property follows for any cubic graph which contains C_5 as an induced subgraph: it is impossible to extend a valid 2-subcoloring to all edges incident to the cycle C_5. See also Figure 1.

Corollary 1 *For any r-regular graphs G, $\frac{r}{2} \leq \chi'_s(G) \leq r$. Moreover, if G is triangle-free, $\frac{r}{2} + 1 \leq \chi'_s(G) \leq r$.*

2.1 Trees, cycles

For trees and cycles, the subchromatic index can be determined explicitely as follows:

Proposition 3

(i) *For any tree T, $\chi'_s(T) = \chi^*(G) \leq 2$; $\chi'_s(T) = 2$ if and only if T is not a star;*

(ii) *$\chi'_s(C_3) = 1$, $\chi'_s(C_n) = \chi^*(G) = 2$ for all $n \geq 4$.*

Proof: Color greedily. □

2.2 Cacti

A *cactus* is a connected graph in which every block (maximal 2-connected subgraph) is an edge or a cycle. Equivalently, a graph G is a cactus if and only if every two cycles in G are edge-disjoint.

Proposition 4 *For all cacti G, $\chi'_s(G) \leq \chi^*(G) \leq 3$. Moreover, an edge 3-subcoloring can be found in linear time.*

Proof: Let T be a breadth-first search (bfs) tree of G, rooted at vertex v. We claim that

$$\text{all edges of } G \text{ outside } T \text{ form a matching.} \tag{1}$$

During searching the graph, its vertices are be arranged into levels, based on the distance from the initial vertex. Since G is a cactus, the tree T misses from each odd cycle the edge connecting the two vertices at the highest level, while similarly in an even cycle one of the two edges incident with the vertex at the highest level remains outside T.

Then, given two edges e, e' of $E(G) \setminus E(T)$, either e is separated from e' by the lowest vertex of the cycle containing e, or vice-versa.

Now, color T with two colors and the matching $E(G) \setminus E(T)$ with the third color showing $\chi'_s(G) \leq 3$. Since a bfs-tree can be performed in linear time, Proposition 4 follows. □

We leave as an open problem whether cacti with subchromatic at most 2 by allow a simple structural description. Since all cacti have treewidth bounded by 2, their subchromatic index can be computed in polynomial time as we will show later in section 3.3.

Observe, that in any edge 2-subcolored graph G each vertex of degree at least 3 is either a center of a monochromatic star or it belongs to a monochromatic triangle. Let us further direct the edges of monochromatic stars $K_{1,k},\ k \geq 2$ towards its center. (The other edges remain undirected.) Clearly, no vertex of degree at least 3 in G is of outdegree 2 or more. Also each directed cycle is of even length, since the colors of stars must alternate. It follows straightforwardly that each component of the subgraph of G induced by vertices of degree at least 3 may only contain at most one cycle. (By the argument with directions, a little more discussion is needed to encompass also triangles.) Figure 1 shows some cacti G that do not fulfill these necessary conditions and hence have $\chi'_s(G) = 3$.

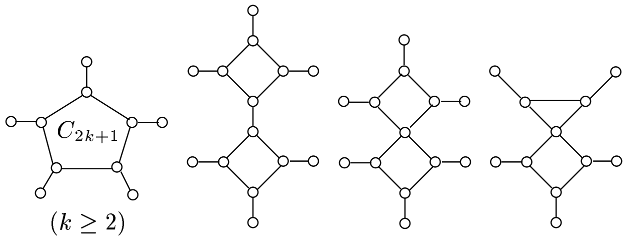

Figure 1: Some cacti with subchromatic number exact 3

Moreover, if some vertices of degree at least 3 induce a cycle in a graph of subchromatic index 2, its subcoloring can be almost uniquely determined:

Corollary 2 *Let vertices of degree at least three induce a cycle C in a graph G of $\chi'_s(G) = 2$. Then either C forms a monochromatic triangle in any edge 2-subcoloring of G, or C is of even length and colors of its edges alternate.*

2.3 Unicyclic graphs

A (connected) graph is *unicyclic* if it contains exactly one cycle. As we show now all edge 2-subcolorable unicyclic graphs have a simple structure and hence can be easily recognized in linear time.

Theorem 1 *For any unicyclic graph G, $\chi'_s(G) \leq 3$. Moreover, $\chi'_s(G) = 3$ if and only if the only cycle C of G has length $2k+1$, $k \geq 2$ and all vertices of C are of degree at least 3 in G.*

Proof: Since unicyclic graphs are cacti, the first part follows from Proposition 4. We have shown above that if an odd cycle C_{2k+1}, $k \geq 2$ contains no vertex of degree 2, G cannot be edge 2-subcolorable.

It suffices to construct an edge 2-subcoloring in all other cases. First consider the case when C is a triangle. We make it monochromatic (say white) and then distribute all remaining edges of G into two color classes, such that two edges belong to the same class if their distance from C has the same parity modulo 2. (The edge-distance being viewed as the distance between the corresponding vertices in the line graph.)

Now assume that C_{2k+1} contains a vertex of degree 2. Let denote the vertices of C by $v_1, v_2, \ldots, v_{2k+1}$, where $\deg(v_1) = 2$. We use the white color on the pair of edges incident with v_1 and color the other edges of the cycle alternately by black and white.

The remaining edges form a forest. We color them in a greedy manner, such that the edges incident with a vertex v_i, i odd should be all white, whereas for even i these should be all black. Similarly as above, inside the same tree of the forest edge colors alternate in between different levels.

For an even cycle C, we first use alternate coloring of C. Then we choose colors of the remaining edges by the same parity principle as in the previous case.

Observe first, that in both color classes, the edges outside C induce a disjoint union of stars. Moreover, this edge subcoloring is also valid if we encounter the edges of C. $\square$

Clearly, C can be found in G in linear time, Theorem 1 implies that unicyclic graphs with subchromatic index at most 2 can be recognized in linear time.

2.4 Complete bipartite graphs

Since complete bipartite graphs contain no triangle, subchromatic index coincides with the star chromatic index and we get the following results:

Proposition 5 ([11, 19]) *For any $n \geq 1$,*

$$\chi'_s(K_{n,n}) = \chi^*(K_{n,n}) = \begin{cases} n, & n \leq 4 \\ 4, & n = 5 \\ \lceil \frac{n}{2} \rceil + 2, & n \geq 6 \end{cases}$$

2.5 Cubes

Recall that the d-dimensional Cube Q_d has all $0, 1$ d-tuples as vertices and two such d-tuples are adjacent in Q_d if and only if they differ in exactly one position. Note that Q_d is d-regular, bipartite and has 2^d vertices.

Proposition 6 ([19])

(i) *For $k \geq 2$, $\chi'_s(Q_{2^k-2}) = \chi^*(Q_{2^k-2}) = 2^{k-1}$;*

(ii) *For $k \geq 3$, $2^{k-3} + 2 \leq \chi'_s(Q_{2^{k-2}}) = \chi^*(Q_{2^{k-2}}) \leq 2^{k-3} + k - 2$;*

2.6 Complete graphs

Proposition 7 *For any $n \geq 1$, the subchromatic index of the complete graph K_n is bounded by:*

$$\frac{n-1}{2} \leq \chi'_s(K_n) \leq \chi^*(K_n) = \left\lceil \frac{n}{2} \right\rceil + 1.$$

Moreover the lower bound is attained whenever $n = 3^k$ for some integer k.

Proof: The upper bound by $\chi^*(K_n) = \lceil \frac{n}{2} \rceil + 1$ was given in [1]. To prove the low bound, we first construct optimal subcolorings for the case when n is a k-th power of 3. We proceed by induction on k. Let the vertices of K_n be denoted by $v_0, \ldots, v_{n-1}$.

For each $i \in \{1, \ldots, \frac{n-1}{2} : 3 \not| i\}$ we define a color class $E_i = \{(v_j, v_{j+i}), (v_j, v_{j-i}), (v_{j+i}, v_{j-i}) : j = 0, 3, 6, \ldots, n-3\}$ where all indices are counted modulo n.

These $\frac{n}{3}$ color classes cover all edges by disjoint sets of packing triangles. The yet uncolored edges connect only vertices that are at distance divisible by 3 and induce three vertex disjoint copies of the graph $K_{n/3}$. By induction hypothesis each of them can be colored independently by $\frac{n/3-1}{2} = \frac{n-3}{6}$ colors. In total we get a subcoloring using $\frac{n}{3} + \frac{n-3}{6} = \frac{n-1}{2}$ distinct edge colors.

A valid subcoloring of K_9 is depicted in Fig. 2 □

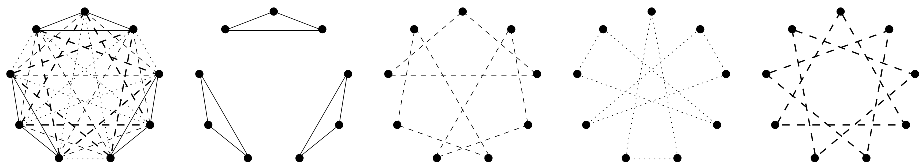

Figure 2: Showing $\chi'_s(K_9) = 4$

This allows us to classify the subchromatic index for all complete graphs of order at most ten. We first show that $\chi'_s(K_6) = 4$. Assume for a contradiction that a valid 3-edge-subcoloring of K_6 exists. Then each color class may have at most 6 edges, hence at least two of these two classes must have at least 5 edges since $|E_{K_6}| = 15$. By a case study it is easy to determine that a color class with at least five edges may only induce in K_6 a subgraph of one of the possible three types: $K_3 \cup K_3$ or $K_3 \cup P_3$ or $K_{1,5} \cup K_1$. (Here $\cup$ stands for disjoint union.) But it is impossible to find a pair of not necessarily distinct types that would be edge disjoint, a contradiction.

We summarize the values of chromatic index of small complete graphs the following table:

Graph G	K_1	K_2	K_3	K_4	K_5	K_6	K_7	K_8	K_9	K_{10}
$\chi'_s(G)$	0	1	1	2	3	4	4	4	4	5

The values for K_7 and K_8 are majorized by $\chi'_s(K_9)$, the remaining values follow from the monotone property of χ'_s and other bounds which were discussed above.

3 Computational complexity

Formally we define EDGE k-SUBCOLORABILITY as a decision problem which for a given graph G (the instance) answers the question: "Is $\chi'_s(G) \leq k$?"

In the following sections we will show, that the problem is NP-complete in general (obviously EDGE k-SUBCOLORABILITY $\in$ NP), but for a restricted class of graphs of bounded treewidth a linear time algorithm exists. This also imply that cacti of subchromatic index at most 2 can be recognized in linear time.

3.1 NP-hardness of edge 2-Subcoloring

In this section we prove the following negative result.

Theorem 2 EDGE 2-SUBCOLORABILITY *is* NP*-complete, even when restricted to triangle-free graphs of maximum degree three.*

As all graphs with maximum degree at most 2 are edge 2-subcolorable, Theorem 2 is best possible with respect to degree constraint.

Proof: We prove Theorem 2 by showing a reduction from the NOT-ALL-EQUAL 3-SATISFIABILITY (*NAE-3SAT*) problem, which has been shown to be NP-complete by Schaefer [18] (see also [14, Problem LO3]).

This problem decides whether a Boolean formula Φ in conjunctive normal form satisfying such that each clause is a disjunction of three not necessarily distinct variables allows a satisfying assignment for Φ such that each clause in Φ contains at least one negatively valued literal. We denote the class of all formulas that allow such an assignment by *NAE-3SAT*.

Let a formula Φ be an instance for the *NAE-3SAT* problem. Assume that Φ consists of m clauses $C_1, C_2, \ldots, C_m$ over variables $x_1, x_2, \ldots, x_n$ such that every clause C_j contains exactly three variables, $C_j = (x_{j_1} \vee x_{j_2} \vee x_{j_3})$.

We will construct a triangle-free graph $H = H(\Phi)$ of maximum degree three such that H has an edge 2-subcoloring if and only if $\Phi \in$ *NAE-3SAT*.

Clause gadget. Consider the graph G_C depicted in Figure 3 (left) with three labelled vertices a, b, c.

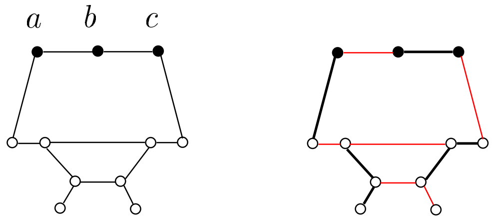

Figure 3: Clause gadget G_C (left) with an edge 2-subcoloring (right)

Fact 2 (i) *The graph G_C is edge 2-subcolorable. In any edge 2-subcoloring, the edges (a,b), (b,c) receive different colors.*

(ii) *Every coloring of the two edges (a,b), (b,c) with two distinct colors can be extended into an edge 2-subcoloring of the entire graph G_C, such that the two edges incident with the vertex a and the two edges incident with c receive different colors (cf. Figure 3 (right)).*

Fact 2 can be seen quickly by inspection, see also Corollary 2.

Variable gadget. Let $k \geq 2$ be an integer. Let G_V^k be the graph depicted in Figure 4 with k labelled vertices $a_1, \ldots, a_k$.

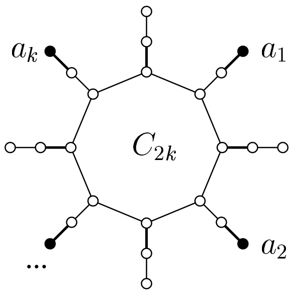

Figure 4: Variable gadget G_V^k

Fact 3 *The graph G_V^k is edge 2-subcolorable. In any edge 2-subcoloring, the thick edges receive the same color.*

Again, Fact 3 can be seen easily by using Corollary 2.

Now, for the construction of H we take the incidence graph G_Φ. Vertices of G_Φ represent variables and clauses of Φ, and edges represent the incidence relation. Since the same variable may appear in the same clause, multiple edges may appear in G_Φ.

For the construction of H we replace each clause-representing vertex v_j of degree three with a unique copy of G_C, such that the three edges incident with v are one-to-one incident with the vertices a, b and c (i.e. each edge chooses exactly one vertex).

Let further each vertex u_i representing a variable x_i with k_i occurrences be replaced by a unique copy G_i of $G_V^{k_i}$. Similarly as in the previous case, the k_i edges incident with u_i become one-to-one incident with the vertices $a_1, \ldots, a_{k_i}$.

The construction of H is completely described. Since all gadgets have maximum degree three and no triangles, and since labelled vertices in the gadgets have degree one or two, H has maximum degree three and no triangles, as well.

Suppose now that the edges of H can be subcolored by two colors, say red and blue. For each i, define $\phi(x_i) = \mathsf{true}$ if the thick edges of G_i are red; and $\phi(x_i) = \mathsf{false}$ otherwise. By Fact 3, ϕ is well defined. For each variable gadget, the edges leaving the gadget G_i from vertices $a_1, \ldots, a_k$ must obtain the same color, complementary to the color of the thick edges of G_i. On the other side, the edges pendant from a clause gadget cannot have all the same color, since then a monochromatic P_4 would appear by Fact 2. Hence, in each clause at least one variable is positively valued and at least one is valued negatively by ϕ and $\Phi \in$ *NAE-3SAT*.

In the opposite direction assume that $\Phi \in$ *NAE-3SAT* for an assignment ϕ. We derive an edge 2-subcoloring of H as follows. If $\phi(x_i) = \mathsf{true}$ (false), color the thick edges of the variable gadget G_i red (blue, respectively). Then extend this coloring into an edge 2-subcoloring of G_i. This is always possible by Fact 3. As was mentioned in the previous paragraph, then the edges stemming from the clause gadgets (i.e. the original edges of G_Φ) allow a unique 2-subcoloring extension. Finally, complete the 2-subcoloring of H on the clause gadgets according to Fact 2 (ii).

This argument completes the proof of Theorem 2. $\square$

Corollaries. Since for triangle-free graphs G, $\chi'_s(G) = \chi^*(G)$, Theorem 2 implies

Corollary 3 *Deciding if the star index of a given graph is two is* NP-*complete, even for triangle-free graphs with maximum degree three.*

We remark that it was first proved in [16] that deciding if the star index of a triangle free graph is two is NP-complete. However, the graph constructed in [16] does not have bounded degree while our NP-complete result for the star index is best possible with respect to degree constraint.

In [12] it was shown that recognizing vertex 2-subcolorable graphs is NP-complete, even for triangle-free planar graphs with maximum degree 4. Theorem 2 and Fact 1 imply

Corollary 4 *Recognizing vertex* 2*-subcolorable graphs is* NP*-complete, even for line graphs (of triangle-free graphs) with maximum degree* 4*.*

An $(r_1, \ldots, r_k)$*-subcoloring* of a graph $G = (V, E)$ is a partition $V_1, \ldots, V_k$ of V such that each V_i consists of disjoint cliques each of which has cardinality at most r_i. In [2] it was shown that all cubic graphs are $(2, 2)$-subcolorable, and in [17] it was shown that deciding if a cubic graph is $(1, 3)$-subcolorable is NP-complete. Theorem 2 and Fact 1 imply that a similar result holds if we restrict ourselves to line graphs.

Corollary 5 *Recognizing vertex* $(3, 3)$*-subcolorable graphs is* NP*-complete, even for line graphs (of triangle-free graphs) with maximum degree* 4*.*

3.2 NP-hardness of edge 3-Subcoloring

This section deals with the proof that EDGE 3-SUBCOLORING is NP-complete. Given a graph G we construct a graph H as follows. Take three copies G_1, G_2, G_3 of G, take a triangle (v_1, v_2, v_3) and, for each $i = 1, 2, 3$, connect v_i with all vertices in G_i. Finally, take two vertices x, y and connect x to v_1, v_2, v_3 and y. See Fig. 5.

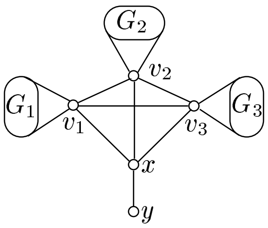

Figure 5: The graph H obtained from the copies G_1, G_2, G_3 of the given graph G

In the graph H, the set of edges between v_i and all vertices of G_i is called the (v_i, G_i)-star. We now point out that $\chi'_s(G) \leq 2$ if and only if $\chi'_s(H) \leq 3$.

Assume that $\chi'_s(G) \leq 2$ and consider an edge 2-subcoloring of G with colors c_1, c_2. In H, color the three copies G_i with this coloring and the (v_i, G_i)-stars with the third color c_3. Color the triangle (v_1, v_2, v_3) with color c_1 and the 4-star at x with color c_2. This yields an edge 3-subcoloring for H.

Assume that $\chi'_s(H) \leq 3$ and consider an edge 3-subcoloring of H with colors c_1, c_2, c_3.

CLAIM. One of the (v_i, G_i)-stars, $i = 1, 2, 3$, is monochromatic.
PROOF OF THE CLAIM. Assume that the claim is false. Then no two edges of the triangle (v_1, v_2, v_3) have the same color: For, if (v_1, v_2) and (v_2, v_3) are colored with c_1, say, then the (v_1, G_1)-star and the (v_3, G_3)-star must be colored with c_2 and c_3. This implies that the edge (v_1, v_3) is also colored with c_1 and the (v_2, G_2)-star is also colored with c_2 and c_3. Now, the edges (x, v_i), $i = 1, 2, 3$, must have color c_2 or c_3 and there exists a P_4 with color c_2 or c_3, a contradiction.

Thus, let without loss of generality that (v_1, v_2) is colored with c_1,

(v_2, v_3) with c_2 and (v_1, v_3) with c_3. If c_1 does not appear in the (v_3, G_3)-star then, by assumption, the (v_3, G_3)-star is colored with c_2 and c_3. Hence c_2 cannot appear in the (v_2, G_2)-star; otherwise there is a P_4 colored with c_2. Therefore, the (v_2, G_2)-star is colored with c_1 and c_3, implying c_1 cannot appear in the (v_1, G_1)-star; otherwise there is a P_4 colored with c_1. Thus, the (v_1, G_1)-star is colored with c_2 and c_3. But then there exists a P_4 colored with c_3. This contradiction shows that c_1 must appear in the (v_3, G_3)-star, and by symmetry, c_2 must appear in the (v_1, G_1)-star and c_3 must appear in the (v_2, G_2)-star.

In particular, for each v_i, each color is appeared in the star at v_i minus the edge (v_i, x).

Thus, the edge (x, y) is colored differently with each of the edges (v_i, x), $i = 1, 2, 3$; otherwise there is a monochromatic P_4. Now, as only two colors are available for the edges $(x, v_1), (x, v_2)$ and (x, v_3), at least two of these edges must have the same color. By symmetry, let (x, v_1) and (x, v_2) have color c, say. Then $c = c_1$; otherwise there is a monochromatic P_4 in the K_4 induced by the v_i and x. Hence both the (v_1, G_1)-star and the (v_2, G_2)-star are colored with c_2 and c_3. But, since c_2 or c_3 appears in the (v_3, G_3)-star, there exists a monochromatic P_4. This last contradiction proves the claim.

By the claim we may assume that the (v_1, G_1)-star is colored with c_1. If G_1 has less than three vertices, clearly $\chi'_s(G) \leq 2$. If G_1 has at least three vertices, then c_1 cannot appear in G_1; otherwise there is a P_4 colored with c_1. Thus, the restriction of the edge 3-subcoloring of H on G_1 is an edge 2-subcoloring for G.

From the reduction above and Theorem 2 we obtain

Theorem 3 EDGE 3-SUBCOLORABILITY *is* NP-*complete.* □

3.3 Graphs of bounded treewidth

We note first, that for a fixed k the EDGE k-SUBCOLORABILITY problem can be expressed in Monadic Second Order Logic (MSOL), hence the existence of a linear-time algorithm for graphs of bounded treewidth [10].

Since this general method in inapplicable due to hidden constants, we follow the usual scheme for constructing linear-time algorithms for graphs of bounded treewidth, cf. [6, 7] and outline the main aspects of the dynamic programming algorithm.

A *nice tree decomposition* of width at most t of a graph G is a rooted tree T, where nodes X_i of T represent subsets of vertices of G according to the following rules:

- For each edge $(u, v) \in E(G)$ there exists a node X_i such that $\{u, v\} \subseteq X_i$.
- For each vertex $u \in V(G)$ the nodes X_i containing u induce a connected subtree in T.
- The size of each node $|X_i| \leq t + 1$.
- Each node X_i has at most two children, and
  - it is called a *leaf node* if it has no children and $|X_i| = 1$,
  - or X_i has one child X_j, then X_i is either an *introduce node* if $X_i = X_j \cup \{u\}$ for some vertex $u \notin X_j$, or a *forget node* when $X_i = X_j \setminus \{u\}$ for some $u \in X_j$,
  - or X_i has two children $X_j, X_{j'}$, then it is called a *join node* and it holds that $X_i = X_j = X_{j'}$.

Theorem 4 *For any fixed k and t, the* Edge k-Subcoloring *problem can be solved in linear time for graphs of treewidth at most t.*

Proof: For the dynamic programming we compute with each node X_i of T a table $\texttt{Tab}_i$ of the following contents: Each entry $(\phi, r) \in \texttt{Tab}_i$ consists of an edge k-subcoloring ϕ of the graph G_i, the subgraph of G induced by the vertex set X_i and of a ranking $r : X_i \times \{1, \ldots, k\} \to \{0, 1, 2, 3, 4\}$ of the following meaning: For any edge k-subcoloring ψ extending ϕ to the subgraph of G induced by the union of descendants of X_i, we define

- $r(u, c) = 0$ if no edge of color c in ψ contains the vertex u.
- $r(u, c) = 1$ if u is incident with exactly one edge (u, v) of color c, and v is incident to no other edge of color c in ψ.
- $r(u, c) = 2$ if u is incident with exactly one edge (u, v) of color c, but v is incident also with other edges of this color in ψ.
- $r(u, c) = 3$ if u belongs to a monochromatic triangle of color c in ψ.
- Finally $r(u, c) = 4$ if u is the center of a monochromatic star $K_{1,k},\ k \geq 2$ of color c in ψ.

The evaluation of $\mathtt{Tab}_i$ proceeds follows:

1. If X_i is a leaf node, we let $\mathtt{Tab}_i = (\emptyset, r(u,c) = 0)$, for $\{u\} = X_i, 1 \leq c \leq k$.

2. If X_i is a forget node with child X_j, then we store in $\mathtt{Tab}_i$ all pairs (ϕ, r) where both ϕ and r are restrictions to the subgraph induced by X_i (or to the subgraph induced by X_i) of some $(\phi', r') \in \mathtt{Tab}_j$.

3. If X_i is an introduce node with child X_j and $\{u\} = X_i \setminus X_j$, we consider all entries $(\phi', r') \in X_j$, and all possible extensions ϕ of ϕ'. Here a pair (ϕ, r) is feasible if for every color c

   3a) either u is incident with no edge of color c, then $r(u,c) = 0$,

   3b) or u is incident with only one edge (u,v) of color c, then $r(u,c) = r(v,c) = 1$ if $r'(v,c) = 0$, or $r(u,c) = 2$ and $r(v,c) = 4$ if $r'(v,c) \in \{1,4\}$,

   3c) or u is incident with two edges $(u,v),(u,w)$ of color c, where $\phi(v,w) = c$ and $r'(v,c) = r'(w,c) = 1$, then we let $r(u,c) = r(v,c) = r(w,c) = 3$,

   3d) or finally u is incident with $q \geq 2$ edges (u, v_s) of color c, where $r'(v_s, c) = 0$ for $1 \leq s \leq q$. Then we let $r(u,c) = 4$ and $r(v_s, c) = 2$ for $1 \leq s \leq q$.

   If not specified above we let $r(v,c) = r'(v,c)$ for all other $v \in X_j$, and store all feasible pairs (ϕ, r) in $\mathtt{Tab}_i$.

4. If X_i is a join node with children $X_j, X_{j'}$ we will keep in $\mathtt{Tab}_i$ all pairs (ϕ, r) for which exists $(\phi', r') \in \mathtt{Tab}_j$ and $(\phi'', r'') \in \mathtt{Tab}_{j'}$ such that $\phi = \phi' = \phi''$ and moreover for each color c and for every vertex $u \in X_i$ at least one of the following cases apply:

   4a) either $r(u,c) = \max\{r'(u,c), r''(u,c)\}$, when $\min\{r'(u,c), r''(u,c)\} = 0$,

   4b) or $r(u,c) = \max\{r'(u,c), r''(u,c)\}$ if there exists unique vertex $v \in X_i$, such that (u,v) is of color c, and $r'(u,c), r''(u,c) \in \{1,2\}$.

   4c) or $r(u,c) = 3$ if

      4ca) either there are $v, w \in X_i$ such that u, v, w form a monochromatic triangle in ϕ (then $r(u,c) = r'(u,c) = \ldots = r''(w,c)$).

4cb) or there is a vertex $v \in X_i$ such that $\phi(u,v) = c$ and $r'(u,c) = r'(v,c) \neq r''(u,c) = r''(v,c)$ where $\{r'(u,c), r''(u,c)\} = \{1,3\}$

4d) or finally $r(u,c) = 4$ if

4da) either $\max\{r'(u,c), r''(u,c)\} = 4$ and $\min\{r'(u,c), r''(u,c)\} \in \{1,4\}$

4db) or $r'(u,c) = r''(u,c) = 1$ and no edge of color c is incident with u in ϕ.

To argue the correctness of these rules consider any edge k-subcoloring ψ of the subgraph induced by the subgraph of G induced by the union of descendants of some node X_i. We show that for ψ there exists a record $(\phi, r) \in \mathtt{Tab}_i$, if the table entries were evaluated recursively according to the above rules. For leaf and forget nodes the statement is correct straightforwardly.

Let X_i be an introduce node for the vertex u. If u has no edge incident of color c in ψ, we get $r(u,c) = 0$.

If in ψ u is incident exactly with one edge (u,v) of color c, then the value of $r(u,c)$ depends, whether v is incident with some other edges or not. If v does not appear on X_i, we keep the value $r'(c,u)$ according the rule 3a). Else, the edge (u,v) is colored also by ϕ and the value of $r(u,c)$ depends whether v has been before incident with an edge of color c by the rule 3b). Note that in this case v cannot be a member of a monochromatic triangle nor a leaf of a monochromatic star of color c.

If in ψ the vertex u become a part of a monochromatic triangle u, v, w, then (v,w) is the only edge of color c in ψ incident with v (or w resp.). This is the only case how a monochromatic triangle may appear and is captured by the case 3c).

Finally the case 3d) shows how a monochromatic star may appear with a new vertex, clearly all q leaves of this star cannot be incident with any other edge of the same color.

Assume now that X_i is a join node. We again distinguish five cases according to the presence of edges colored in ψ by a color c around a vertex u. If there is no such an edge, we get $r(u,c) = 0$ and the same must hold also by on the children nodes (hence rule 4a).

If u is incident with exactly one edge (u,v) of color c, and this is also the only edge of color c incident with v, either $v \in X_i$ and the case 4b) apply or $v \notin X_i$ and we follow 4a). (Here v appears either in the subtree rooted either in X_j, or in the subtree below $X_{j'}$, but not in both. Clearly ϕ cannot contain any other edge incident with u of color c.)

Similarly, if the edge (u, v) belongs to a monochromatic star of color c in ψ, either the center v and also no other edge of the star appears on X_i — case 4a), or v belongs to X_i and the star also appears in some of the two children of X_i — case 4b).

If u is a member of a triangle in ψ, then either the entire triangle appears in only one subtrees below X_j or $X_{j'}$ — case 4a), or it completely lies in X_i — case 4ca) or possibly only one edge (u, v) of the triangle appears in X_i. This last case is captured in 4cb), here the edge (u, v) must be recognized and the only edge of color c incident with u and v in one of the subtrees, while in the other subtree we must find the remaining vertex of the triangle.

Finally, if u is a center of a monochromatic star of color c, either the star is completely placed in one of the subtrees and case 4a) apply. Or the star appears as a union of a star with another star of with a new edge — case 4da) or the star is formed out of two edges, each coming from different subtrees below X_j and $X_{j'}$ and we get the case 4db).

The dynamic programming algorithm evaluates the tables $\texttt{Tab}_i$ in a bottom-up manner. An edge k-subcoloring of the entire graph G exists if and only if the table $\texttt{Tab}_r$ for the root node X_r is nonempty. For each node, the table may contain at most $k^{O(t^2)} \cdot 5^{(t+1)k}$ entries, each of length $O(t^2 \log k + kt)$. Both these values are bounded by a constant, since the treewidth t and the number of colors k are also bounded by a constant.

The evaluation of each table can be performed in time depending on k and t only hence the entire complexity of the dynamic programming algorithm depends only on the number of nodes of the nice tree decomposition T. As it is mentioned in [7] a nice decomposition of width at most t containing at most $O(|V(G)|)$ nodes exists for any graph of treewidth at most t, and can be found in linear time [8] $\square$

If we restrict the rankings r only to values $\{0, 1, 2, 4\}$, the dynamic programming will check for the existence of a edge k-subcoloring without monochromatic triangles, i.e. for a star partition with at most k subsets. Hence we can conclude that

Corollary 6 *For any fixed parameters k and t, the test whether $\chi^*(G) \leq k$ can be performed in time linear in $|V(G)|$ for any graph of treewidth at most t.*

4 Conclusion

The concept of edge subcoloring of graphs is introduced for the first time in this paper, motivated by the study of vertex subcolorings.

Among many interesting open questions we pose the following.

1. What is the exact value for $\chi'_s(K_n)$?
   (Known for $n \leq 10$; see section 2)

2. What is the computational complexity of EDGE 2-SUBCOLORABILITY for planar graphs?
   (Note that for all planar graphs G, $\chi'_s(G) \leq \chi^*(G) \leq 5$; cf. [16]. Moreover the complexity of finding the ordinary chromatic index is not yet determined for planar graphs. It is widely expected to be a nontrivial problem since already the fact that any bridgeless cubic planar graph has chromatic index 3 is equivalent to the four color theorem, see e.g. [13].)

3. What is the computational complexity of EDGE K-SUBCOLORABILITY for fixed $k \geq 4$?
   (We have proved that EDGE 2-SUBCOLORABILITY is NP-complete for triangle-free graphs with maximum degree 3; see Theorem 2 and that EDGE 3-SUBCOLORABILITY is NP-complete; see Theorem 3)

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