Geometric graphs with no three disjoint edges

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Abstract

A *geometric graph* is a graph drawn in the plane so that the vertices are represented by points in general position and edges are represented by straight line segments. We show that a geometric graph on n vertices with no three pairwise disjoint edges has at most $2.5n+1$ edges. This result is tight up to a constant.

1 Introduction

An *(abstract) graph* G is a pair $(V(G), E(G))$ where $V(G)$ is the set of *vertices* and $E(G)$ is the set of *edges* $\{u, v\}$ each joining two vertices $u, v \in V(G)$.

A *geometric graph* is a graph G drawn in the plane by straight line segments. It is defined as a pair $(V(G), E(G))$, where $V(G)$ is a finite set of points in general position in the plane, i.e. no three points are collinear, and $E(G)$ is a set of line segments with endpoints in $V(G)$. $V(G)$ and $E(G)$ are the *vertex set* and the *edge set* of G, respectively. Let H and G

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be two geometric graphs, we say that H is a *(geometric) subgraph* of G if $V(H) \subseteq V(G)$ and $E(H) \subseteq E(G)$.

A *topological graph* is defined similarly. It is a graph drawn in the plane in such a way that edges are Jordan curves. Two of these curves share at most one point and no curve passes through a vertex. Obviously, geometric graphs are a subclass of topological graphs. We say that two edges *cross* each other if they have an interior point in common. Two edges are *disjoint* if they have no point in common.

We investigate properties of subclasses of geometric or topological graphs with some geometrical constrains. One of the simplest questions is how to characterize graphs with no crossing edges. These graph are known as *plane* graphs and have been studied for more than hundred years.

Kupitz, Erdős and Perles initiated and many others continued in the investigation of the following general problem. Given a class $\mathcal{H}$ of so-called forbidden geometric subgraphs, determine or estimate the maximum number $t(\mathcal{H}, n)$ of edges that a geometric graph with n vertices can have without containing a subgraph belonging to $\mathcal{H}$.

There are many nice results for various forbidden classes — k pairwise crossing edges, k pairwise "parallel" edges, k pairwise disjoint edges, self-crossing paths, even cycles and many others. For a survey of results on geometric graphs see Pach [9].

We focus on geometric graphs with no $k+1$ pairwise disjoint edges. For $k \geq 1$, let $\mathcal{D}_k$ denote the class of all geometric graphs consisting of k pairwise disjoint edges. Denote $d_k(n) = t(\mathcal{D}_{k+1}, n)$ the maximum number of edges of a geometric graph on n vertices with no $k+1$ pairwise disjoint edges.

Let's look at the history of this problem. One of the first investigations on geometric graphs, besides planar graphs, was motivated by repeated distances in the plane. Erdős asked how many times can the maximum distance among n points in the plane be repeated. Connect each pair of points with the maximum distance by an edge. It's clear that the resulting graph cannot have two disjoint edges. The convex hull of endpoints of two disjoint edges forms either a triangle or a quadrilateral. In both cases there is a distance longer than the length of the edge. That's a contradiction. The former question turns to the following: How many edges can have a geometric graph with no two disjoint edges? Erdős [3] proved the following theorem:

Theorem 1 *$d_1(n) = n$.*

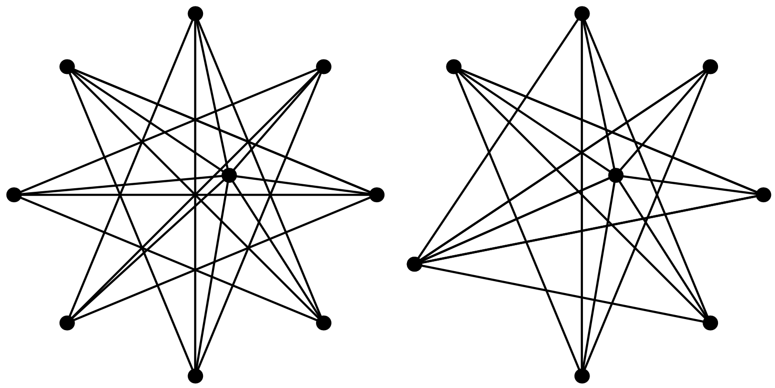
Alon and Erdős (1989) proved $d_2(n) \leq 6n$. One year later O'Donnell and Perles (1990) improved it to $d_2(n) \leq 3.6n + c$. Later Goddard et al. [5] (1993) showed $d_2(n) \leq 3n$. At the end Mészáros [8] improved that to $d_2 \leq 3n - 1$. Combining some of the ideas of the proof of Goddard et al. [5] with a discharging method we show the following upper bound:

Theorem 2 $d_2(n) \leq \lfloor 2.5n \rfloor + 1$.

The best known lower bound is due to Perles:

Theorem 3 *(Perles)* $d_2(n) \geq \lceil 2.5n \rceil - 3$.

Examples of such a graph for $n = 9$ and for $n = 8$ are given in the following figure. The construction for odd n can be easily generalized and the graph for even n is obtained from the previous one by contraction of two neighboring vertices on the convex hull.



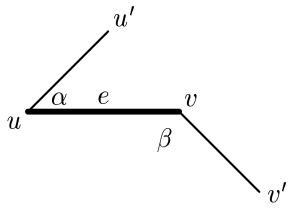
The first general upper bound $d_k(n) = O(n(\log n)^{k-3})$ was given again by Goddard et al. [5]. In 1993 Pach and Törőcsik [11] introduced the order relations on disjoint edges and as an application of Dilworth's Theorem they showed that $d_k(n) \leq k^4 n$. That was the first upper bound linear in n. Tóth and Valtr [13] added a concept of zig-zag and improved the bound to $d_k(n) \leq k^3(n+1)$. Later Tóth [12] further improved the bound to $d_k(n) \leq 256k^2 n$. Original constant in Tóth's proof was a bit bigger. This one is due to Felsner [4].

It is believed that the truth is about $d_k(n) \sim ckn$. It is also an interesting problem, if this is true for geometric graphs whose edges can be intersected by a line. That would give general upper bound $d_k(n) \leq c(k \log k)n$. Just bisect the vertex set of the graph and count edges in both parts recursively.

2 Preliminaries

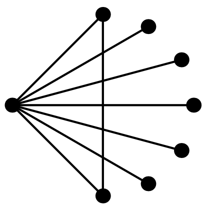
We show the proof of theorem 1 because it is a beautiful illustration of a discharging method and it is very simple. Later we use a similar but more complicated approach to prove theorem 2 — the upper bound for $d_2(n)$.

Proof of theorem 1 (Perles): For each vertex mark one edge incident to it. For vertices of degree one, there is no choice. At the other vertices, mark the right edge at the largest angle. If there remain an unmarked edge $e = uv$, we have the situation as in the following figure:



There must be edges uu' and vv' because we marked the right edge at the largest angle at every vertex. Angles α and β are less than the largest angles at the vertices u, v, so they are less than π. Edges uu', vv' are disjoint and hence there is no unmarked edge. Thus, the number of edges is less than the number of vertices.

On the other hand there exist graphs achieving this bound:



■

Definition 1 *A vertex v is pointed if all edges incident with it lie in a halfplane whose boundary contains the vertex v.*

Definition 2 *A vertex v which is not pointed is cyclic. This means that in every open halfplane determined by a line passing through the vertex v there is an edge incident with v.*

Definition 3 *We say that an edge xy is to the left of an edge xz if the ray $\overrightarrow{xz}$ can be obtained from the ray $\overrightarrow{xy}$ by a clockwise turn of less than π. Similarly we define when an edge is to the right of another edge.*

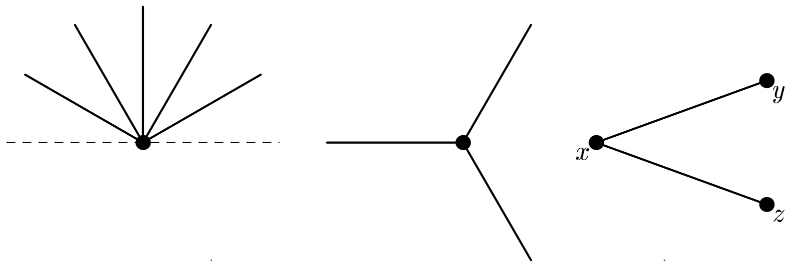
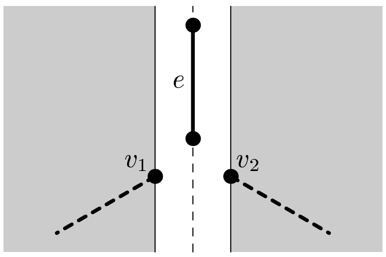


Figure 1: On the left is an example of a pointed vertex, in the middle of a cyclic vertex and on the right are edges xy, xz where the edge xy is to the left of the edge xz.

3 Cyclic vertices

Lemma 1 *A geometric graph with two cyclic vertices and an edge, whose continuation strictly separates these two cyclic vertices, contains three pairwise disjoint edges.*

Proof:



In the picture there are two cyclic vertices and an edge which separates them strictly. We can find an edge in each grey halfplane, because the vertices v_1,v_2 are cyclic. This yields three pairwise disjoint edges. ∎

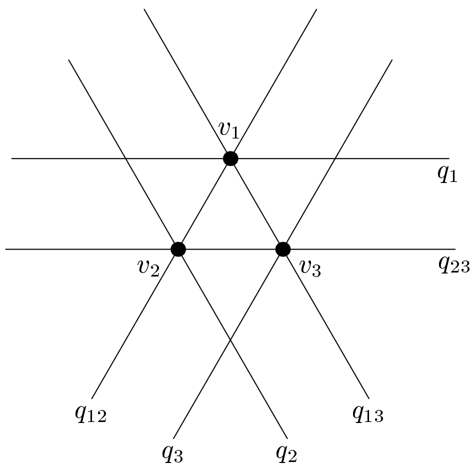


Figure 2: Position of three cyclic vertices

Lemma 2 *A geometric graph with three cyclic vertices contains three pairwise disjoint edges.*

Proof: Denote the cyclic vertices by v_1,v_2 and v_3. Let q_i be the line passing through v_i parallel to the line passing through the other two cyclic vertices (for $i = 1, 2, 3$). See figure 2.

There must be an edge e_i incident with v_i lying in the open halfplane determined by the line q_i, not containing the other cyclic vertices. At least two edges from e_i, $i = 1, 2, 3$ must cross. Otherwise we have three disjoint edges. W.l.o.g. edges e_2 and e_3 cross. Let us discuss where can the third edge lie. See figure 3.

If e_1 lies in region R_1, use lemma 1. In the other cases, there must be an edge in the open halfplane determined by the line q_{12}, not containing other cyclic vertices, because vertex v_2 is cyclic. See figure 4. This edge and edges e_1, e_3 are pairwise disjoint. ∎

We remark that for $k \geq 4$ there is a geometric graph G with k cyclic vertices without 4 pairwise disjoint edges (see figure 5).

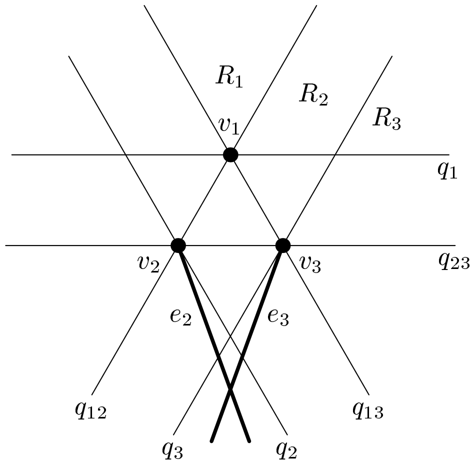


Figure 3: Edge e_1 can w.l.o.g lie only in regions R_1, R_2 and R_3.

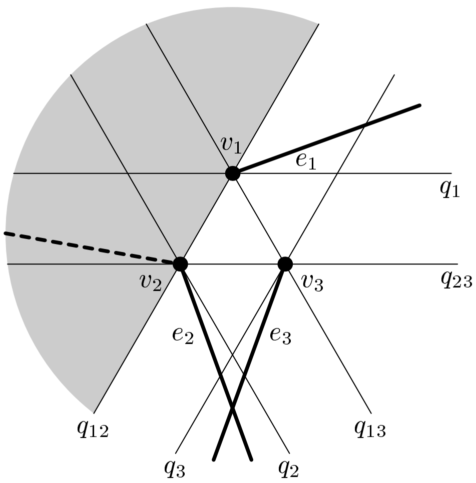


Figure 4: If edge e_1 lies in regions R_2, R_3, there must be an edge in open halfplane determined by q_{12}, which doesn't cross edges e_1 and e_3. These edges are pairwise disjoint.

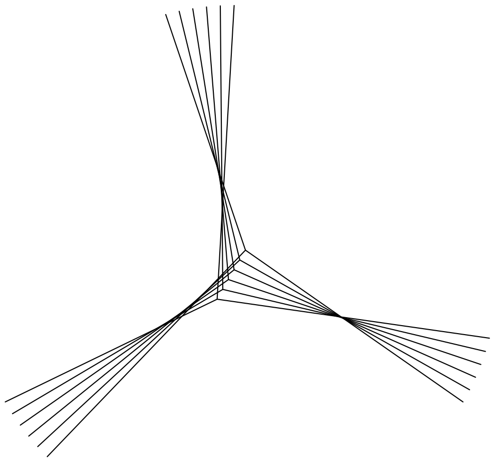


Figure 5: For $k \geq 4$, there are geometric graphs with k cyclic vertices and without 4 pairwise disjoint edges. This figure is for $k = 6$.

4 The upper bound

4.1 Sketch of the proof

In this section we prove theorem 2. Let $G = (V, E)$ be a geometric graph with no three disjoint edges. Denote the number of cyclic vertices in G by γ. We know by lemma 2 that $\gamma \leq 2$. We construct three subgraphs $G_i = (V_i, E_i)$, $i = 1, 2, 3$ as follows. For each pointed vertex in G delete the rightmost edge. Denote the resulting graph by G_1. For each pointed vertex in G_1 delete the leftmost edge (if there is any). Denote the resulting graph by G_2. Deleting in the second round is for each pointed vertex in G_1 (not in G)! The graph G_3 is obtained from G_2 by deleting at most one edge.

We show that the graph G_2 has an edge h which is contained in each pair of disjoint edges (lemma 4). If G_2 has no pair of disjoint edges, we set $G_3 := G_2$. Otherwise there is a cyclic vertex in G_1. After deleting the edge h from G_2, we obtain a graph G_3 with no two disjoint edges. We have deleted at most $2n - \gamma$ edges to get the graph G_2. If there is a cyclic vertex in G_1, we have deleted at most $2n - \gamma - 1$ edges, but then we might need to delete one more edge to get G_3. In both cases we deleted at most $2n - \gamma$ edges from G to get G_3. Finally we use a discharging method to show, that

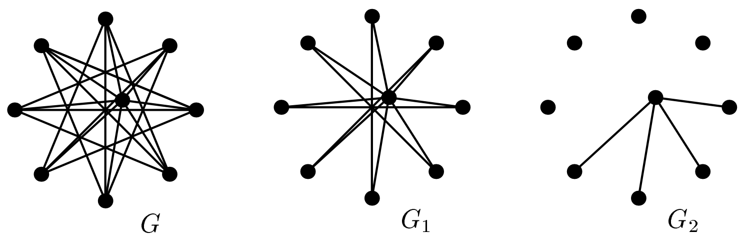
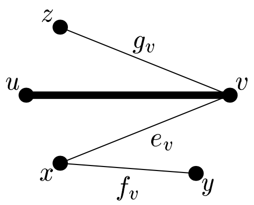


Figure 6: (Example) Graphs G, G_1 and G_2. In the first round delete the rightmost edge at each pointed vertex of G and obtain graph G_1. In the second round delete the leftmost edge at each pointed vertex of G_1 and obtain graph G_2.

graph G_3 has at most $(n+3\gamma)/2$ edges (lemma 8). We then conclude that G has at most $(2n-\gamma)+(n+3\gamma)/2 \le 2.5n+1$ edges. We will need many auxiliary lemmas to prove, that the discharging method works.

4.2 The proof

Lemma 3 *For each vertex v and the edge $e = uv \in E_2$ containing v there exist vertices $x, y, z \in V$ and edges $e_v(e), f_v(e), g_v(e) \in E$ such that $e_v(e) = vx$ is to the left of the edge uv, the edge $f_v(e) = xy$ is to the right of the edge $e_v(e)$ and the edge $g_v(e) = vz$ is to the right of the edge uv. If the vertex v is pointed, the edges $e_v(e)$, $g_v(e)$ are determined uniquely as the edges $e_v \in E_1 - E_2$ and $g_v \in E - E_1$ thrown by the vertex v. If the vertex x is pointed, the edge $f_v(e)$ is determined uniquely as the edge $f_v \in E - E_1$ thrown by the vertex x.*

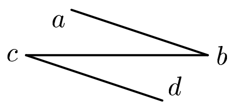


Proof: Let us start in the vertex v. If v is pointed then there must be an edge e_v to the left of the edge uv, because we had deleted the leftmost edge in the second round. If v is cyclic then e_v exists by the definition of cyclic vertex. Similarly, there must be edges f_v and g_v to the right of the edges e_v

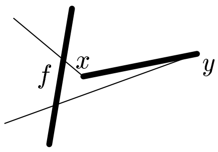
and uv because we had deleted the rightmost edges in the first round or by the definition of cyclic vertex. It can happen, that $y = z$ or that f_v crosses g_v. ∎

We use the following arguments to show, that some edges are disjoint:

Observation 1 (Z argument) *If a, b, c, d is a path in G and ab lies to the right of bc and cd lies also to the right of bc (the shape of letter Z), then the edges ab, cd are disjoint. Similarly if the edges lie to the left of edge bc.*



Observation 2 (both ends in one halfplane argument) *Let $e = xy$ and f be edges. If there exist rays beginning in the vertices x, y which intersect the continuation of the edge f from the same side, then the edges e, f are disjoint.*



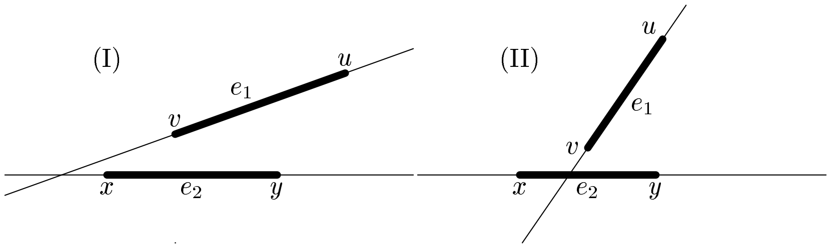
When we consider some of the edges e_v, f_v, g_v in the following proofs, we know by lemma 3 that they exist. Sometimes we omit verification of the fact that three edges from the proof are pairwise disjoint. It is a direct application of previous arguments and we leave it to the reader.

In the pictures the edges of G_2 and G_3 are drawn by thick lines. When we want to show three disjoint edges in some picture, they are drawn by grey color.

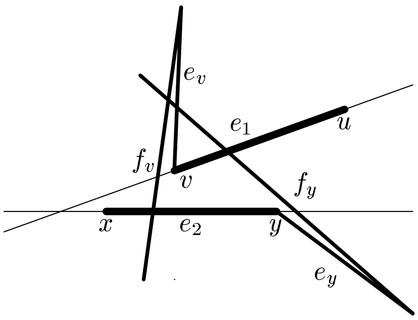
Lemma 4 (about two disjoint edges in G_2) *There exists an edge $h \in G_2$ which is contained in each pair of disjoint edges in G_2. If G_2 has a pair of disjoint edges then there is a cyclic vertex in G_1.*

Proof: (by contradiction) We try to prove that G_2 has no pair of disjoint edges, but finally we find out that there is a case with such a pair. There is an edge h contained in each such a pair.

There are two possible positions of two disjoint edges. Either continuation of one edge intersect the second edge or not. See the following figure:

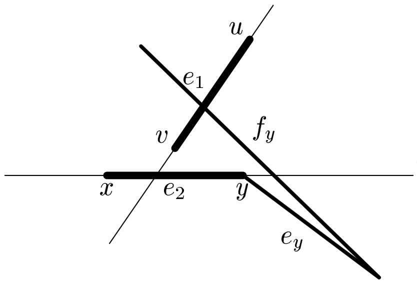


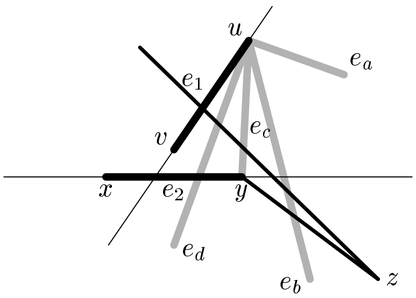
Let's look at the case I. The edge f_v cannot be disjoint with the edge e_2, otherwise we have 3 disjoint edges—e_1, e_2, f_v. If the edge f_v leads to the vertex y, there are two disjoint edges e_1, f_v in one halfplane determined by the edge e_2. Then consider the edge g_x incident to the vertex x lying in the opposite halfplane and we get 3 disjoint edges. Similarly for the edge f_y. See the following figure:



Denote the continuations of the edges f_v, f_y by p, q. Assume w.l.o.g. that the lines p and q cross above the edge e_1 (or at infinity). Then the edges f_v, e_y and e_1 are pairwise disjoint.

Let's look at the case II. The edge f_y cannot be again disjoint with the edge e_1 because otherwise we have 3 disjoint edges — e_1, e_2, f_y. The case when f_y leads to an end vertex of the edge e_1 can be handled almost in the same way as the case of proper crossing. See the following figure on the left:





Where can the edge e_u lie? Possible cases according to the angle with the edge e_1 are in the previous figure on the right.

(a) If the ray determined by e_u crosses neither the edge e_y nor the edge e_2 we have 3 disjoint edges — e_a, e_2 and f_y. In case when f_y leads to the vertex u, we have to consider the edges e_u, e_y which are disjoint because of Z argument and the edge e_v which lies in the opposite halfplane determined by the edge e_1.

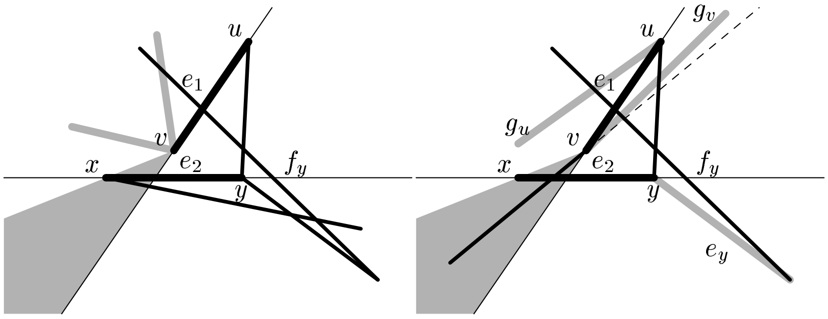
(b) If the ray crosses the edge e_y or passes through the vertex z then the edges e_1 and e_2 lie in one halfplane determined by the edge e_u. Moreover, the edge f_u lies in the opposite halfplane. This yields 3 disjoint edges.

(c) The vertex y is cyclic, because the edge e_2 doesn't cross the edge f_y, but its continuation crosses it between the crossings with the other edges incident with y. Note that both edges e_c, $e_y \in E_1 - E_2$ so the vertex y is cyclic in G_1 too. We will come back to this case later.

(d) If the ray crosses the edge e_2 there are two disjoint edges e_y, e_u in one halfplane determined by the edge e_1. In the opposite halfplane, there is the edge e_v. Again we have 3 disjoint edges.

In all possible cases except for the case (c) we found 3 disjoint edges. That contradicts the assumption that G has no 3 disjoint edges.

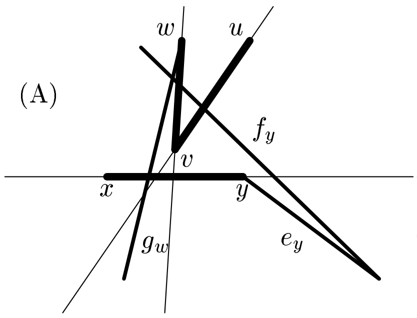
Look at the case (c) in more detail. If the edge e_v is disjoint with the edge xy, we have 3 disjoint edges: Either e_v is disjoint with the edge f_y and we take the edges e_v, f_y and xy, or e_v crosses the edge f_y and then we take e_v, uy and g_x.

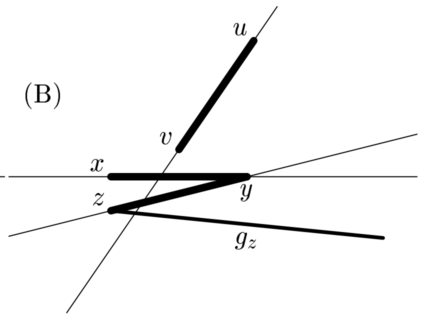


Otherwise e_v crosses the edge xy and in the figure lies in the grey area. In that case the vertex v is cyclic or the edge g_v lies between the edge e_1 and the dashed line which is continuation of e_v. In the latter case the edges g_u, g_v and e_y are pairwise disjoint (previous figure on the right).

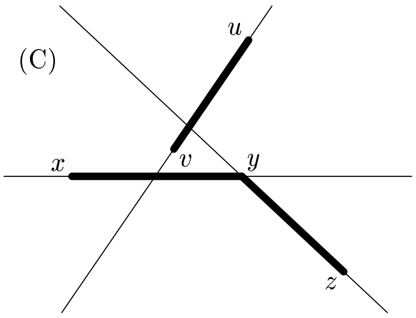
We have proved that each pair of disjoint edges in G_2 must look like in the case (c) where vertices v, y are cyclic. Each edge of disjoint pair contains one cyclic vertex and moreover if we orient each edge towards the cyclic vertex, we observe that the other cyclic vertex must lie in the left halfplane determined by the oriented edge.

We show that we can always choose one of the edges e_1, e_2 as the edge h contained in each pair of disjoint edges. Suppose that there is another pair of disjoint edges, otherwise we are done. By the lemma 2 we know that there are at most two cyclic vertices in the graph G. Thus each edge from a pair of disjoint edges must be incident to one of the vertices v and y. If one edge of the other disjoint pair or its continuation crosses an edge from e_1, e_2 (not only in one cyclic vertex), then the previous observation cannot be satisfied—the second cyclic vertex will always be in the right halfplane determined by this edge. If any edge from the second pair or its continuation doesn't cross properly any of the edges e_1, e_2, then these edges of G_2 must be in one of the following three positions A, B, C or there are two disjoint edges which don't match case (c). In the following figures it is the position of thick edges. Consider the new edge incident with the cyclic vertex and rotate it to make sure that these are all possible cases (there is one more case, but it is isomorphic to the position C).





In the case (A) there are two disjoint edges uv, e_y in one halfplane determined by the edge vw. In the opposite halfplane there is the edge g_w. In the case (B) there are two disjoint edges xy, uv in one halfplane determined by the edge yz. In the opposite halfplane there is an edge g_z. In both cases, we have 3 disjoint edges.



In the last case (C) we claim that the edge uv is the desired edge h. From the previous part of the proof we know that all edges of disjoint pair must be incident to one of the cyclic vertices v, y. If there is an edge of another pair of disjoint edges incident to v different from $h = uv$, it must look like the case (c) and it must satisfy the observation about the cyclic vertices. But then we can always find three edges of G_2 which are in the position A or B. That's a contradiction because positions A and B cannot occur.

■

The previous lemma says that there is an edge h contained in each pair of disjoint edges in G_2. If there is some pair of disjoint edges in G_2, delete the edge h. Denote the resulting graph by G_3.

Corollary 1 *The graph G_3 has no two disjoint edges. Therefore each vertex is pointed in G_3 or all edges of G_3 form a star with a cyclic vertex in G_3 as a root.*

Corollary 2 *Any geometric graph G on n vertices with no three pairwise disjoint edges has at most $3n$ edges.*

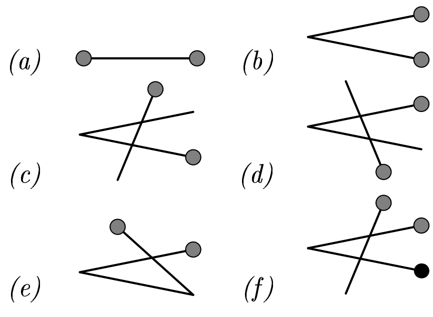
Proof: We have deleted at most $2n - \gamma$ edges to get graph with no disjoint edges. By the theorem 1 the graph G_3 has at most n edges. Thus, the graph G has at most $3n$ edges. ∎

For each vertex v which is neither isolated nor cyclic in G_1 we say that the second endpoint of the edge $e_v \in E_1 - E_2$ is the *partner vertex* $\bar{v}$ of the vertex v. We say that two vertices *share* their partner, if they have common partner vertex (not all vertices have a partner vertex).

The idea of the discharging technique is to show that partner vertices are isolated in G_3 or cyclic in G. Then if the partner vertices are not shared, the number of vertices contained in some edge of G_3 is less or equal to the number of isolated vertices in G_3 or cyclic in G. That would yield the bound $n/2$ on the number of edges in G_3.

In the following lemma we show, that the partner vertices are not too shared. Its prove is a bit technical and based on the case study.

Lemma 5 (sharing of partner vertices) *For any edges in the graph G_3 which are in one of the following positions a, b, c, d, e, the grey vertices don't share their partner. In the position f the grey vertices don't share their partner or the black vertex's partner is isolated in G_3 or cyclic in G or the black vertex is cyclic in G.*

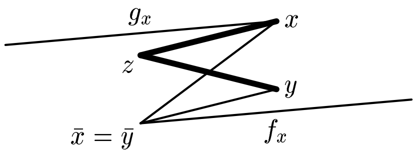


Lemma 6 (auxiliary) *Let $e = zx$, $f = zy \in E_3$ be two adjacent edges. The edges e_x and e_y are not disjoint.*

Proof: If e_x and e_y are disjoint, then consider these two edges and the edge e_z. These edges are pairwise disjoint in G. ∎

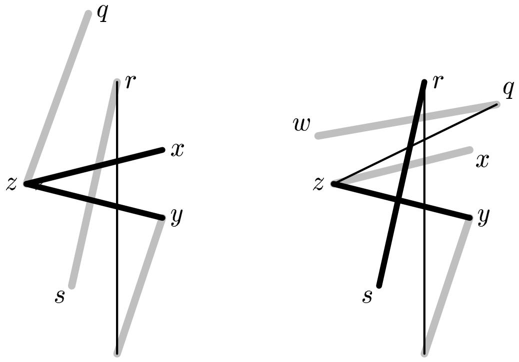
Proof of lemma 5a: If there is an edge xy connecting two vertices in G_3 then the edges e_x, e_y lie in the opposite halfplanes determined by the edge xy. ∎

Proof of lemma 5b:



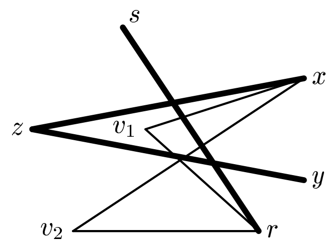
If the vertices x and y share their partner $\bar{x} = \bar{y}$, then there are 3 disjoint edges g_x, zy and f_x. It's not necessary that all edges which cross in the figure must cross (i.e. e_x and zy), but the same choice of disjoint edges works. Similarly by the proves of parts c, d, e and f. ∎

Proof of lemma 5c:

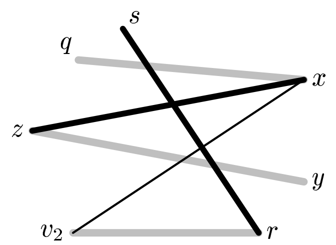


Consider the edge $zq = e_z$. The edges zq and rs are either disjoint or not. The first case is on the left, the second which includes case $r = q$ is on the right. In both cases there are 3 disjoint edges in G. ∎

Proof of lemma 5d: There are two possible positions of the shared vertex — v_1,v_2:

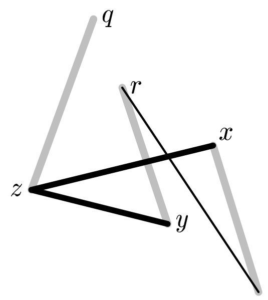


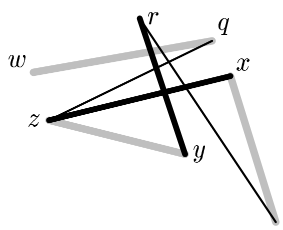
In the first case (shared vertex v_1), use lemma 6, which says, that this cannot happen otherwise we have 3 disjoint edges. In the second case consider the edges g_x, zy and e_r.



They are again pairwise disjoint. ∎

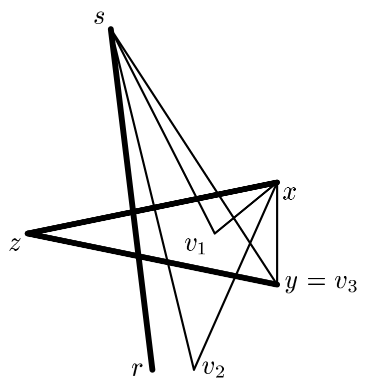
Proof of lemma 5e: This proof is almost the same as the proof of lemma 5c. Only the figures are a bit different.





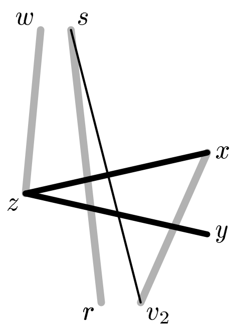
Consider the edge $zq = e_z$. The edges zq and ry are either disjoint or not. The first case is on the left, the second which includes case $r = q$ is on the right. In both cases there are 3 disjoint edges. ∎

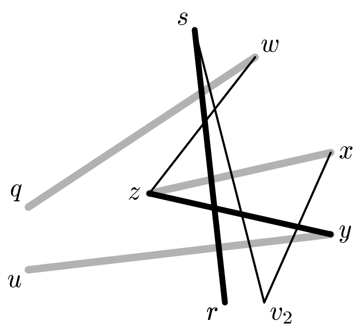
Proof of lemma 5f: There are three possible positions of the shared vertex $\bar{x} = \bar{y}$. Denote them by v_1, v_2 and v_3. See the following figure:



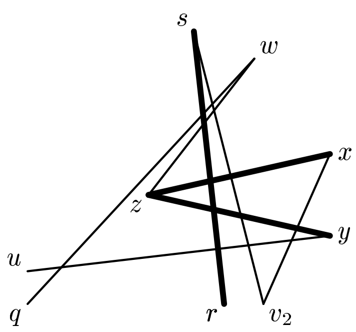
(v_1) Use lemma 6, which yields 3 disjoint edges.

(v_2) Assume that the vertex y is not cyclic in G otherwise we are done. Let us start in the picture on the left. Consider the edge $zw = e_z$. If the edges wz, rs are disjoint, we have 3 disjoint edges. Otherwise look at the picture on the right. The case $s = w$ is handled in the same way.

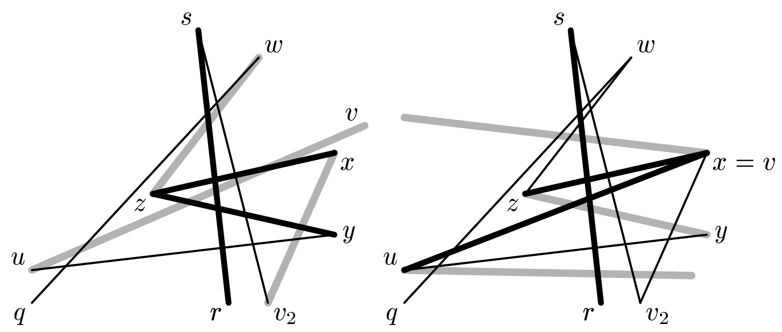




The vertex y is not cyclic in G hence it has a partner vertex u. Consider the edges $wq = f_z$ and $yu = e_y$. If they are disjoint, we again have 3 disjoint edges, else look at the following figure.

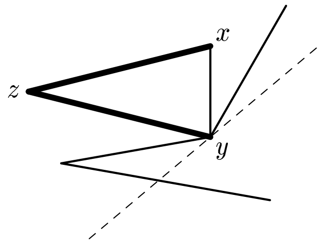


The partner vertex u is either isolated in G_3 as we claim in the lemma or there must be an edge in G_3 containing vertex u. All edges in G_3 must intersect (lemma 4). All possible positions of this edge are in the following figure:



On the right is the case when $x = v$. There is an edge to the right of the edge uy because the edge uy belongs to the graph G_1, but not to G. The graph G_1 was obtained from G by deleting of the rightmost edge at each pointed vertex. In both cases we have 3 disjoint edges.

(v_3) The vertex $y = v_3$ is either cyclic or we have 3 disjoint edges — f_y, xz, g_y. Edge g_y lies to the right of the edge xy because $xy \in E_1$ and to get graph G_1 the rightmost edge was deleted at the vertex y.



■

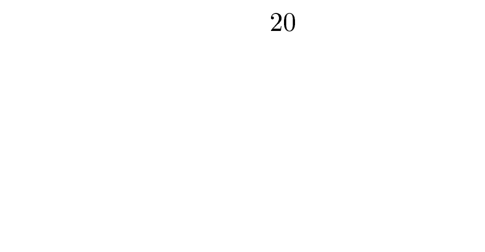
In the following two corollaries we summarize the results on sharing partner vertices.

Corollary 3 *In each star $S \subseteq G_3$ there are no two vertices $v_i, v_j \in S$ which share their partner $\bar{v}_i = \bar{v}_j$.*

Proof: Apply sharing lemma 5b to each pair of leaves. A leaf and the root of the star cannot share their partner either, because they are joined by edge (lemma 5a). ■

Corollary 4 *Let S, S' be two different stars in G_3.*

1. *If the root vertices of these stars are not joined by an edge in G_3, then there are at most two leaves $u \in S$ and $v \in S'$ that can share their partner $\bar{u} = \bar{v}$. Vertex u is the leftmost leaf in one star and vertex v is the rightmost leaf in the second star (grey vertices in the following figure on the left).*
2. *If the root vertices are joined by an edge in G_3, then any two vertices of stars S, S' cannot share a partner vertex (figure on the right).*



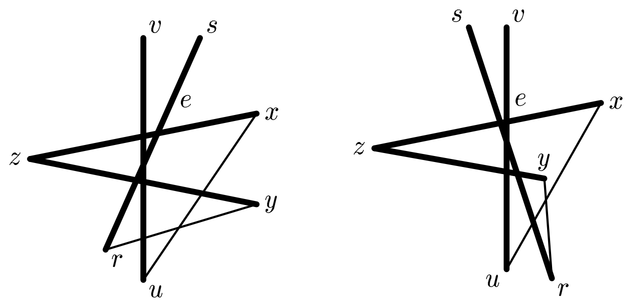
Proof: 1. Apply one of the sharing lemmas 5b, 5c, 5d to each other pair of leaves. 2. Use the result of case 1. and for the other pairs apply lemma 5e or 5a. ■

Now we show, that there are many isolated vertices in G_3.

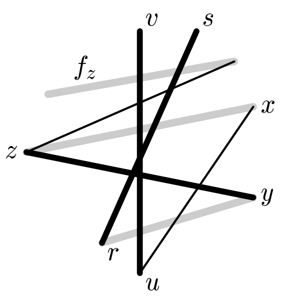
Lemma 7 *Let zx, $zy \in G_3$ be two adjacent edges. At least one vertex of x, y has the isolated or cyclic partner vertex or is cyclic.*

Proof: (by contradiction) The edges $xu = e_x$ and $yr = e_y$ are neither disjoint by lemma 6 nor lead to a common vertex by lemma 5b.

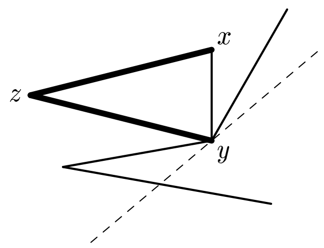
Let us assume that x, y are not cyclic in G and that both partners of vertices x and y are not isolated in G_3. So there must be edges in G_3 containing partner vertices. These edges must intersect all the other edges of G_3 (corollary 1), so except for the special cases, when some vertices are equal, there are only two possible positions. They are on the following figure:

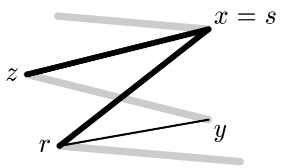


We choose one edge from rs, uv and one edge from xu,yr in such a way, that we have two disjoint edges. Denote the first chosen edge by e. Then consider the edge e_z. If it is disjoint with the edge e then we have 3 disjoint edges. Otherwise consider the edge f_z and take 3 disjoint edges from the following figure:



There are some special cases, when some of the vertices are equal. In case $v = s$, the former approach works. In the following picture on the left we show the case when $u = y$. The vertex y is either cyclic or we have 3 disjoint edges. The edge g_y lies to the right of the edge xy because $xy \in E_1$ and to get the graph G_1 the rightmost edge was deleted at the vertex y. The same approach will work also for the cases $v = z$ and $s = z$. On the right is the case when $s = x$. We again have 3 disjoint edges.





■

Corollary 5 *In any star $S_k \subseteq G_3$ all leaves but at most one are cyclic or have an isolated or cyclic partner vertex.*

Proof: Apply lemma 7 to the pair of leaves, that both don't satisfy the condition yet. ■

Lemma 8 *The graph G_3 has at most $(n + 3\gamma)/2$ edges.*

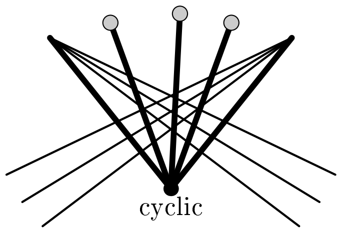
Proof: We use the discharging method. Give 1 dollar to each vertex and additional 3 dollars to each cyclic vertex. We need to pay for all edges of G_3, where each edge costs 2 dollars (1 dollar per each end). So each vertex v must pay $\deg v$ dollars — 1 dollar for each edge incident with it. How will the vertices pay for all the edges?

- Isolated vertices don't pay anything. Someone can borrow from them.
- Vertices of degree one pay for themselves.
- Vertices with $\deg v \geq 2$ borrow $(\deg v - 1)$ dollars from the partners of their neighbors or cyclic neighbors and pay for one edge themselves.

It remains to show, that this works. Let us look at the vertex of $\deg v \geq 2$. Corollary 5 says that in each star in G_3 all leaves but at most one have either isolated or cyclic partner or are cyclic. So v has always someone to ask to borrow.

We must also show, that the partner vertices are not lending to more than one vertex and that cyclic vertices are not lending to more than 3 vertices. But the first condition would mean that the partner vertex is shared by some vertices. In case when edges of G_3 form only one star (corollary 1), use corollary 3, which says that no two vertices can share their partner vertex. Otherwise use corollary 4, which says that the only leaves of two stars, which can share a partner vertex are the leftmost in one maximal star and the rightmost in the second maximal star. Moreover this holds only in the case, when the roots of considered stars are not joined by edge. If these two leaves share a partner and the partner is supposed to pay twice, apply lemma 5f, which says, that root of one star can borrow from the last neighbor or his partner and pay for the leftmost edge on his own. Then there are no problems with sharing partner vertices between these two stars.

Cyclic vertices cannot lend more than they have either. Each cyclic vertex has 1 dollar to pay as a partner vertex. By the previous part, partner vertices cannot be shared. Than each cyclic vertex has 2 dollars to lend directly to the vertices which need help. There cannot be more then two such vertices. See the following figure:



All neighbors of cyclic vertex in G_3 except for the outer ones must have the degree one. Otherwise there are two disjoint edges in the graph G_3.

Vertices with the degree one pay for themselves and needn't any help.

Altogether vertices have $(n + 3\gamma)$ dollars, we paid for all edges so we have at most $(n + 3\gamma)/2$ edges. ∎

We have deleted at most $2n - \gamma$ edges to get the graph G_3. The graph G_3 has at most $(n + 3\gamma)/2$ edges. Thus the graph G has at most $2.5n + \gamma/2$ edges. That finishes the proof of the theorem 2.

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