Antisymmetric flows on planar graphs

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Abstract

We prove that every oriented planar graph admits a homomorphism to the Paley tournament $P(271)$ and hence that every oriented planar graph has an antisymmetric flow number and a strong oriented chromatic number of at most 271.

1 Introduction

Let $G = (V, E)$ be an oriented graph (that is a digraph with no opposite edges) and M an abelian group. An M-*antisymmetric flow* (M-ASF) on G is a flow $f : E \to M$ for which no two arcs e and e' take opposite values $f(e) = -f(e')$. In particular, setting $e = e'$ gives that every M-ASF is a nonzero flow. This notion arose initially as a dual concept to that of oriented colorings or the equivalent idea of orientation preserving homomorphisms. [3], [5] (the exact duality is given in Theorem 3 below).

Two conditions are evidently necessary for the existence of an M-ASF on G; G cannot have an isthmus (which excludes *any* nonzero flow) or an oriented 2-cut (a 2-cut in which both arcs are oriented in the same direction), since on such a cut every flow must takes opposite values on the two arcs. These conditions are also sufficient and in fact we have

Theorem 1 ([1]) *Every oriented graph without an isthmus or an oriented 2-cut has a* $\mathbf{Z}_3{}^3 \times \mathbf{Z}_6{}^6$*-ASF.*

The *antisymmetric flow number* of an oriented graph G is the least $|M|$ for which G admits an M-ASF. Thus the above shows that every oriented

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graph without obvious obstructions has antisymmetric flow number at most $3^3 \times 6^6 = 157464$. It is known that for oriented planar graphs this bound can be reduced to 672 [4]. This is a consequence of more general results bounding the antisymmetric flow number in terms of the acyclic chromatic number. Here we prove

Theorem 2 *Every oriented planar graph without an isthmus or an oriented 2-cut has a $\mathbf{Z}_{271}$-ASF.*

This result is a consequence of the next two theorems. If M is an abelian group and $B \subset M$ with $B \cap -B = \emptyset$ we define an oriented graph C_B by letting $V(C_B)$ be the group generated by B and $E(C_B) = \{(v,w)|\ w - v \in B\}$. Thus C_B is an oriented Cayley graph. Our definitions ensure that C_B has no loops or opposite arcs.

The following result is a straightforward consequence of the definitions and the standard proof of flow-coloring duality

Theorem 3 *An oriented planar graph G has an M-ASF if and only if its dual $G^\star$ admits a homomorphism to C_B for some $B \subset M$ with $B \cap -B = \emptyset$.*

If $M = \mathbf{Z}_p$ with $p \equiv 3 \bmod 4$ and B is the set of nonzero quadratic residues mod p, then C_B is the Paley tournament $P(p)$. In the remainder of the paper we prove

Theorem 4 *Every planar graph admits a homomorphism to $P(271)$,*

which, together with Theorem 3, gives Theorem 2. This theorem also gives directly a bound of 271 for the *strong* oriented chromatic number of a planar graph [3].

2 R_k graphs

Although we will be concerned with oriented graphs, the terms in this section for are defined for general digraphs. Let $G = (V, E)$ be a digraph. For $A \subseteq V$, we define $in_G(A)$ (resp. $out_G(A)$) to be the set of vertices which have an arc from (resp. to) every vertex of A. In particular observe that (vacuously) $in_G(\emptyset) = out_G(\emptyset) = V$. For $v \in V$, $d_G(v)$ denotes the degree of v in G (ignoring orientations). For a digraph G and vertex sets A and B we define a function ν by

$$\nu_G(A, B) = |in(A) \cap out(B)|.$$

We will usually omit the subscript G in these notations when the digraph G is clear from context. Define an digraph G to be *n-tuply R_k* if

$$\nu(A,B) \geq n \quad \forall A,B \subseteq V \text{ with } A \cap B = \emptyset,\ |A|+|B| \leq k, \tag{1}$$

When $|G| \geq k$, the last inequality in (1) can be replaced by an equation.

We do not use the following result in the sequel.

Lemma 5 *If $n \geq 1$ and G is n-tuply R_k, then it is $(n+k)$-tuply R_{k-1}.*

Proof:- Suppose that G is n-tuply R_k and let A, B be disjoint subsets of V with $|A|+|B| \leq k-1$. Let W be a subset of $in(A) \cap out(B)$ chosen to have either k or the maximum possible number of elements, whichever is smaller.

Since G is n-tuply R_k, there is some vertex $v \in out(W)$. Again by the n-tuple R_k property, $in(A) \cap out(B \cup \{v\})$ has at least n elements and this set is disjoint from W. Hence $|W \cup (in(A) \cap out(B \cup \{v\})| \geq |W|+n$, Whence $|W| = k$ and $|in(A) \cap out(B)| \geq n+k$. $\square$

Remarks: If G is n-tuply R_k then clearly $V(G)$ must have some subset S of k elements and, for each partition $S = S_1 \cup S_2$, $|in(S_1) \cap out(S_2)| \geq n$. Since these sets are disjoint from each other and from S we conclude $|G| \geq k + n2^k$. In conjunction with the above lemma this shows that an R_k graph has at least $k-1+(k+1)2^{k-1}$ vertices. For $k = 2,3,4,5$ we get bounds of 7, 18, 43 and 100 respectively. The first of these is sharp ($P(7)$ is R_2). The graph $P(19)$ is R_3 and we can show that there is no 4-tuply R_2 (and hence no R_3) graph of order 18. If such a graph existed then the matrix obtained from its incidence matrix by substituting ones on the diagonal would be a Hadamard matrix of order 18, which is impossible.

Sopena [5] has introduced a very similar notion to (1-tuple) R_k. He defines a graph to be T_k if, in our terminology, $\nu(A,B) > 0$ for every pair of disjoint sets (A,B) for which $A \cup B$ induces a clique. Of course these two definitions coincide for tournaments.

We define

$$\mathcal{E}_{k,n} = \{(A,B) \mid \ A,B \subseteq V,\ A \cap B = \emptyset,\ |A|+|B| = k,\ \nu(A,B) < n\},$$

so that (if $|G| \geq k$) G is n-tuply R_k exactly when $\mathcal{E}_{k,n} = \emptyset$. It is useful to consider graphs for which are "almost" n-tuply R_k, that is when $\mathcal{E}_{k,n}$ is small.

For a sequence $m_1, \ldots m_i$ of nonnegative integers, we define $\mathcal{E}_{k,n}[m_1, \ldots m_i]$ $(0 \le i \le k)$ inductively by $\mathcal{E}_{k,n}[\ \ ] = \mathcal{E}_{k,n}$ and, for $i > 0$,

$$\begin{aligned}&\mathcal{E}_{k,n}[m_1, m_2, \ldots m_i] = \{(A,B) \mid A, B \subseteq V,\ A \cap B = \emptyset,\ |A| + |B| = k - i,\\ &\max[\ |\{x \in V \setminus (A \cup B) \mid (A \cup \{x\}, B) \in \mathcal{E}_{k,n}[m_1, m_2, \ldots m_{i-1}]\}|,\\ &|\{x \in V \setminus (A \cup B) \mid (A, B \cup \{x\}) \in \mathcal{E}_{k,n}[m_1, m_2, \ldots m_{i-1}]\}|\ ] \ge m_i\ \}\end{aligned}$$

In the next section we prove

Lemma 6 *For $G = P(271)$, $\mathcal{E}_{4,9} = \mathcal{E}_{5,1}[3] = \mathcal{E}_{5,2}[5] = \emptyset$,*

and investigate the set $\mathcal{E}_{5,1}[2,2]$.

3 Computations

The computations necessary to prove our results are greatly reduced by the symmetry of the Paley tournaments. If G is an oriented graph, $A \subseteq V(G)$ and ψ is a permutation of $V(G)$, then we define $\psi(A) = \{\psi(v) |\ v \in A\}$. If such a permutation preserves (reverses) the direction of arcs then it is an *(anti-) automorphism* of G. If, for some for some automorphism ψ, $B = \psi(A)$, then we say that A and B are equivalent and write $A \sim B$. Similarly for ordered pairs of vertex sets we define $(A', B') \sim (A, B)$ to mean $A' = \psi(A)$ and $B' = \psi(B)$ for some automorphism ψ.

If ψ is an automorphism and ϕ is an antiautomorphism of G then clearly

$$\nu(A, B) = \nu(\psi(A), \psi(B)) = \nu(\phi(B), \phi(A)).$$

An easy induction then gives

Lemma 7 *If ψ is an automorphism and ϕ is an antiautomorphism of G, then*

$$\begin{aligned} & (A,B) \in \mathcal{E}_{k,n}[m_1, m_2, \ldots m_i] \\ \Longleftrightarrow\ & (\psi(A), \psi(B)) \in \mathcal{E}_{k,n}[m_1, m_2, \ldots m_i] \\ \Longleftrightarrow\ & (\phi(B), \phi(A)) \in \mathcal{E}_{k,n}[m_1, m_2, \ldots m_i]. \end{aligned}$$

In $P(p)$ the map $\phi(v) = av + b$, where $a, b \in \mathbf{Z}_p$, $a \neq 0$ is an (anti) automorphism if a is a (non) quadratic residue. The automorphism group is arc transitive and so, in particular, there is an automorphism mapping any two distinct vertices to $\{0, 1\}$.

It follows from this and Lemma 7 that, when $G = P(271)$ and $k - i \geq 3$, every $(A, B) \in \mathcal{E}_{k,n}[m_1, m_2, \ldots m_i]$ is equivalent to either (A, B) or $(-B, -A)$ where $|A| \geq |B|$ and $\{0, 1\} \subseteq A$. Thus to prove the first equation of Lemma 6 ($\mathcal{E}_{4,9} = \emptyset$), it suffices to show that $\nu(A, B) \geq 9$ for disjoint pairs (A, B) of the form $(\{0, 1\}, \{x, y\})$, $(\{0, 1, x\}, \{y\})$ and $(\{0, 1, x, y\}, \emptyset)$. This is readily verified by a machine computation.

We now define subsets $\tilde{\mathcal{E}}_{5,n}[m_1, \ldots m_i]$ of each $\mathcal{E}_{5,n}[m_1, \ldots m_i]$ with $0 \leq i \leq 2$ by

$$\tilde{\mathcal{E}}_{5,n}[m_1, \ldots m_i] = \{(A, B) \mid (A, B) \in \mathcal{E}_{5,n}[m_1, ., m_i],\ |A| \geq |B| - \epsilon,\ \{0, 1\} \subseteq A\},$$

where $\epsilon = 1$ for $i = 0$, $\epsilon = 0$ for $i > 0$. We set $\tilde{\mathcal{E}}_{5,n} = \tilde{\mathcal{E}}_{5,n}[\ \ ]$. Every $(A, B) \in \mathcal{E}_{5,n}[m_1, ., m_i]$ with $|A| \geq |B|$ is thus equivalent to a pair in $\tilde{\mathcal{E}}_{5,n}[m_1, \ldots m_i]$.

For $i = 1, 2$, $\tilde{\mathcal{E}}_{5,n}[m_1, \ldots m_i]$ is clearly just the set of pairs of disjoint vertex sets (A, B) for which $|A| + |B| = 5 - i$, $|A| \geq |B|$, $\{0, 1\} \subseteq A$ and at least one of the sets

$$\{x \mid (A \cup \{x\}, B) \in \tilde{\mathcal{E}}_{5,n}[m_1, \ldots m_{i-1}]\}$$

or

$$\{x \mid (A, B \cup \{x\}) \in \tilde{\mathcal{E}}_{5,n}[m_1, \ldots m_{i-1}]\}$$

has m_i or more elements.

For $G = P(271)$, a routine machine calculation gives $\tilde{\mathcal{E}}_{5,1}$. It contains 20, 60, 138 and 10 pairs with $|A| = 2, 3, 4$ and 5, respectively. Similarly $\tilde{\mathcal{E}}_{5,2}$ contains 258, 774, 912 and 480 pairs with $|A| = 2, 3, 4$ and 5, respectively. Now $\tilde{\mathcal{E}}_{5,n}[m]$ is given explicitly as the union of the sets

1. $\{(\{0, 1\}, \{a, b\}) \mid \{a, b\} \subseteq B$ for m or more pairs (A, B) in $\tilde{\mathcal{E}}_{5,n}$ with $|A| = 2$ or for m or more pairs (A, B) in $\tilde{\mathcal{E}}_{5,n}$ with $|A| = 3$
2. $\{(\{0, 1, b\}, \{a\}) \mid a \in B$ and $b \in A$ for m or more pairs (A, B) in $\tilde{\mathcal{E}}_{5,n}$ with $|A| = 3$ or for m or more pairs (A, B) in $\tilde{\mathcal{E}}_{5,n}$ with $|A| = 4$
3. $\{(\{0, 1, a, b\}, \emptyset) \mid \{a, b\} \subseteq A$ for m or more pairs (A, B) in $\tilde{\mathcal{E}}_{5,n}$ with $|A| = 4$ or for m or more pairs in $\tilde{\mathcal{E}}_{5,n}$ with $|A| = 5$

Using this, a further machine computation shows that $\tilde{\mathcal{E}}_{5,1}[3] = \tilde{\mathcal{E}}_{5,2}[5] = \emptyset$ and hence that $\mathcal{E}_{5,1}[3] = \mathcal{E}_{5,2}[5] = \emptyset$. This completes the proof of Lemma 6.

We finish the section by calculating $\tilde{\mathcal{E}}_{5,1}[2, 2]$, which we will use in the proof of Lemma 10 in the next section. Firstly, a computation shows that $\tilde{\mathcal{E}}_{5,1}[2]$ contains the following 32 pairs.

1. $(\{0,1\},\{104,232\})$, $(\{0,1\},\{186,222\})$.

2. $(\{0,1,x\},\{y\})$ for each of the following (x,y): (14,149),(15,253), (19,219), (26,59), (29,104), (29,203), (29,265), (50,117),(51,124), (56,5), (58,18), (65,24), (86,243), (94,207), (103,127), (106,21), (121,63), (147,73), (168,203), (192,246), (199,30), (202,5), (213,30), (248,59).

3. $\{(\{0,1,14,262\},\emptyset)$, $\{(\{0,1,28,192\},\emptyset)$, $\{(\{0,1,106,147\},\emptyset)$, $\{(\{0,1,123,242\},\emptyset)$, $\{(\{0,1,142,248\},\emptyset)$, $\{(\{0,1,213,251\},\emptyset)$.

whence in turn

$$\tilde{\mathcal{E}}_{5,1}[2,2] = \{(\{0,1,29\},\emptyset)\} \cup \{(\{0,1\},\{x\}) \mid x \in \{5,30,59,203\}\}. \qquad (2)$$

The four pairs in the second set are equivalent to

$$(\{276,277\},\{0\}),\ (\{241,242\},\{0\}),\ (\{212,213\},\{0\}),\ (\{68,69\},\{0\}), \quad (3)$$

respectively, and any pair of the form $(\{x,y\},\{0\})$ in $\mathcal{E}_{5,1}[2,2]$ must be a nonzero quadratic multiple of one of these. In the first pair neither of the residues in the first set are quadratic and in the other pairs both are. We will use these observations later.

4 Reducible Configurations

A *configuration* is an unoriented graph C together with a function ρ from a subset of its vertices to $\mathbb{N}$. An oriented graph G *contains* a configuration C if for some orientation $\tilde{C}$ of C, there is an imbedding ι of $\tilde{C}$ into H for which $d_H(\iota(v)) = \rho(v)$ for all v in the domain of ρ. When defining configurations, the term *degree* of a vertex v will always refer to $\rho(v)$.

A configuration is *reducible* with respect to an oriented graph H if, whenever oriented graph G contains it and maps homomorphically to H, then there is a smaller oriented graph G' which also maps homomorphically to H.

Lemma 8 *Let $m, m_1, m_2 \geq 1$ and let H be an oriented graph for which $|H| \geq 5$ and $\mathcal{E}_{4,m} = \mathcal{E}_{5,1}[m_1] = \mathcal{E}_{5,2}[m_2] = \emptyset$.*

Let W be the configuration consisting of a wheel with central vertex v of degree n having mutually disjoint vertex sets V_i, ($4 \leq i \leq 6$), not all empty, *none of which contain v, such that the vertices in V_i have degree i, no two vertices of V_6 are adjacent, every vertex in $z \in V_6$ is adjacent to a vertex $w = w(z) \not\in V_4 \cup \{v\}$ of degree 4, the mapping $z \to w(z)$ is injective,*

$$|V_4| + m_1|V_5| + m_2|V_6| \leq m - 1 \tag{4}$$

and

$$|V_4| + |V_5| + 2|V_6| \geq n - 4, \tag{5}$$

then W is reducible with respect to H.

Proof:- Let G be a planar graph containing the configuration W. We suppose for a contradiction that G is of smallest order not admitting a homomorphism to H. Let $V_4' = \{w(z) \mid z \in V_6\}$. By assumption, every $z \in V_6$ is adjacent to $w(z)$. We assume temporarily that, apart from this, no two vertices of $V_4 \cup V_5 \cup V_6 \cup V_4'$ are adjacent.

We fix some notation. Throughout the proof, x, y and (w, z) will denote, respectively, vertices in V_4, vertices in V_5 and adjacent pairs $z \in V_6$ and, $w = w(z)$. We let $V_{4,6}$ denote the set of all such pairs. For $x \in V_4$ we let v, x_1, x_2, x_3 be the neighbors of x taken in order around x. For $y \in V_5$ we let v, y_1, y_2, y_3, y_4 be the neighbors of y taken in order around y. For $(w, z) \in V_{4,6}$ we let v, z_1, z_2, z_3, z_4, w be the neighbors of z taken in order around z and w_1 be the neighbor of w opposite z.

We construct a new planar graph G' by deleting the vertices in $V_4 \cup V_5 \cup V_4' \cup V_6$ and adding new arcs as follows. For each $x \in V_4$ add an arc (in either direction) between x_1 and x_3. For each $y \in V_5$ add arcs between y_1 and y_4 in such a way that the circuit $y_1y_2y_3y_4$ includes a directed path of length 3 (it is easy to check that this is always possible), and between y_1 and y_3, oriented to make the path vy_1y_3 directed. For each $(w, z) \in V_{4,6}$ add arcs between z_1 and z_4, and z_1 and z_3 so that the paths $z_2z_1z_4$ and vz_1z_3 are directed.

Since the V_i are not all empty, $|G'| < |G|$ and so there is a homomorphism $\varphi : G' \to H$. Observe that two vertices must have different images under φ if they are joined by an arc or by a directed path of length 2. Thus, our placement of our arcs ensures that $\varphi(v)$ differs from $\varphi(y_3)$ ($\forall y \in V_5$) and (since one of the paths vz_1z_2, vz_1z_4 is directed) at least two of each $\{\varphi(z_2), \varphi(z_3), \varphi(z_4)\}$ ($(w, z) \in V_{4,6}$). Also φ takes distinct values on each of the sets $\{x_1, x_2, x_3\}$ ($x \in V_4$), $\{y_1, y_2, y_3, y_4\}$ ($y \in V_5$) and $\{z_1, z_2, z_3, z_4\}$ ($(w, z) \in V_{4,6}$). In addition we will require that

$$\varphi(v) \neq \varphi(x_2) \quad (x \in V_4) \tag{6}$$

$$\varphi(v) \neq \varphi(y_2) \quad (y \in V_5) \tag{7}$$

$$\varphi(v) \not\in \{\varphi(z_2), \varphi(z_3), \varphi(z_4)\} \quad ((w,z) \in V_{4,6}) \tag{8}$$

These conditions guarantee that φ takes distinct values on each of the sets $\{v, x_1, x_2, x_3\}$ $(x \in V_4)$, $\{v, y_1, y_2, y_3, y_4\}$ $(y \in V_5)$ and $\{v, z_1, z_2, z_3, z_4\}$ $((w,z) \in V_{4,6})$. We will then further require that

$$(\{\varphi(u) \mid u \in in_G(y)\}, \{\varphi(u) \mid u \in out_G(y)\}) \not\in \mathcal{E}_{5,1} \quad \forall y \in V_5 \tag{9}$$

$$\left(\{\varphi(u) \mid u \in in_{(G \setminus \{w\})}(z)\}, \{\varphi(u) \mid u \in out_{(G \setminus \{w\})}(z)\}\right) \not\in \mathcal{E}_{5,2} \quad \forall (w,z) \in V_{4,6} \tag{10}$$

Since, for H, $\mathcal{E}_{5,1}[m_1] = \mathcal{E}_{5,2}[m_2] = \emptyset$, and in view of the above discussion, the conditions (6-10) exclude at most $|V_4| + |V_5| + |V_6| + (m_1 - 1)|V_5| + (m_2 - 1)|V_6| = |V_4| + m_1|V_5| + m_2|V_6| \leq m - 1$ possible values of $\varphi(v)$. Since also $\mathcal{E}_{4,m} = \emptyset$ and, by (5), $d_{G'}(v) \leq 4$, we can choose $\varphi(v)$ to satisfy (6-10).

Now we extend φ back to G. First we define φ on each z $((w,z) \in V_{4,6})$. By (10), there are at least two choices for $\varphi(z)$ so we choose so that $\varphi(z) \neq \varphi(w_1)$. Now, since $\mathcal{E}_{4,m} = \emptyset$ and φ takes distinct values on the neighbors of w, φ can be defined on w. Similarly φ can be extended to the $x \in V_4$ and (using (9)) to the $y \in V_5$. This gives the required contradiction.

The restricted result that we have proved so far shows that two adjacent vertices of degree ≤ 5 and a vertex of degree 6 with 2 neighbors of degree ≤ 5 are both reducible. If we now relax the non-adjacency assumptions made at the beginning of the proof, we find that W must contain one of these configurations, hence the lemma holds in general. □

Corollary 9 *If H satisfies the hypotheses of Lemma 8, then the following configurations are reducible with respect to H. In particular, they are reducible with respect to $P(271)$.*

1. *a vertex of degree 2 or 3.*
2. *two adjacent vertices of degree ≤ 5.*
3. *a vertex of degree 6 with 2 neighbors of degree ≤ 5.*
4. *a vertex of degree 7 with 2 neighbors of degree ≤ 5. and a third of degree 4.*
5. *a vertex of degree 6 with adjacent neighbors of degree 4 and 6.*

6. *a vertex of degree* 7 *with adjacent neighbors of degree* 4 *and* 6 *and another neighbor of degree* ≤ 5.

Proof: It is easy to prove that vertices of degree 2 or 3 are reducible. The other cases follow from Lemmas 6 and 8 with $m = 9$, $m_1 = 3$ and $m_2 = 5$. □

Lemma 10 *A vertex of degree 5 with 3 consecutive neighbors of degree* 6 *is reducible with respect to* $P(271)$.

Proof: Let G be an oriented graph containing a vertex of degree 5 with 3 consecutive neighbors of degree 6. We suppose G to be of minimal order admitting no homomorphism to $P(271)$. Let a be the vertex of degree 5 and b its middle neighbor of degree 6. Let the neighbors of b be, in order, a, v, α, β, α' and v'. Thus v and v' are both neighbors of a of degree 6. Let the remaining neighbors of a be γ and γ', adjacent to v and v' respectively. If $\phi : G \to P(271)$ is homomomorphism, then so is $-\phi : -G \to P(271)$,, where $-G$ is obtained from G by reversing arcs and $(-\phi)(v) = -\phi(v)$. Thus, by substituting $-G$ for G if necessary, we may assume that at least two of the vertices $\{v, \gamma, \gamma'\}$ are in $out(a)$.

Now consider $G \setminus \{a, b\}$. This graph has a 7-gonal face bounded by the path $v\alpha\beta\alpha'v'\gamma'\gamma$. Let G' be the planar graph obtained by then adding arcs between α and α', γ and α', α and γ and α' and γ' so that $v'\alpha'\alpha$ and $v\gamma\alpha'$ are directed and $\alpha\gamma$ (resp. $\alpha'\gamma'$) has the same orientation as $b\alpha$ (resp. $b\alpha'$) in G. Since $|G'| < |G|$, there is a homomorphism $\psi : G' \to H$.

We first change the value of $\psi(v)$, if necessary, so that

$$\psi(v) \neq \psi(\beta), \psi(\gamma') \tag{11}$$

and, given this, that

$$(\{\psi(u)\ |u \in in_G(a) \cap \{v, \gamma, \gamma'\}\}, \{\psi(u)\ |u \in out_G(a) \cap \{v, \gamma, \gamma'\}\}) \not\in \mathcal{E}_{5,1}[2, 2]. \tag{12}$$

The first condition excludes at most two values of $\psi(v)$. We show that (12) excludes at most six more. Suppose first that $\gamma \in in_G(a)$, whence $\{v, \gamma'\} \subseteq out_G(a)$. We may assume by symmetry that $\psi(\gamma) = 0$ and $\psi(\gamma') = \pm 1$. If $\psi(\gamma') = 1$ then, by (3) and the remarks after it, (12) fails for six values of $\psi(v)$ (namely $(241/242)^{\pm 1}$, $(212/213)^{\pm 1}$ and $(68/69)^{\pm 1}$). If $\psi(\gamma') = -1$, then (12) fails only if $\psi(v) = -(276/277)^{\pm 1}$. The same argument applies if the roles γ and γ' are interchanged.

Finally we suppose that $\{\gamma, \gamma'\} \subseteq out_G(a)$, in which case we may assume that $\{\psi(\gamma), \psi(\gamma')\} = \{0, 1\}$. Now (2) shows that (12) fails only if $\psi(v) = 29$ when $v \in out_G(a)$ and only if $\psi(v) \in \{5, 30, 59, 203\}$ when $v \in in_G(a)$.

Thus, in all cases, (12) excludes at most 6 possible values of $\psi(v)$, so that (11) and (12) together exclude at most 8. Since $d_{G'}(v) = 4$ and $\mathcal{E}_{4,9} = \emptyset$ for $P(271)$, we can choose $\psi(v)$ to satisfy both these conditions.

We next want to change, if necessary, the value of $\psi(v')$ so that

$$\psi(v') \neq \psi(v), \psi(\beta), \psi(\gamma). \tag{13}$$

The placement of arcs guarantees $\psi(v) \neq \psi(\alpha')$ and $\psi(v') \neq \psi(\alpha)$ so that, by (11) and (13), ψ takes distinct values on the sets $in_G(a) \cup out_G(a) \setminus \{b\}$ and $in_G(b) \cup out_G(b) \setminus \{a\}$. We then further require that

$$(\{\psi(u)|\ u \in in_G(a) \setminus \{b\}\}, \{\psi(u)|\ u \in out_G(a) \setminus \{b\}\}) \not\in \mathcal{E}_{5,1}[2] \tag{14}$$

and

$$(\{\psi(u)|\ u \in in_G(b) \setminus \{a\}\}, \{\psi(u)|\ u \in out_G(b) \setminus \{a\}\}) \not\in \mathcal{E}_{5,2} \tag{15}$$

In view of (12), and the fact that $\mathcal{E}_{5,2}[5] = \emptyset$, (13-15) exclude at most $3 + 1 + 4 < 9$ possible values of $\psi(v')$ and so can be satisfied.

We now remove the arcs we have added and extend ψ back to $G \setminus \{a\}$. By (15), we have at least two choices for $\psi(b)$ and the arcs between α and γ and between α' and γ' ensure that neither of these is in $\{\psi(\gamma), \psi(\gamma')\}$. Hence, by (14) we may choose $\psi(b)$ so that

$$(\{\psi(u)|\ u \in in(a)\}, \{\psi(u)|\ u \in out(a)\}) \not\in \mathcal{E}_{5,1}.$$

Finally, by definition of $\mathcal{E}_{5,1}$, we can extend ψ to G, contradicting our assumption. $\square$

Lemma 11 *Every planar triangulation contains a configuration reducible with respect to* $P(271)$.

Proof: Throughout this proof we will refer to vertices of degree at most 5 as *small* and of degree at least 7 as *large*.

Suppose, for a contradiction, that some triangulation G contained none of the configurations listed in Corollary 9. We give each vertex v of G a charge of $6 - d(v)$ and then discharge by the rule that each large vertex v which has $k \geq 1$ small neighbors receives a charge of $(d(v) - 6)/k$ from each of them. This immediately gives that the new charge on any non-small vertex is at most zero. We now prove that the same is true of small vertices. Let v be small. By Corollary 9(1), $4 \leq d(v) \leq 5$. Let w be a neighbor of v of degree $n \geq 7$ which has s $(0 \leq s \leq 2)$ neighbors of degree 6 in common with v. Using Corollary 9(2, 3), w has $s+3$ consecutive neighbors which include v but no other small vertex. Hence, by Corollary 9(2), it has at most $\lfloor (n-s-1)/2 \rfloor$ small neighbors and so receives charge of at least $(n-6)\lfloor (n-s-1)/2 \rfloor^{-1}$ from v. When $d(v)=4$ and $n=7$, Corollary 9(4, 6) gives the better result that w has at most two small neighbors when $s=0$ and at most one when $s>0$, whence v gives w a charge of at least $1/2$ and 1, respectively.

Hence, when $d(v)=5$, v gives w a charge of at least $1/3$ in all cases and at least $1/2$ unless $n=7$, $s=0$. When $d(v)=4$, v gives w a charge of at least $1/2$ in all cases, at least $2/3$ except when $n=7$, $s=0$ and at least 1 when $s=2$ or $s=1$ and $n \neq 8$.

If $d(v)=5$. By Corollary 9(2) and Lemma 10, v has at least two large neighbors, and when there are exactly two these must both share a neighbor of degree 6 with v. Thus, in any case, v gives a total charge of at least 1 to it neighbors

If $d(v)=4$, then by Corollary 9(5), v has at most two neighbors of degree 6 and, if there are two, then these must be opposite. Hence v loses a charge of at least two unless it has a single neighbor u of degree 6, flanked by two neighbors w, w' of degree 8 and a neighbor x of degree 7 opposite u.

In this case, if the common neighbor of x and w is not small then w has 5 consecutive neighbors which include v but no other small vertex and so, as above, has at most two small neighbors altogether and receives charge of at least 1 from v. We are thus finished in this case and similarly when the common neighbors of x and w' is not small. Thus the common neighbors of x and w and of x and w' are both small, but this contradicts Corollary 9 (4), whence, in all cases, the new charge on each vertex is at most zero, which contradicts Euler's formula. $\square$

This lemma implies Theorem 4 since there can be no smallest counterexample.

5 Concluding Remarks

The methods used here may generalize to primes $p < 271$ but are liable to be much more difficult; for $200 < p < 271$, $\mathcal{E}_{4,9}$ and $\mathcal{E}_{5,2}[6]$ are nonempty.

The smallest oriented graph known which admits a homomorphism from every oriented planar graph has order 80. This is not a Cayley graph so it implies nothing about antisymmetric flow numbers but it does suggest that the upper bound of 271 is still a considerable overestimate. In the other direction, the oriented planar graph constructed in [2] has an oriented chromatic number of at least 17, hence its dual has antisymmetric flow number at least 17. No graph (planar or otherwise) is known to have an antisymmetric flow number greater than this.

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