GAPS AND DUALITIES IN HEYTING CATEGORIES

J. NEŠETŘIL, A. PULTR AND C. TARDIF

ABSTRACT. We present an algebraic treatment of the correspondence of gaps and dualities in partial ordered classes induced by the morphism structures of certain categories which we call Heyting (such are for instance all cartesian closed categories, but we present examples of others, too). This allows to extend the results of [12] to a wide range of more general structures. Also, we introduce a notion of combined dualities and discuss the relation of their structure to that of the plain ones.

INTRODUCTION

The object of the theory of homomorphism duality is to characterise a family $\mathcal{C}$ of obstructions to the existence of a homomorphism into a given structure B. In a large sense, such a class $\mathcal{C}$ always exists; for instance, the class of all the structures not admitting a homomorphism to B has this property. However, it is desirable to seek a more tractable family of obstructions to make this characterisation meaningful.

When the family $\mathcal{C}$ of obstructions is finite (or algorithmically "well behaved") then such theorems clearly provide an example of *good characterisations* (in the sense of Edmonds [2]). Any instance of such good characterisation is called a *homomorphism duality.* This concept was introduced by Nešetřil and Pultr [10] and applied to various graph-theoretical good characterisations (see [9] and references there). The simplest homomorphism dualities are those where the family of obstructions consists of just one structure. In the other words such homomorphism dualities are described by a pair A, B of structures as follows.

(Singleton) Homomorphism Duality Scheme:
C admits a homomorphism into B if and only if A does not admit a homomorphism into C.

The authors would like to express their thanks for the support by DIMATIA and the project LN 00A056 of the Ministry of Education of the Czech Republic.

The (singleton) homomorphism duality may capture general theorems such as Farkas Lemma (see [3]) and Menger-type theorems ([4]). For undirected graphs there is only one singleton duality (Nešetřil and Pultr [10]), but there are many in the directed case (Komárek [6], and Nešetřil and Tardif [11] present a complete list). In [12] the problem is solved in a surprising generality for all finite relational structures. In view of the scarcity of examples that arise in the category of undirected graphs, and in view of the difficulty of the dichotomy problem even for directed graphs it seemed unlikely that the framework for such a generalisation would be found in this context. Yet, paradoxically, this is precisely what happened. The absence of good characterisations for undirected graphs is explained by an apparently unrelated result, that is, the density theorem of Welzl ([13]), which states that the class of undirected non-bipartite graphs is dense with respect to the homomorphism order.

The arguments in [12], formulated in terms of finite relational structures, have a much more general range. One considers the partially ordered class (in fact, lattice) obtained from the preorder

$$A \leq B \text{ iff there exists a morphism } f : A \to B$$

on the class of objects of the category in question; then, all the reasoning is based on the lattice structure and its two special properties, namely that

(1) there is a Heyting operation (that is, an operation adjoint to the meet), and
(2) the elements are suprema of systems of connected ones.

In categories we are usually concerned with (they are typically such that the coproducts are disjoint unions) the second property is ubiquitous. The first ("Heyting") condition is somewhat more special; still, it is being satisfied quite frequently. For instance, trivially, it holds true whenever the original category is cartesian closed. But cartesian closedness is not necessary and we can present non-trivial examples of Heyting categories that are not cartesian closed.

The paper is divided into five sections. In Preliminaries we recall the necessary (very simple) facts about the Heyting operations and about categories, and introduce the notion of Heyting category. Section 2 is devoted to the facts from [12] proved now more generally in the lattice-theoretical context. In Section 3 we define combined dualities (as an extension of finitary dualities) and prove an analogy of the correspondence from Section 2. In Section 4 we discuss a number of cartesian closed categories of combinatorial character, and in Section 5 we present some Heyting categories that are not cartesian closed (for instance, transitive relations with strictly monotone maps, or partial unary algebras).

1. Preliminaries

1.1. In our context, a lattice L can be a proper class.

A *Heyting algebra* is a lattice with an additional operation $\to$ satisfying

$$a \wedge b \leq c \quad \text{iff} \quad a \leq b \to c.$$

Since $b \wedge a \leq b$ we have

$$b \leq a \to b \tag{1.1.1}$$

and since $a \to b \leq a \to b$ we have

$$a \wedge (a \to b) \leq b \quad \text{(modus ponens)}, \tag{1.1.2}$$

and combining these two formulas we obtain

$$a \wedge (a \to b) = a \wedge b. \tag{1.1.3}$$

1.2. An element $a \neq \perp$ of a lattice is *connected* if

$$a = a_1 \vee a_2 \quad \text{implies} \quad a = a_i \text{ for some of the } i.$$

We will say that a Heyting algebra L has *connected decompositions* (briefly, CD) if

(CD) every $x \in L$ is a supremum of a *set* of connected elements.

1.3. We shall use only basic notions from category theory (such as can be found in the introductory chapters of [8]): category, functor, (natural) transformation, natural equivalence (this last will be indicated by $\cong$, and the reader will mostly need only know that it is a one-one correspondence).

If $\mathcal{C}$ is a category, the symbol $\mathcal{C}(A, B)$ will be used for the set of all morphisms $f : A \to B$ in $\mathcal{C}$.

Recall that a product of two objects X_1, X_2 consists of an object X (often indicated as $X_1 \times X_2$) and morphisms $p_i : X \to X_i$ such that for every couple of morphisms $f_i : Y \to X_i$ there is a unique $f : Y \to X$ such that $p_i f = f_i$. Dually, a coproduct (or sum) of two objects X_1, X_2 consists of an object X (often indicated as $X_1 + X_2$) and morphisms $\iota_i : X_i \to X$ such that for every couple of morphisms $f_i : X_i \to Y$ there is a unique $f : X \to Y$ such that $f \iota_i = f_i$.

1.4. For a category $\mathcal{C}$ define a partially ordered class $\widehat{\mathcal{C}}$ as the set of objects of $\mathcal{C}$ ordered by

$$A \leq B \quad \text{iff} \quad \exists f : A \to B$$

factorized by the relation $A \sim B$ iff $A \leq B \leq A$. If there is no danger of confusion, the class consisting of A only is usually denoted by the same symbol.

If the category $\mathcal{C}$ has products then $A_1 \times A_2$ is the meet (infimum) of A_1 and A_2 in $\widehat{\mathcal{C}}$, and if $\mathcal{C}$ has coproducts, $A_1 + A_2$ is the join (supremum) of A_1, A_2. Thus, if $\mathcal{C}$ has products and coproducts, $\widehat{\mathcal{C}}$ is a lattice.

Note. In the categories relevant for our purposes (graphs, special graphs, relational systems, hypergraphs, unary algebras, etc.) the coproducts are typically disjoint unions and we easily see that $\widehat{\mathcal{C}}$ has CD.

1.5. A category with products is said to be *cartesian closed* if there is a functor $[-,-] : \mathcal{C}^{\mathrm{op}} \times \mathcal{C} \to \mathcal{C}$ (the superscript "op" indicates that $[-,-]$ is contravariant in the first variable) such that

$$\mathcal{C}(A \times B, C) \cong \mathcal{C}(A, [B, C]).$$

The functor $[-,-]$ will be called the *exponentiation* in $\mathcal{C}$.

1.6. A category $\mathcal{C}$ is said to be *Heyting* if the partially ordered class $\widehat{\mathcal{C}}$ is a Heyting lattice.

Note that a cartesian closed category is Heyting, with the Heyting operation $A{\to}B = [A, B]$. This should not be confused with the notation $A \to B$ (denoting the existence of a morphism) insome of the papers mentioned in the Introduction: there it is the order of the $\widehat{\mathcal{C}}$ and here it will be indicated by $A \leq B$).

2. Gaps and dualities

Throughout this section, L is a Heyting algebra (possibly carried by a proper class) with connected decompositions.

2.1. A *gap* in L is a couple (c, d) such that $c < d$ and $c \leq x \leq d$ implies that either $x = c$ or $x = d$. We will usually say "$c < d$ is a gap" instead of "(c, d) is a gap".

2.1.1. Lemma. *Let $a < b$ be a gap and let $a \leq c < c \vee b$. Then $c < c \vee b$ is a gap.*

Proof. Let $c \leq x \leq c \vee b$. Then $a \leq x \wedge b \leq b$ and hence either $a = x \wedge b$ and $x = x \wedge (c \vee b) = (x \wedge c) \vee (x \wedge b) = c \vee a = c$, or $x \wedge b = b$, hence $x \geq b$ and since also $x \geq c$ we conclude $x \geq c \vee b$. $\square$

2.1.2. Lemma. *Let $c < d$ be a gap. Then there is precisely one connected b such that $d = c \vee b$.*

Proof. Let $d = \bigvee_{i \in J} d_i$ with d_i connected and let $b = d_i$ be some of the summands such that $b \not\leq c$. Then $c \lneq c \vee b \leq d$ and hence $c \vee b = d$. Now let also $c \vee b' = d$ with b' connected. Then $b = b \wedge (c \vee b') = (b \wedge c) \vee (b \wedge b')$ and since $b \neq b \wedge c$ we have $b = b \wedge b'$ and $b \leq b'$. Similarly $b' \leq b$. $\square$

2.2. A *duality* in L is a couple (a, b) such that

$$a \leq x \quad \text{iff} \quad x \not\leq b.$$

2.2.1. Observation. If (a, b) is a duality then a is connected.
(Indeed, if $a = a_1 \vee a_2 \not\leq a_i$, $i = 1, 2$, then $a = a_1 \vee a_2 \leq b$ and $a \not\leq a$.)

2.3.1. Lemma. *Let $c < d$ be a gap and let a be the unique connected element such that $d = c \vee a$. Then $(a, a \to c)$ is a duality. Consequently, if d is connected, $(d, d \to c)$ is a duality.*

Proof. If $a \leq x$ we cannot have $x \leq a \to c$ since else $a = a \wedge c \leq c$. On the other hand, if $x \not\leq a \to c$ then $x \wedge a \not\leq c$, and since $c < (x \wedge a) \vee c \leq d = a \vee c$ we have $(x \wedge a) \vee c = a \vee c$. Thus, $(x \wedge a) \vee (c \wedge a) = (a \vee c) \wedge a = a$ and since a is connected and $a \neq c \wedge a$, $a = x \wedge a$ and $a \leq x$. $\square$

2.3.2. Lemma. Let (a, b) be a duality. Then $(a \wedge b, a)$ is a gap.
Proof. Let $a \wedge b \leq x \leq a$. If $a \not\leq x$ we have $x \leq b$ and $x \leq a \wedge b \leq x$. $\square$

2.4. Proposition. *The formulas*

$$\alpha(a, b) = (a \wedge b, a), \quad \beta(c, d) = (d, d \to c)$$

constitute a one-one correspondence between dualities and gaps with connected d.

Proof. By 2.3.2 and 2.2.1, $(a \wedge b, a)$ is a gap with connected a, and by 2.3.1, $(d, d \to c)$ is a duality. Further, $\beta\alpha(a, b) = (a, a \to (a \wedge b))$. Trivially, $b \leq a \to (a \wedge b)$; if $a \to (a \wedge b) \not\leq b$ we have $a \leq a \to (a \wedge b)$ and $a \leq a \wedge b$, that is, $a \leq b$ and $a \not\leq a$. Thus, $\beta\alpha(a, b) = (a, b)$. Finally, $\alpha\beta(c, d) = (d \wedge (d \to c), d) = (d \wedge c, d) = (c, d)$ by (1.1.3). $\square$

2.4.1. Note. The existence of the Heyting operation (which for finite lattices is equivalent to distributivity) is essential. It is not just that lacking $\to$ we could not have the simple formula: for instance, in the Chinese lantern $\{\perp < 1, 2, \ldots, n < \top\}$ with $n \geq 3$ there is no duality while we have n gaps $\perp < i$ with connected i. The relation of distributivity to the links between (connected) gaps and dualities in finite lattices is not quite clear and may be of some interest.

2.5. Proposition. *Let $c < d$ be a gap and let a be the unique connected element such that $d = c \vee a$. Then $e = c \wedge a$ forms a gap $e < a$ such that $e \leq c \leq a \to e$.*

Proof. By 2.3, $(a, a \to c)$ is a duality and hence, by 2.4, $a \wedge (a \to c) < a$ is a gap. By (1.1.3), $a \wedge (a \to c) = a \wedge c$. $\square$

2.6. Proposition. *The gaps in L are exactly the couples $c < d$ such that for some duality (a, b),*

$$a \wedge b \leq c \leq b \quad \textit{and} \quad d = a \vee c.$$

Proof. If $c < d$ is a gap consider the duality $(a, a \to c)$ as in 2.5. Then $a \wedge (a \to c) \leq c \leq (a \to c)$. On the other hand, let for a duality (a, b), $a \wedge b \leq c \leq b$ and $d = a \vee c$. Let $c \leq x \leq d = a \vee c$. If $a \vee c \not\leq x$ then $a \not\leq x$ and hence $x \leq b$; consequently $x = x \wedge (c \vee a) \leq c \vee (a \wedge b) = c$. $\square$

3. Combined dualities

3.1. A *combined duality* is a couple $((a_i)_{i \in J}, b)$ such that

(1) if $i \neq j$ then $a_i \not\leq a_j$,
(2) $(\forall i,\ a_i \not\leq x)$ iff $x \leq b$.

Note. Suppose we have a system satisfying just the condition (2). If J is finite, it can be easily modified, omitting some of the a_i, to a combined duality with equivalent (2). If J is infinite, however, the first assumption is essential.

3.2. Lemma. *Let $((a_i)_{i \in J}, b)$ be a combined duality. Then all the a_i are connected.*

Proof. Let $a_{i_0} = c \vee d$ with $a_{i_0} \not\leq c, d$. Then for all i, $a_i \not\leq c, d$, hence $c, d \leq b$ and finally $a_{i_0} = c \vee d \leq b$ contradicting $a_{i_0} \leq a_{i_0}$. $\square$

3.3. Proposition. *I. Let $((a_i)_{i\in J}, b)$ be a combined duality. Let either J be finite or L admit infima of sets of the size of the J. Then there are dualities (a_i, b_i), $i \in J$, such that $b = \bigwedge_{i\in J} b_i$.*

II. On the other hand, if (a_i, b_i), $i \in J$, are dualities and $b = \bigwedge_{i\in J} b_i$ and $a_i \nleq a_j$ for $i \neq j$ then $((a_i)_{i\in J}, b)$ is a combined duality.

Proof. I. By 3.2 a_i are connected. We have $a_i \nleq b$ (else $a_i \nleq a_i$) and hence $a_i < b \vee a_i$. Now each of these $a_i < b \vee a_i$ is a gap. Indeed, let $b \leq x \leq b \vee a_i$. If $x \nleq b$ there is a j such that $a_j \leq x$. If $j \neq i$ we had a non-trivial decomposition $a_j = (b \wedge a_i) \vee (a_i \wedge a_j)$ so that necessarily $i = j$. Thus, $a_i \leq x$ and $b \leq x$ and we have $b \vee a_i \leq x$. Hence by 2.5 there are gaps $c_i < a_i$ such that $c_i \leq b \leq a_i \!\to\! c_i$. By 2.3, $(a_i, a_i \!\to\! c_i)$ are dualities and hence

$$\forall i\ a_i \nleq x \quad \text{iff} \quad \forall i\ x \leq a_i \!\to\! c_i \quad \text{iff} \quad x \leq \bigwedge (a_i \!\to\! c_i).$$

II. The second statement is obvious. $\square$

4. Some cartesian closed categories

This section contains several examples of cartesian closed categories relevant for combinatorics. The cartesian structure in these categories is often well-known (for the exponentiations from 4.1 and 4.2 see, e.g., [7]); we describe just the exponentiation mechanisms and leave the checking to the reader.

4.1. Systems of binary relations. In $\mathbf{Rel}(n)$, the category of sets (X, R), $R = (R_1, \ldots, R_n)$ with n binary relations, with the standard (all relations preserving) homomorphisms, one has the exponentiation

$$[(X, R), (Y, S)] = (\{\varphi \mid \varphi : X \to Y \text{ all maps}\}, T)$$

with $(\varphi, \psi) \in T_i \quad \text{iff} \quad (x, y) \in R_i \;\Rightarrow\; (\varphi(x), \psi(y)) \in S_i$.

For maps one defines $[f, g](\alpha) = g\alpha f$. We have this formula also in most of the following examples; it will not be unnecessarily repeated.

4.1.1. Symmetric relations. The subcategory $\mathbf{SymRel}(n)$ generated by the (X, R) where all the R_i are symmetric is obviously closed in $\mathbf{Rel}(n)$ under product and the exponentiation and hence inherits the cartesian structure.

4.1.2. Note that $\mathbf{Rel}(1)$ is the category of oriented (directed) graphs, and that $\mathbf{SymRel}(1)$ is the category of graphs with loops allowed. Observe, however, that if the relations in the paragraphs above are antireflexive, so are the products and exponentiations, so that we obtain the cartesian structures also on the loop-free graphs, more usual in combinatorics.

4.2. The category

$$A\text{-}\mathbf{Graph}$$

of A-*graphs* is defined as follows:

the objects X are couples $(\mathsf{V}(X), \mathsf{E}(X))$ where $\mathsf{V}(X)$ is a set, and $\mathsf{E}(X)$ is a subset of $\mathsf{V}(X)^A$;

the morphisms $f : X \to Y$ are maps $f : \mathsf{V}(X) \to \mathsf{V}(Y)$ such that for every $\alpha \in \mathsf{E}(X)$ the composition $f \cdot \alpha$ is in $\mathsf{E}(Y)$.

The category **Sym**A**-Graph** is the full subcategory of A**-Graph** generated by the X such that for every $\alpha \in \mathsf{E}(X)$ and every permutation π of A, $\alpha\pi$ is in $\mathsf{E}(X)$.

We obviously have the product given by the formula

$$\begin{aligned}
&\mathsf{V}(X_1 \times X_2) = \mathsf{V}(X_1) \times \mathsf{V}(X_2),\\
&\mathsf{E}(X_1 \times X_2) =\\
&\qquad\{(\alpha_1 \times \alpha_2) \cdot \Delta \mid \alpha_i \in \mathsf{E}(X_i),\ \Delta \text{ the diagonal } A \to A \times A\}.
\end{aligned}$$

For A-graphs Y, Z define an A-graph $[Y, Z]$ by setting

$$\begin{aligned}
\mathsf{V}([Y, Z]) = \{\varphi : \mathsf{V}(Y) \to \mathsf{V}(Z) \mid (\textit{all}\text{ maps})\},&\\
\varphi \in \mathsf{E}([Y, Z]) \quad \text{iff for each } \beta \in \mathsf{E}(Y)&\\
(a \mapsto \varphi(a)(\beta(a))) : A \to Z \text{ is in } \mathsf{E}(Z).&
\end{aligned}$$

This cartesian closedness structure restricts to **Sym**A**-Graph**.

Note. Of course, this example can be generalized for systems of relations of various arities.

4.3. Hypergraphs. The category **HGraph** of *hypergraphs* is defined as follows:

the objects (*hypergraphs*) X are couples $(\mathsf{V}(X), \mathsf{E}(X))$ where $\mathsf{V}(X)$ is a set (the set of *vertices* of X) and $\mathsf{E}(X)$, the set of *hyperedges* of X, is any subset of $\mathfrak{P}(\mathsf{V}(X))$;

the morphisms $f : X \to Y$ are maps $f : \mathsf{V}(X) \to \mathsf{V}(Y)$ such that for every $U \in \mathsf{E}(X)$ the image $f[U]$ is in $\mathsf{E}(Y)$.

The product in **HGraph** is given by the formula

$$\mathsf{V}(X_1 \times X_2) = \mathsf{V}(X_1) \times \mathsf{V}(X_2),$$
$$\mathsf{E}(X_1 \times X_2) = \{U \mid p_i[U] \in \mathsf{E}(X_i)\}.$$

The exponentiation $[Y, Z]$ is defined by setting

$$\mathsf{V}([Y,Z]) = \{\varphi : \mathsf{V}(Y) \to \mathsf{V}(Z) \mid (\textit{all} \text{ maps})\},$$
$$\Phi \in \mathsf{E}([Y,Z]) \quad \text{iff for any} \quad B \in \mathsf{E}(Y)$$
$$\text{and any two surjections} \quad \alpha : M \twoheadrightarrow \Phi,\ \beta : M \twoheadrightarrow B \in \mathsf{E}(Y),$$
$$\{\alpha(m)(\beta(m)) \mid m \in M\} \in \mathsf{E}(Z).$$

4.3.1. Hypergraphs with bounded hyperedges. Let α be an infinite cardinal. Denote by $\mathbf{HGraph}_\alpha$ the full subcategory of **HGraph** generated by the hypergraphs X such that for each $A \in \mathsf{E}(X)$, $|A| < \alpha$. Thus,

$$\mathbf{HGraph}_{\omega_0} = \mathbf{HGraph}_{\text{fin}}$$

is the category of hypergraphs with finite hyperedges.

Obviously,

(1) if X, Y are in $\mathbf{HGraph}_\alpha$ then so is the product $X \times Y$, and
(2) if we set, for a general hypergraph X,

$$X_{<\alpha} = (\mathsf{V}(X), \{A \in \mathsf{E}(X) \mid |A| < \alpha\})$$

then for any $Y \in \mathbf{HGraph}_\alpha$ and any X, the morphisms $Y \to X$ coincide with the morphisms $Y \to X_{<\alpha}$.

Consequently

The category $\mathbf{HGraph}_\alpha$ *is cartesian closed with the exponentiation* $[X, Y]_{<\alpha}$.

Note. This is, of course, a special case of the general categorical fact that if $\mathcal{A}$ is cartesian closed and $\mathcal{B}$ a coreflective subcategory closed under finite products, then $\mathcal{B}$ is cartesian closed.

Here we have such a coreflection given by the system $(X_{<\alpha} \to X)_X$ carried by the identities.

4.4. The categories of functors $\mathbf{Set}^{\mathsf{A}}$. A much stronger fact, namely that these categories are topoi (see, e.g., [5]), is well known and widely used. Since, however, the mechanism differs from the previous cases, we will be more explicit.

Let A be a small category. For functors $F, G : \mathsf{A} \to \mathbf{Set}$ denote by

$$\langle F, G \rangle$$

the set of all transformations $F \to G$. For objects $a \in A$ define $A(a)(c) = \mathsf{A}(a,c)$, the set of all morphisms $a \to c$ in A, and for a morphism $\varphi : c \to d$ define $A(a)(\varphi) : A(a)(c) \to A(a)(d)$ by setting $A(a)(\varphi)(\alpha) = \varphi\alpha$. Obviously each $A(a)$ is a functor $\mathsf{A} \to \mathbf{Set}$.

Further, if $f : b \to a$ is a morphism in A, define a transformation $A(f) : A(a) \to A(b)$ by setting $A(f)_a(\alpha) = \alpha f$.

For functors $G, H : \mathsf{A} \to \mathbf{Set}$ define

$$[G,H](a) = \langle A(a) \times G, H\rangle \text{ for objects } a \in \mathsf{A}, \text{ and}$$
$$[G,H](f)(\tau) = \tau \cdot (A(f) \times \mathrm{id}) \text{ for morphisms } f : a \to b.$$

If we define, for a transformation $\tau : F \times G \to H$, a transformation

$$\widetilde{\tau} : F \to \langle A(-) \times G, H\rangle = [G,H]$$

by setting, for $x \in F(a)$, $(\widetilde{\tau}_a(x))_b(\alpha, y) = \tau_b(F(\alpha)(x), y)$ (one has to prove, of course, that each individual $\widetilde{\tau}_a(x)$ is a transformation $A(a) \times G \to H$, and that $\widetilde{\tau}$ is a transformation as a whole), and if we define for $\theta : F \to [G,H]$ a transformation $\overline{\theta} : F \times G \to H$ by setting $\overline{\theta}_a(x,y) = (\theta_a(x))_a(\mathrm{id}, y)$ we find that $\overline{\widetilde{\tau}} = \tau$ and $\widetilde{\overline{\theta}} = \theta$ and that the correspondences $\tau \mapsto \widetilde{\tau}$ and $\theta \mapsto \overline{\theta}$ constitute a natural equivalence. Thus,

the functors $[G,H]$ constitute a cartesian exponentiation in $\mathbf{Set}^{\mathsf{A}}$.

4.4.1. A particular case is for instance *the category of multigraphs* with A constituted by two objects a, b and non-identical morphisms $\alpha, \beta : a \to b$.

Other examples are *arbitrary varieties of unary algebras*, obtained from suitable monoids A. For instance, in the simplest case of sets with single unary operations the exponentiation is very transparent: we have ($\mathbb{N}$ is the set of natural numbers endowed with the successor operation $(i \mapsto i+1)$)

$$[(Y,\beta),(Z,\gamma)] = (\{\varphi \mid \varphi : \mathbb{N} \times (Y,\beta) \to (Z,\gamma)\}, \nu)$$

with $\nu(\varphi)(i,y) = \varphi(i+1,y)$.

4.5. Note. The reader may have observed that none of our examples had the exponentiation $[A,B]$ given as the set of *the morphisms* $A \to B$ (endowed by suitable structures). Cartesian categories with such exponentiations are not rare (for instance, the categories of reflexive relations have the property). They are not very interesting in our context, though. We have

Fact. *Let $\mathcal{C}$ be a cartesian closed category and let there be a forgetful functor $U : \mathcal{C} \to \mathbf{Set}$ such that $G \in \mathcal{C}$ for some object G, $U(X) \cong \mathcal{C}(G, X)$ and that we have $U([A, B]) = \mathcal{C}(A, B)$. Then the partially ordered class $\widehat{\mathcal{C}}$ has at most two elements.*

Proof. We have $\mathcal{C}(A, B) \cong U([A, B]) \cong \mathcal{C}(G, [A, B]) \cong \mathcal{C}(G \times A, B)$. Thus, there is always a morphism $A \to G \times A$ (take $B = G \times A$ and the morphism corresponding to the identity $G \times A \to G \times A$), and consequently also $A \to G \times A \to G$. On the other hand, unless $U(A)$ is void (which, by the faithfulness, can happen at most for one isomorphism type), we have also $G \to A$. $\square$

5. Some more Heyting categories

5.1. A Heyting category is not necessarily cartesian closed: for instance every connected $\mathcal{C}$ (that is, a $\mathcal{C}$ with trivial $\widehat{\mathcal{C}}$) is Heyting. To obtain a less trivial example consider the obvious fact that a product of Heyting categories is Heyting; take a cartesian closed $\mathcal{C}_1$ and a connected $\mathcal{C}_2$, and form $\mathcal{C}_1 \times \mathcal{C}_2$.

5.2. A more interesting example. Consider the category $\mathcal{C}$ of transitive relations with (strictly) monotone maps. It is obvious that $\widehat{\mathcal{C}}$ is isomorphic to the ordinal $\omega + 1$ (also it is a special case of the simplicial sets, see([8]) The poset $\omega + 1$ is Heyting, but we will show that $\mathcal{C}$ is not cartesian closed.

Suppose there is an exponentiation $[Y, Z]$ with a natural equivalence $\varepsilon : \mathcal{C}(X \times Y, Z) \cong \mathcal{C}(X, [Y, Z])$. Consider

$$P = (\{0\}, \emptyset), \quad A = (\{0, 1\}, \{(0, 1)\}),$$
$$B = (\{0, 1, 2\} \times \{0, 1\}, \{(0, 0), (1, 1)), ((1, 0), (2, 1))\},$$

and the morphisms

$$\xi_i : P \to A, \quad \xi_i(0) = i,$$
$$\varphi_i : A \times A \to B, \quad \varphi_(j, k) = (j + i, k).$$

Then $\varphi_0(\xi_1 \times \mathrm{id}) = \varphi_1(\xi_0 \times \mathrm{id})$ and hence we can define, for $i = 0, 1, 2$,

$$\alpha_i = \varphi_j(\xi_k \times \mathrm{id}) \quad \text{with} \quad i = j + k.$$

Set $x_i = \varepsilon(\alpha_i(0)$. Then

$$\begin{aligned}\varepsilon(\varphi_i)(j) &= (\varepsilon(\varphi_i \cdot \xi_j)(0) = \mathcal{C}(\xi_j, \mathrm{id})(\varepsilon(\varphi_i))(0) = \\ &= \varepsilon(\mathcal{C}(\xi_i \times \mathrm{id}, \mathrm{id})(\varphi_i))(0) = \varepsilon(\varphi_i \cdot (\xi_j \times \mathrm{id}))(0) = x_{i+j}\end{aligned}$$

so that $x_0 < x_1 < x_2$. Hence $x_0 < x_2$ and there is a $\psi : A \to [A, B]$ such that $\psi(0) = x_0$, that is, $\psi \cdot \xi_0 = \varepsilon(\alpha_0)$, and $\psi(1) = x_2$, that is, $\psi \cdot \xi_1 = \varepsilon(\alpha_2)$. Then

$$\varepsilon^{-1}(\psi)(0,0) = (\varepsilon^{-1}(\xi_0 \times \mathrm{id}))(0,0) = \varepsilon^{-1}(\psi \cdot \xi_0)(0) = \alpha_0(0,0) = (0,0)$$

and

$$\varepsilon^{-1}(\psi)(1,1) = (\varepsilon^{-1}(\xi_1 \times \mathrm{id}))(0,1) = \varepsilon^{-1}(\psi \cdot \xi_1)(0) = \alpha_2(0,1) = (2,1),$$

while $(0,0) \not< (2,1)$.

5.3. Another class of examples of Heyting categories that are not cartesian closed is provided by the following trivial fact.

Proposition. *Let $\mathcal{A}$ be a cartesian closed category with the exponentiation $[X, Y]$ and let $\mathcal{B}$ be a full subcategory closed under product. Let there be a functor $F : \mathcal{A} \to \mathcal{B}$ such that*

(a) *for each A in $\mathcal{A}$ there is a morphism $F(A) \to A$, and*
(b) *for each B in $\mathcal{B}$ there is a morphism $B \to F(B)$.*

Then $\mathcal{B}$ inherits the Heyting structure by way of $F([X, Y])$.

Proof. If B is in $\mathcal{B}$ and A is general we have $B \to A$ iff $B \to F(A)$ ($\Rightarrow$ because of

$$B \longrightarrow F(B) \xrightarrow{F(f)} F(A)$$

using (b), and $\Leftarrow$ immediately from (a).) Thus, $B_1 \times B_2 \to B_3$ iff $B_1 \to [B_2, B_3]$ iff $B_1 \to F([B_2, B_3])$. $\square$

5.4. Partial unary algebras. Denote by

$$\mathbf{PA}(n \times 1)$$

the full subcategory of $\mathbf{Rel}(n)$ generated by the partial unary algebras (X, R) (that is, the (X, R) such that for any i, if xR_iy and xR_iy' then $y = y'$; we then write $y = R_i(x)$).

Proposition. *The category* $\mathbf{PA}(n \times 1)$ *is Heyting.*

Proof. Denote by $\mathbb{T}_n$ the system of all finite words in $1, 2, \ldots, n$, including the empty word $\emptyset$. For an object (X, R) of $\mathbf{Rel}(n)$ set

$$\widetilde{T_R} = \{t : \mathbb{T}_n \rightharpoonup X \mid \text{partial maps satisfying (1) and (2)}\},$$

where

(1) if t is defined for vw then it is defined for v, and
(2) $t(j_1 \cdots j_k) R_i t(i j_1 \cdots j_k)$.

On $\widetilde{R_n}$ define the relational system $\widetilde{R} = (\widetilde{R}_1, \ldots, \widetilde{R}_n)$ by setting

$$t \widetilde{R}_i \tau \quad \text{iff} \quad \tau(i_1 \cdots i_k) = t(i_1 \cdots i_k i).$$

Obviously, $(\widetilde{X_R}, \widetilde{R})$ is in $\mathbf{PA}(n \times 1)$.

Define $p = p_{(X,R)} : (\widetilde{X_R}, \widetilde{R}) \to (X, R)$ by setting $p(t) = t(\emptyset)$ (if $t\widetilde{R}_i\tau$ we have in particular $\tau(\emptyset) = t(i)$ and hence $p(t) = t(\emptyset) R_i t(i) = p(\tau)$).

For a homomorphism $f : (X, R) \to (Y, S)$ define $\widetilde{f} : (\widetilde{X_R}, \widetilde{R}) \to (\widetilde{Y_S}, \widetilde{S})$ by setting $\widetilde{f}(t) = f \cdot t$. Obviously this is a homomorphism and we see that we have obtained a functor $\mathbf{Rel}(n) \to \mathbf{PA}(n \times 1)$. If (X, R) is in $\mathbf{PA}(n \times 1)$ we can define $q : (X, R) \to (\widetilde{X_R}, \widetilde{R})$ by setting

$$q(\emptyset) = x, \quad q(x)(i_1 i_2 \cdots i_k) = R_{i_1} R_{i_2} \cdots R_{i_k}(x) \text{ whenever defined.}$$

(The condition (1) is obviously satisfied, and $q(x)(iw) = R_i(q(x)(w))$, hence also (2). Now if xR_iy, that is, $y = R_i(x)$, we have $q(y)(i_1 \cdots i_k) = R_{i_1} \cdots R_{i_k} R_i(x) = q(x)(i_1 \cdots i_k i)$ so that q is a homomorphism.)

Thus, by 4.3, $\mathbf{PA}(n \times 1)$ is Heyting. $\square$

However

the category $\mathbf{PA}(n \times 1)$ *is not cartesian closed.*

Indeed, consider $A = (\{0\}, \emptyset)$, $B = (\{0,1\} \times \{0,1\}, R_j = \{((0,i),(1,i)), i = 0,1\} \mid j = 1, \ldots, n\}$ and $C = (\{0,1\}, S_j = \{(0,1)\}, j = 1, \ldots, , n)$, and the maps $f_i : A \to B$ sending 0 to $(0,i)$. Then the coequalizer of f_1, f_2 is the homomorphism $g : B \to C$ defined by $g(i,j) = i$ while $g \times \mathrm{id}_A$ is not the coequalizer of $f_i \times \mathrm{id}_A$.

References

[1] D. Duffus, N. Sauer, *Lattices arising in categorial investigations of Hedetniemi's conjecture*, Discrete Math. **152** (1996), 125–139.

[2] J. Edmonds, *Paths, Trees and Flowers*, Canad. J. Math. **17** (1965), 449-467.

[3] W.Hochstättler, J. Nešetřil: Linear programming duality and morphisms, Comment. Math. Univ. Carol. 40,3 (1999), 577–592.

[4] W.Hochstättler, J. Nešetřil: A note on maxflow-mincut and homomorphic equivalence of matroids, J. Alg. Comb. **12,3** (2000), 295–300.

[5] P.T. Johnstone, *T*opos Theory, Academic Press (1977).

[6] P. Komárek, *Some good charactersizations for directed graphs*, Čas. Pěst. Mat. **109** (1984), 348-354.
[7] L. Lovász, *Operations with structures,* Acta Math. Acad. Sci. Hung., **18** (1972), 321–328.
[8] S. Mac Lane, *Categories for the Working Mathematician,* Springer-Verlag, New York 1971.
[9] J. Nešetřil, *Aspects of structural combinatorics (Graph homomorphisms and their use),* Taiwanese J. Math., **3,4** (1999), 381–423.
[10] J. Nešetřil, A. Pultr, *On classes of relations and graphs determined by subobjects and factorobjects*, Discrete Math. **22** (1978), 287–300.
[11] J. Nešetřil, C. Tardif, *Density via duality*, Theor. Comp. Sci. **287,2** (2002), 585-595.
[12] J. Nešetřil, C. Tardif, *Duality theorems for finite structures (characterising gaps and dualities)* J. Comb. Th. B, **80,1** (2000), 80–97.
[13] E. Welzl, *Color-families are dense*, J. Theor. Comput. Sci. **17** (1982), 29-41.

Department of Applied Mathematics and ITI, MFF, Charles University, CZ 11800 Praha 1, Malostranské nám. 25
E-mail address: `nesetril@kam.ms.mff.cuni.cz, pultr@kam.ms.mff.cuni.cz`

Department of Mathematics and Computer Science, Royal Military College of Canada, Kingston, Ontario K7K 7B4
E-mail address: `Claude.Tardif@rmc.ca`