Group coloring is Π_2^P-complete

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Abstract

The group chromatic number of a graph G is the smallest integer k such that for each Abelian group A of order at least k, each orientation of G and each edge-labeling $\varphi : E(G) \to A$, there exists a vertex-coloring $c : V(G) \to A$ with $c(v) - c(u) \neq \varphi(uv)$ for all oriented edges uv of G. We show that the decision problem whether a given graph has group chromatic number at most k is Π_2^P-complete for each integer $k \geq 3$.

1 Introduction

Group colorings of graphs have been introduced by Jaeger, Linial, Payan and Tarsi [4]. For plane graphs, this concept is dual (in the usual sense) to group connectivity which generalizes the intensively studied concept of nowhere-zero flows in graphs. For an Abelian group A, a graph G is said to be A-*colorable* if for every orientation of G and for every *edge-labeling* $\varphi : E(G) \to A$, there is a vertex coloring $c : V(G) \to A$ such that $c(w) - c(v) \neq \varphi(vw)$ for each oriented edge $vw \in E(G)$. Note that the choice of an orientation of edges of G is not essential since reversing the orientation of the edge e can be replaced by changing the forbidden difference $\varphi(e)$ to $-\varphi(e)$. A plane graph is A-colorable iff its dual is A-connected [4]. We remark that it is unknown whether the property of being A-colorable (A-connected) depends on the structure or only on the order of the group A (like in the case of nowhere-zero flows [12]). The least number $\chi_g(G)$ such that G is A-colorable for each

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Abelian group A of order at least $\chi_g(G)$ is the *group chromatic number* of G.

Group colorings have attracted a lot of attention from the combinatorial point of view. Some theorems for ordinary colorings, e.g., Brooks' theorem, can be translated to group colorings [6], some cannot. However, the concept of group coloring is closer to the concept of list-coloring [5]: The group chromatic number of a graph G with the average degree d is at least $\Omega(d/\log d)$. Hence, as in the concept of list coloring, the group chromatic number, the choice number, respectively, of graphs with large average degree is large. Lai and Zhang [7] proved that the group chromatic number of planar graphs is bounded by five and the author with Pangrác and Voss [5] constructed a planar graph with the group chromatic number five. This is the same bound as in the case of list-coloring [11]. Planar graphs without 3-cycles and 4-cycles have the group chromatic number at most three and planar graphs without 4-cycles at most four [5] matching the same bounds for choosability [8, 10]. On the other hand, the choice number of planar bipartite graphs is at most three [1] but there is a planar bipartite graph with the group chromatic number equal to four [5].

In this paper, we address the complexity of group coloring. We show that for each Abelian group A of order at least three it is Π_2^P-complete to decide whether a given graph G is A-colorable. The problem to decide whether the group chromatic number of a graph G is at most k is also Π_2^P-complete for each integer $k \geq 3$.

2 Definitions and notation

All groups considered in this paper are finite and Abelian. The neutral element is always denoted by 0. Elements of the group are sometimes also referred to as colors. If an orientation and edge-assignment φ of a graph G is given, then a vertex coloring c is *proper* if $c(v) - c(u) \neq \varphi(uv)$ for every oriented edge uv of G. Hence, a graph G is A-colorable if there is a proper coloring for each orientation of it and each edge-assignment $\varphi : E(G) \to A$. An edge uv with $c(v) - c(u) \neq \varphi(uv)$ is said to be *properly* colored. On the other hand, if $c(v) - c(u) = \varphi(uv)$, the colors of u and v *conflict* with each other.

Let us state a simple lemma about group coloring which is used later in this paper:

Lemma 1 *Let G be a graph and v a fixed vertex of it. If a graph G is A-colorable, then for each orientation and each edge-labeling, there exists a proper coloring c of G with $c(v) = 0$.*

We first show that a variant of group coloring where sizes of lists are not the same for all vertices of a graph is Π_2^P-complete and then we derive the Π_2^P-completeness of the original problem. A *list-size-assigning* function is a mapping $\psi : V(G) \to \{1, \ldots, |A|\}$. An *$A$-list-$\psi$-assignment* for a graph G and an Abelian group A is a mapping $L : V(G) \to 2^A$ with $|L(v)| = \psi(v)$ for every vertex $v \in V(G)$. If the group A is clear from the context, the list assignment L is just said to be a *list-ψ-assignment*. The graph G is said to be *A-ψ-choosable* if for each orientation of G, each edge-labeling $\varphi : E(G) \to A$ and each list-ψ-assignment L, there is a vertex coloring $c : V(G) \to A$ such that $c(v) \in L(v)$ for every vertex $v \in L(v)$ and $c(w) - c(v) \neq \varphi(vw)$ for every oriented edge $vw \in E(G)$. We also say that G can be colored from lists L or that it is *A-L-colorable* in such a situation. If ψ is a constant function equal to ℓ, then the list assignment L is said to be *list-ℓ-assignment* and the graph G is said to be *A-ℓ-choosable*. Note that a graph is A-$|A|$-choosable iff it is A-colorable. Group choosability has sometimes a somewhat strange behavior, e.g., a cycle of even length is A-2-choosable for every group A of odd order and it is not A-2-choosable for any group A of even order.

The complexity class Π_2^P is defined to be the class of the complements of problems which can be solved by non-deterministic algorithms running in polynomial time which have access to the oracle solving problems from the class NP [2, 9]. Only very few natural combinatorial problems are known to be Π_2^P-complete, e.g., the problem to decide whether a graph is 3-choosable is Π_2^P-complete [3], to compute the generalized Ramsey number [2]. Similarly as ordinary satisfiability of formulas is the basic NP-complete problem, the decision problem whether a given Π_2-formula is true is the basic Π_2^P-complete problem. A formula Ψ is a *Π_2-3CNF-formula* if it is of the following form:

$$\Psi = \forall x_1 \cdots \forall x_m \exists y_1 \cdots \exists y_n \Psi_0(x_1, \ldots, x_m, y_1, \ldots, y_n)$$

where Ψ_0 is a 3CNF-formula with variables $x_1, \ldots, x_m, y_1, \ldots, y_n$, i.e., a formula Ψ_0 is in the conjunctive normal form (CNF) and each clause of it has size exactly three. Formulas Ψ where Ψ_0 is of this restricted form are called *Π_2-3CNF-formulas*. The *size* of a Π_2-3CNF-formula is the number of its clauses. We remind that occurrences of variables in clauses of Ψ_0 are called *literals*. A literal is *positive* if it is of the form x_i, $i = 1, \ldots, m$, or y_i, $i = 1, \ldots, n$. Similarly, a literal is *negative* if it is of the form $\neg x_i$, $i = 1, \ldots, m$, or $\neg y_i$, $i = 1, \ldots, n$.

3 Group choosability is hard

In this section, we construct, for a given Π_2-3CNF-formula Ψ and an integer k, a graph G which is A-ψ-choosable for a certain list-size-assigning function ψ for each Abelian group A of order k if Ψ is true and for no Abelian group A of order k if Ψ is false. This shows that A-choosability is Π_2^P-complete if sizes of lists for vertices form a part of the input. In the next section, these results are used to deduce that the decision problem whether a given graph is A-colorable is Π_2^P-complete.

3.1 Preliminary results on group choosability

First, let us prove a simple lemma on group choosability which later helps us to simplify some of our arguments:

Lemma 2 *Let G be a directed graph, T a spanning tree of G, A a fixed Abelian group and ψ a list-size-assigning function. Then, the graph G is A-ψ-choosable if and only if G can be colored from every A-ψ-lists for every edge-labeling $\varphi : E(G) \to A$ with $\varphi(uv) = 0$ for all $uv \in E(T)$.*

Proof: Clearly, it is enough to prove that if G can be colored from any A-ψ-lists for each edge-labeling $\varphi : E(G) \to A$ with $\varphi(uv) = 0$ for all $uv \in E(T)$, then G is A-ψ-choosable. Fix an orientation, an edge-labeling $\varphi : E(G) \to A$ and an A-ψ-list assignment L of G. We can assume that T is a branching, i.e., T has a root r and all edges of T are oriented in the direction from the root. We modify the edge-labeling φ so that the resulting edge-labeling is equal to 0 on the edges of the tree T. In order to do this, we process one edge after another in the order determined by their distance from the vertex r. The edges incident with the root r are processed first.

Let us consider an edge uv of the tree T and modify the edge-labeling φ and the list assignment L as follows:

- Decrease the label of each edge e leading to the vertex v by $\varphi(uv)$ and increase the label of each edge e going out from the vertex v by $\varphi(uv)$. Observe that the label of the edge uv after the modification is equal to 0.

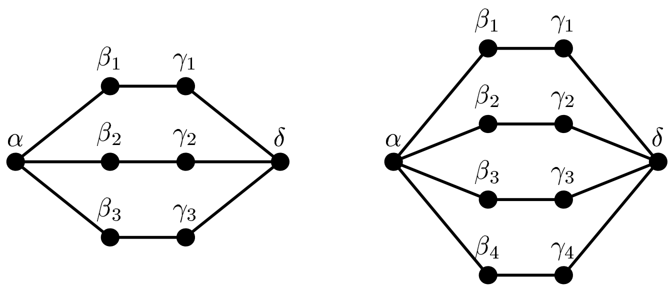

Figure 1: The graphs P_4^3 and P_4^4.

- Decrease each element in the list $L(v)$ of the vertex v by $\varphi(uv)$.

It is easy to see that there is a proper vertex-coloring $c : V(G) \to A$ with respect to the original edge-labeling and list assignment if and only if there is a proper vertex-coloring with respect to the modified ones. The modification does not change edge-labels of edges closer to the root r in the tree T than the edge uv and hence after all edges are processed, the edge-labeling φ has been modified so that $\varphi(uv) = 0$ for all edges $uv \in E(T)$.

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The core of our gadgets is the following graph denoted as P_4^k: Consider k copies of the four-vertex path P_4. Let $\alpha_i\beta_i\gamma_i\delta_i$, $i = 1, \ldots, k$, be these paths. We identify all the vertices $\alpha_1, \ldots, \alpha_k$ to a single vertex α and the vertices $\delta_1, \ldots, \delta_k$ to a single vertex δ. Next, define the list-size-assigning function ψ as follows: $\psi(\alpha) = \psi(\beta_1) = \ldots = \psi(\beta_k) = \psi(\gamma_1) = \ldots = \psi(\gamma_k) = 2$ and $\psi(\delta) = k$. The graphs P_4^3 and P_4^4 are depicted in Figure 1. A property of the graph P_4^k which is crucial for our construction is stated in the next proposition:

Proposition 3 *Let $k \geq 2$ be an integer and A an Abelian group of order k. The graph P_4^{k-1} is A-ψ-choosable.*

Proof: Fix an orientation, an edge-labeling φ and a list-ψ-assignment L. In particular, the size of the list $|L(\delta)|$ of the vertex δ is $k-1$. We may assume that the edges are oriented so that the edges $\alpha\beta_i$ are oriented from α to β_i, the edges $\beta_i\gamma_i$ from β_i to γ_i and $\gamma_i\delta$ from γ_i to δ for $i = 1, \ldots, k-1$. By Lemma 2, we can also assume that $\varphi(\alpha\beta_i) = 0$ and $\varphi(\beta_i\gamma_i) = 0$ for every $i = 1, \ldots, k-1$.

In the proof, we distinguish two cases. The first case is that there is an edge $\alpha\beta_i$ with $L(\alpha) \neq L(\beta_i)$ or an edge $\beta_i\gamma_i$ with $L(\beta_i) \neq L(\gamma_i)$ for some $i = 1, \ldots, k-1$. Assume that there is such an edge $\alpha\beta_i$. Color the vertex α by a color from its list which is not contained in the list $L(\beta_i)$. Color now all the vertices $\beta_{i'}$ for $i' \neq i$ by colors from their lists which do not conflict with the color of the vertex α and then color all the vertices $\gamma_{i'}$ for $i' \neq i$ by colors from their lists which do not conflict with the color of the vertex $\beta_{i'}$. The vertex δ has now $k-2$ colored neighbors and thus it can be colored by a color from its list which conflicts with the color of no vertex $\gamma_{i'}$ with $i' \neq i$. Color now the vertex γ_i by a color from its list which does not conflict with the color of the vertex δ and then the vertex β_i by a color from its list which does not conflict with the color of the vertex γ_i. By the choice of the color of the vertex α, the edge $\alpha\beta_i$ is also properly colored. An analogous argument applies if there is an edge $\beta_i\gamma_i$ with $L(\beta_i) \neq L(\gamma_i)$.

The second (and last) case to deal with is that the lists of all the vertices $\alpha, \beta_1, \ldots, \beta_{k-1}, \gamma_1, \ldots, \gamma_{k-1}$ are the same. Let σ_1 and σ_2 be the two elements of A contained in these lists. We now consider the following two colorings c_1 and c_2 of all the vertices of P_4^{k-1} except for the vertex δ: $c_\ell(\alpha) = \sigma_\ell$, $c_\ell(\beta_i) = \sigma_{3-\ell}$ and $c_\ell(\gamma_i) = \sigma_\ell$ for $\ell = 1, 2$ and $i = 1, \ldots, k-1$. Define now a set B_ℓ for $\ell = 1, 2$ to be the set containing all the elements $\sigma_\ell + \varphi(\gamma_i\delta)$ for $i = 1, \ldots, k-1$. If the graph G does not have a proper coloring from the list-assignment L with respect to the orientation and the edge-labeling φ, both the sets B_1 and B_2 are equal to $L(\delta)$. In particular, all the labels $\varphi(\gamma_1\delta), \ldots, \varphi(\gamma_{k-1}\delta)$ are mutually distinct. Let τ be the only element of A missing in the list $L(\delta)$. By the definition of B_1, the equality implies $B_1 = L(\delta)$ that none of the labels $\varphi(\gamma_1\delta), \ldots, \varphi(\gamma_{k-1}\delta)$ is equal to $\tau - \sigma_1$. Similarly, none of the labels $\varphi(\gamma_1\delta), \ldots, \varphi(\gamma_{k-1}\delta)$ is equal to $\tau - \sigma_2$. Since the labels $\varphi(\gamma_1\delta), \ldots, \varphi(\gamma_{k-1}\delta)$ are mutually distinct, it must hold that $\tau - \sigma_1 = \tau - \sigma_2$. This implies $\sigma_1 = \sigma_2$ which is impossible. Hence, $B_1 \neq L(\delta)$ or $B_2 \neq L(\delta)$ and the graph G can be properly colored from the lists L with respect to the orientation and the edge-labeling φ.

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3.2 Overview of the construction

We aim to construct a graph G and a list-size-assigning function ψ for a given Π_2-3CNF-formula Ψ and a given integer $k \geq 3$ such that G is A-ψ-choosable for each group A of order k if Ψ is true and G is A-ψ-choosable for no group A of order k if Ψ is false. The structure of the graph G depends on the order of the group A but not on its structure. The graph G will be glued from several types of gadgets. The *gadget* is a graph with several (usually two) distinguished *contact vertices* together with a list-size-assigning function defined for all vertices including the contact ones. Contact vertices are always non-adjacent and vertices of the gadget which are not contact are called *inner vertices* of the gadget.

Once the Abelian group A, the orientation of the edges of the gadget, the edge-labeling and the list-assignment for inner vertices are chosen, there exist one or more proper vertex-colorings of the gadget. Such a proper vertex-coloring is said to be *consistent* with the given orientation, the edge-labeling and the list-assignment. If the orientation, the edge-labeling and the list-assignment are clear from the context, the vertex-coloring is just said to be consistent. A contact vertex is *precolored* if it is colored with a color from its list. If all but one contact vertices are precolored, the remaining contact vertex is said to be *rigid* if only a single color from its list can be used to color it, i.e., every consistent coloring which extends the precoloring assigns this vertex the same color. The contact vertex is said to be *free* if there are at least two colors in its list with which it can be colored. Finally, the contact vertex is said to be *colorable* if it is rigid or free. In our construction, free vertices correspond to true literals and rigid vertices to false ones.

3.3 Transfer gadget

First, we construct a so-called *transfer gadget* which is used to transfer the truth value of a literal throughout the constructed graph. The gadget has two contact vertices IN and OUT. For an Abelian group A of order k, the transfer gadget is obtained from the graph P_4^{k-1} by adding a new vertex OUT and joining it to the vertices β_1 and γ_1 of P_4^{k-1}. The other contact vertex IN is the vertex α of the graph P_4^{k-1}. The list-size-assigning function ψ is the same as for the graph P_4^{k-1} except for the vertices $\alpha = \mathrm{IN}$, β_1 and γ_1. For these three vertices and the vertex OUT, the function ψ is set to be three. The transfer gadgets for $k = 3$ and $k = 4$ can be found in Figure 2. In the rest the transfer gadget is denoted by a bold double arrow joining the vertex IN to the vertex OUT (as in Figure 3). We describe some properties of the transfer gadget essential for our construction in the following propositions:

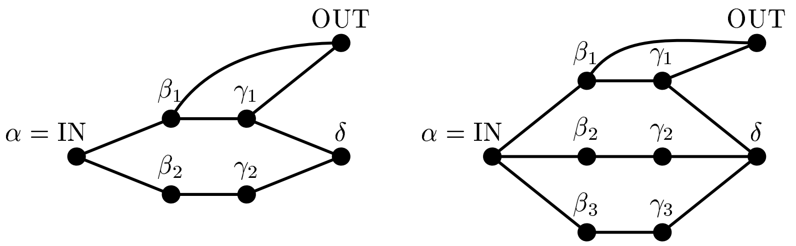

Figure 2: The transfer gadgets for $k = 3$ and $k = 4$.

Proposition 4 *Let A be an Abelian group of order $k \geq 3$, $\rho \in A$ and $B \subseteq A$ with $|B| = 3$ and $0 \in B$. There exist an orientation, an edge-labeling and a list-assignment L of the transfer gadget with $L(\mathrm{OUT}) = B$ such that if the contact vertex IN is precolored with ρ, then the contact vertex OUT is rigid and it must be colored with* 0.

Proof: Let $B = \{0, \rho_1, \rho_2\}$ and let ρ' be an element of A different from both ρ and $\rho_2 - \rho_1$. Let R be a set of $k-2$ elements of A such that $0 \notin R$. Orient the edges $\alpha\beta_i$ from α to β_i, $\beta_i\gamma_i$ from β_i to γ_i and $\gamma_i\delta$ from γ_i to δ for $i = 1, \ldots, k-1$, the edge $\beta_1\mathrm{OUT}$ from β_1 to OUT and the edge $\gamma_1\mathrm{OUT}$ from γ_1 to OUT. Next, we define the edge-labeling φ: The edges $\alpha\beta_2$, $\ldots$, $\alpha\beta_{k-1}$ are assigned $-\rho$ by φ and the edges $\beta_1\gamma_1$, $\beta_2\gamma_2$, $\ldots$, $\beta_{k-1}\gamma_{k-1}$ are assigned 0 by φ. The edges $\gamma_2\delta$, $\ldots$, $\gamma_{k-1}\delta$ are labeled by mutually distinct elements from the set R. The labels of the remaining six edges are defined as follows: $\varphi(\alpha\beta_1) = \rho'$, $\varphi(\gamma_1\delta) = -\rho'$ and $\varphi(\beta_1\mathrm{OUT}) = \varphi(\gamma_1\mathrm{OUT}) = \rho_1$. Finally, we define the list-assignment L. The lists of all the vertices $\beta_2, \ldots, \beta_{k-1}$ and $\gamma_2, \ldots, \gamma_{k-1}$ are the same and they are equal to $\{0, \rho_1\}$. The lists of the remaining four vertices are the following: $L(\delta) = R \cup \{0\}$, $L(\beta_1) = L(\gamma_1) = \{0, \rho_2 - \rho_1, \rho'\}$ and $L(\mathrm{OUT}) = B = \{0, \rho_1, \rho_2\}$. If the vertex IN$= \alpha$ is precolored with ρ, all the vertices $\beta_2, \ldots, \beta_{k-1}$ must be colored with the color ρ_1 and the vertices $\gamma_2, \ldots, \gamma_{k-1}$ with the color 0. This forces the vertex δ to be colored with the color 0 because of the choice of the edge-labeling φ for the edges $\gamma_2\delta, \ldots, \gamma_{k-1}\delta$. Then, the colors of the vertices β_1 and γ_1 must be 0 and $\rho_2 - \rho_1$ (the color ρ' conflicts with the colors of the vertices α and δ, respectively). The color of the vertex OUT is forced to be neither ρ_1 nor ρ_2 as desired.

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Proposition 5 *Let A be an Abelian group of order $k \geq 3$. The following holds for each orientation, each edge-labeling and each list-assignment of the transfer gadget: If the contact vertex OUT is precolored, then the vertex IN is free.*

Proof: Assume for the sake of contradiction that the statement is false. Fix an orientation, an edge-labeling ϕ, a list-assignment L of the transfer gadget and a precoloring of the OUT vertex. Assume that there exist two colors ρ_1 and ρ_2 in the list $L(\mathrm{IN})$ conflicting with any consistent coloring. We now create a list-assignment L' for the vertices of the copy P_4^{k-1} based on a list-assignment L and the color of the vertex OUT. The lists $L'(\beta_1)$ and $L'(\gamma_1)$ are the lists $L(\beta_1)$ and $L(\gamma_1)$ without colors conflicting with the color of the vertex OUT. The list $L'(v)$ for the remaining vertices v except for α is the same as the list $L'(v)$. The list $L'(\alpha)$ is $\{\rho_1, \rho_2\}$. By our assumption, the graph P_4^{k-1} cannot be colored from the lists L' for the restriction of the orientation and the edge-labeling ϕ to it. But this is impossible by Proposition 3.

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Proposition 6 *Let A be an Abelian group of order $k \geq 3$. The following holds for each orientation, each edge-labeling and each list-assignment of the transfer gadget: If the contact vertex IN is precolored, the vertex OUT is colorable.*

Proof: Fix an orientation, an edge-labeling, a list-assignment of the transfer gadget and a precoloring of the vertex IN. Color the remaining vertices in the order α_2, α_3, $\ldots$, α_{k-1}, β_2, β_3, $\ldots$, β_{k-1}, δ, α_1, β_1 and OUT. Observe that each time a vertex of the gadget is to be colored, it always has at least one available color in its list and hence it can be colored.

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Proposition 7 *Let A be an Abelian group of order $k \geq 3$. Fix an orientation, an edge-labeling and a list-assignment of the transfer gadget. There is at most one color ρ in the list of the vertex IN such that if the vertex IN is precolored with ρ, the vertex OUT is rigid. In particular, for any other color $\rho' \neq \rho$ from the list of the vertex IN, if IN is precolored with ρ', the vertex OUT is free.*

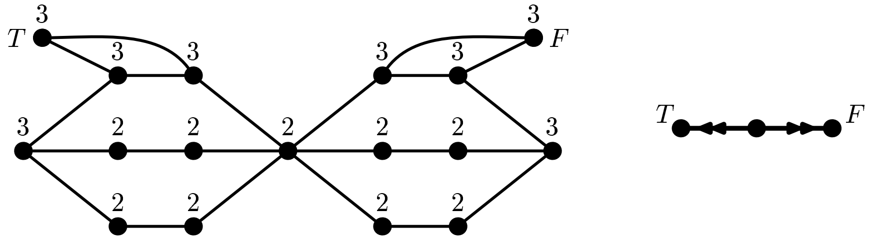

Figure 3: The existential gadget for $k = 4$. The numbers represent sizes of lists and the letters their names. The gadget is in the left part of the figure and its simplified diagram is in the right part.

Proof: If the vertex IN is precolored, then the vertex OUT is always colorable by Proposition 6. Assume for the sake of contradiction that there are two different colors ρ_1 and ρ_2 in the list of the vertex IN such that if the vertex IN is precolored with ρ_1 or ρ_2, then the vertex OUT is rigid. Let σ_1, σ_2, be the colors forced to the vertex OUT if the vertex IN is precolored with the color ρ_1, ρ_2, respectively, and let σ_3 be the remaining element of the list of the vertex OUT. If the vertex OUT is precolored with σ_3, the vertex IN is rigid since it can be colored neither with ρ_1 nor with ρ_2. But this is impossible by Proposition 5.

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3.4 Existential gadget

The *existential gadget* is obtained from two transfer gadgets by identifying their IN vertices. The vertex obtained by identifying these two vertices is called the *middle vertex* of the existential gadget. The list-size-assigning function of the gadgets is modified by assigning the middle vertex two instead of three (the remaining vertices preserve their original sizes of lists). The contact vertices of the existential gadgets are the two OUT vertices of the transfer gadgets and they are denoted by T and F. The existential gadget for groups of order $k = 4$ can be found in Figure 3. The following two propositions summarize properties of the existential gadget which will be needed later:

Proposition 8 *Let A be an Abelian group of order $k \geq 3$. The following holds for each orientation, each edge-labeling, each list-assignment L of the* *existential gadget: There are colors $\alpha_F \in L(F)$ and $\alpha_T \in L(T)$ such that if F is precolored with α_F, then the vertex T is free and, on the other hand, if T is precolored with α_T, then the vertex F is free.*

Proof: Fix an orientation, an edge-labeling and a list-assignment of the existential gadget. Let ρ_1 and ρ_2 be the colors contained in the list of the middle vertex. Let us first consider the following case: If the middle vertex is precolored with the color ρ_1, then both the vertices T and F are free. Let t_0 and f_0 be colors with which the vertices T and F, respectively, can be colored. Set $\alpha_T = t_0$ and $\alpha_F = f_0$. If T is precolored with $\alpha_T = t_0$, then precolor the middle vertex with the color ρ_1 and consider a consistent coloring of the transfer gadget containing T which conflicts neither with the color of the middle vertex nor with the color of T. The vertex F is now free. A symmetric argument applies when F is precolored with the color $\alpha_F = f_0$.

The next case to consider is the following: If the middle vertex is precolored with the color ρ_2, then both the vertices T and F are free with respect to their transfer gadgets. However, this case is analogous to the previous case.

The final case is the following: If the middle vertex is precolored with either of the colors ρ_1 and ρ_2, then one of the vertices T and F is rigid. By Proposition 7, we may assume that if the middle vertex is precolored with the color ρ_1, then the vertex T is rigid and if the middle vertex is precolored with the color ρ_2, then the vertex F is rigid. Let $\sigma_1 \in L(T)$, $\sigma_2 \in L(F)$, be the color which the vertex T, F, is forced to get, if the middle vertex is precolored with ρ_1, ρ_2, respectively. Set now $\alpha_T = \sigma_1$ and $\alpha_F = \sigma_2$. If T is precolored with $\alpha_T = \sigma_1$, then precolor the middle vertex with the color ρ_1 and consider a consistent coloring of the transfer gadget containing T which conflicts neither with the color of the middle vertex nor with the color of T. The vertex F is now free. A symmetric argument applies when F is precolored with the color $\alpha_F = \sigma_2$.

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Proposition 9 *Let A be an Abelian group of order $k \geq 3$. There exist an orientation, an edge-labeling and a list-assignment of the existential gadget such that in every consistent coloring one of the vertices T and F is rigid and forced to be colored with 0.*

Proof: Let ρ_0 be any element of A different from 0. Set the list of the middle vertex of the existential gadget to be the set $\{0, \rho_0\}$ and choose

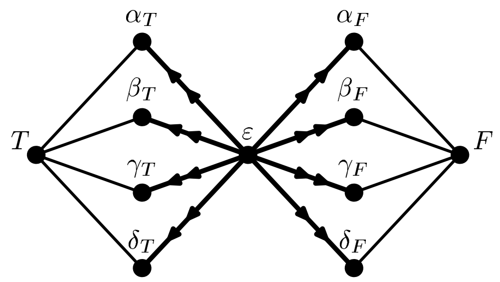

Figure 4: The universal gadget. The letters represent the names of the vertices.

the orientation, the edge-labeling and the list-assignment of each of the two transfer gadget as in Proposition 4 with $\rho = 0$ for the transfer gadget containing the vertex F and with $\rho = \rho_0$ for the transfer gadget containing the vertex T. The middle vertex must clearly be colored either with the color 0 or with the color ρ_0. In the first case, the vertex F is rigid and it must be colored with 0. In the latter case, the vertex T is rigid and it must be colored with 0.

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3.5 Universal gadget

We construct a so-called *universal gadget* in this subsection. The gadget is obtained by identifying the vertices IN of eight copies of the transfer gadget to a single vertex ε. Let α_T, β_T, γ_T, δ_T, α_F, β_F, γ_F and δ_F be the contact vertices OUT of the transfer gadgets. Introduce two new vertices T and F. Join the vertex T by an edge to each of the vertices α_T, β_T, γ_T and δ_T and the vertex F to each of the vertices α_F, β_F, γ_F and δ_F. The contact vertices of the universal gadget are the two vertices T and F. Both T and F are assigned three by the list-size-assigning function while the other vertices preserve their original values. The universal gadget is depicted in Figure 4. We now state and prove two properties of the universal gadget needed in the rest:

Proposition 10 *Let A be an Abelian group of order $k \geq 3$. Fix an orientation, an edge-labeling and a list-assignment of the universal gadget. It is* *possible to precolor the vertex T by a color from its list so that the vertex F is free or it is possible to precolor the vertex F by a color from its list so that the vertex T is free.*

Proof: Let $\{\sigma_1, \sigma_2, \sigma_3\}$ be the list of the vertex ε. Precolor the vertex ε by σ_1 and let t_1 be the number of vertices of α_T, β_T, γ_T and δ_T which are rigid. Similarly, let f_1 be the number of vertices of α_F, β_F, γ_F and δ_F which are rigid. Analogously, let t_2 and f_2 be the number of such rigid vertices if the vertex ε is precolored with σ_2 and t_3 and f_3 if it is precolored with σ_3. By Proposition 7, we have $t_1 + t_2 + t_3 \leq 4$ and $f_1 + f_2 + f_3 \leq 4$. Since $t_1 + t_2 + t_3 + f_1 + f_2 + f_3 \leq 8$, there exists $i \in \{1, 2, 3\}$ such that $t_i + f_i \leq 2$. Assume that $t_i \leq 2$ and $f_i \leq 1$. The case that $t_i \leq 1$ and $f_i \leq 2$ is symmetric.

Precolor the vertex ε by i. At most two of the vertices α_T, β_T, γ_T and δ_T are rigid with respect to the transfer gadgets, i.e., at most two of them are forced to get a certain color. We may assume that these vertices are α_T are β_T. Precolor now the vertex T so its color conflicts neither with the color of α_T nor with the color of β_T. Since the vertices γ_T and δ_T are free by Proposition 7 and the choice of i, we can color each them by a color which does not conflict with the color of T and then we can extend this coloring to inner vertices of their transfer gadgets.

We show that vertex F is free (note that the vertex T is now precolored). By the choice of i, at most one of the vertices α_F, β_F, γ_F and δ_F is rigid. We may assume that this vertex is the vertex α_F. Color the vertex F by any of the two colors which do not conflict with the color of the vertex α_F. Since the vertices β_F, γ_F and δ_F are free by Proposition 7 and the choice of i, we can color each of them by a color which does not conflict with the color of F and then we can extend this coloring to inner vertices of their transfer gadgets. Since the vertex F can be colored by (at least two) colors from its list which do not conflict with the color of the vertex α_F, the vertex F is free as desired.

■

Proposition 11 *Let A be an Abelian group of order $k \geq 3$. There exist an orientation, an edge-labeling and a list-assignment of the universal gadget such that the vertex T is rigid and forced to be colored with 0. Similarly, there exist an orientation, an edge-labeling and a list-assignment of the universal gadget such that the vertex F is rigid and forced to be colored with 0.*

Proof: We construct an orientation, an edge-labeling and a list-assignment which forces the vertex T to get the color 0. The case of the vertex F is completely symmetric. Let us fix two elements $\sigma_1 \neq \sigma_2$ of A different from 0. First we fix orientations, edge-labelings and list-assignments for transfer gadgets as in Proposition 4 with $\rho = 0$ for the transfer gadgets containing α_F, β_F, γ_F and δ_F, with $\rho = \sigma_1$ for the gadgets containing α_T and β_T and with $\rho = \sigma_2$ for the gadgets containing γ_T and δ_T. The edges not contained in transfer gadgets are oriented to the vertices T and F and the edge-labeling φ is extended to these edges as follows:

$$\begin{array}{cccc} \varphi(\alpha_F F) = 0 & \varphi(\beta_F F) = \sigma_1 & \varphi(\gamma_F F) = \sigma_2 & \varphi(\delta_F F) = 0 \\ \varphi(\alpha_T T) = \sigma_1 & \varphi(\beta_T T) = \sigma_2 & \varphi(\gamma_T T) = \sigma_1 & \varphi(\delta_T T) = \sigma_2 \end{array}$$

Finally, the lists of the vertices ε, T and F are set to be $\{0, \sigma_1, \sigma_2\}$.

We now show that the vertex T is forced to get the color 0. If the vertex ε is colored with 0, then the vertices α_F, β_F and γ_F must be colored with 0 by the choice of the orientations, the edge-labelings and the list-assignments in the corresponding transfer gadgets. But then the vertex F cannot be colored because of the edges $\alpha_F F$, $\beta_F F$ and $\gamma_F F$. Hence, the vertex ε must be colored with σ_1 or σ_2. If the vertex ε is colored with σ_1, then the vertices α_T and β_T must get the color 0 by the choice the orientations, the edge-labelings and the list-assignments in the corresponding transfer gadgets. Thus, the edges $\alpha_T T$ and $\beta_T T$ force the vertex T to be colored with the color 0. Similarly, if the vertex ε is colored with σ_2, then the vertices γ_T and δ_T must get the color 0 and the edges $\gamma_T T$ and $\delta_T T$ force the vertex T to be colored with the color 0. In either of the cases, the vertex T must be colored with the color 0 as desired.

■

3.6 Literal gadget

The *ℓ-literal gadget* is formed by ℓ copies of the transfer gadget and ℓ new vertices denoted by $\mathrm{OUT}_1, \ldots, \mathrm{OUT}_\ell$. Let α_i and β_i be the vertices IN and OUT of the i-th copy of the transfer gadget. The ℓ-literal gadget has $\ell + 1$ contact vertices which are the vertices $\mathrm{OUT}_1, \ldots, \mathrm{OUT}_\ell$ and the vertex α_1 denoted further as the vertex IN. In addition to the edges of the transfer gadget, the ℓ-literal gadget contains edges $\beta_i \alpha_{i+1}$ for $i = 1, \ldots, \ell - 1$ and edges $\beta_i \mathrm{OUT}_i$ for $i = 1, \ldots, \ell$. The list-size-assigning function ψ is modified so that $\psi(\alpha_2) = \ldots = \psi(\alpha_\ell) = 2$ and extended to the new vertices OUT_1,

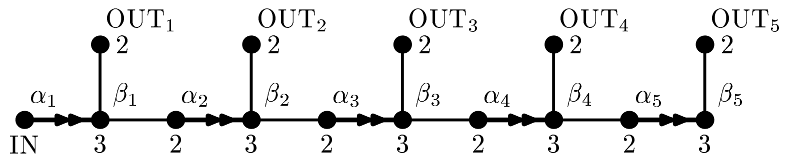

Figure 5: The 5-literal gadget.

$\ldots$, OUT$_\ell$ so that $\psi(\text{OUT}_1) = \ldots = \psi(\text{OUT}_\ell) = 2$. An example of the ℓ-literal gadget with list sizes for $\ell = 5$ can be found in Figure 5. Some properties of the ℓ-literal gadget which are essential for our construction are stated in the following three propositions:

Proposition 12 *Let A be an Abelian group of order $k \geq 3$ and ℓ a positive integer. There exist an orientation, an edge-labeling and a list-assignment of the ℓ-literal gadget such that if the contact vertex IN is precolored with* 0*, then all contact vertices OUT_1, $\ldots$, OUT_ℓ must be colored with* 0*.*

Proof: Fix orientations, edge-labelings and list-assignments in each of the ℓ copies of the transfer gadget as in Proposition 4 for $\rho = 0$. Let φ be the resulting partial edge-labeling and L the list-assignment. Next, orient the remaining edges so that there are leading from the vertices β_i to OUT$_i$ for $i = 1, \ldots, \ell$. Define $\varphi(\beta_i\text{OUT}_i) = \sigma$ for $i = 1, \ldots, \ell$ where $\sigma \neq 0$ is an element of A. In addition, set $\varphi(\beta_i\alpha_{i+1}) = \sigma$ for $i = 1, \ldots, \ell - 1$. The edge-labeling φ is now defined for all edges of the ℓ-literal gadget. Next, define $L(\alpha_i) = \{0, \sigma\}$ for $i = 2, \ldots, \ell$ and $L(\text{OUT}_i) = \{0, \sigma\}$ for $i = 1, \ldots, \ell$. The list assignment L is now defined for all vertices of the ℓ-literal gadget, too. In the next paragraph, we show that if the vertex IN is precolored with 0, then all the contact vertices OUT$_1$, $\ldots$, OUT$_\ell$ must be colored with 0.

If the vertex IN $= \alpha_1$ is precolored with 0, then the vertex β_1 must also get the color 0 by the choice the orientation, the edge-labeling and the list-assignment inside the first transfer gadget. Because $\varphi(\beta_1\alpha_2) = \sigma$ and $L(\alpha_2) = \{0, \sigma\}$, the vertex α_2 must get the color 0, too. Again, the choice of the orientation, the edge-labeling and the list-assignment inside the transfer gadget forces the vertex β_2 to get the color 0 and consequently the vertex α_3 to get the color 0. In this way, we derive that all the vertices $\alpha_2, \ldots, \alpha_\ell$ and $\beta_1, \ldots, \beta_\ell$ must get the color 0. Then, none of the contact vertices OUT$_1$, $\ldots$, OUT$_\ell$ can be colored with the color σ and hence they all must get the color 0.

■

Proposition 13 *Let A be an Abelian group of order $k \geq 3$ and ℓ a positive integer. Fix an orientation, an edge-labeling and a list-assignment of the ℓ-literal gadget. If the vertices $OUT_1, \ldots, OUT_\ell$ are precolored, then the vertex IN is free.*

Proof: It is enough to show that it is possible to color all the inner vertices of the ℓ-literal gadget except for the inner vertices of the first transfer gadget so that there is no conflict. Once we have such a coloring, then the vertex OUT (which is the vertex β_1) of the first transfer gadget is precolored and by Proposition 6, the vertex IN is free.

Color first the vertex β_ℓ by a color from its list which does not conflict with the color of the vertex OUT_ℓ. If the vertex β_ℓ is precolored, then the vertex α_ℓ would free by Proposition 6 if its list had size three. Since its list has size two, it can be colored with a color from its list so that there is a consistent coloring not conflicting with the colors assigned to the vertices α_ℓ and β_ℓ. The vertices $\mathrm{OUT}_{\ell-1}$ and α_ℓ forbid at most two colors at the vertex $\beta_{\ell-1}$ and hence it is possible to color the vertex $\beta_{\ell-1}$ with a color from its list. The same argument as applied to the ℓ-th transfer gadget yields that the vertex $\alpha_{\ell-1}$ can be colored with a color from its list. If we proceed in this way, all the inner vertices of the ℓ-literal gadget except for the inner vertices of the first transfer gadget are precolored so that there is no conflict.

■

Proposition 14 *Let A be an Abelian group of order $k \geq 3$ and ℓ a positive integer. Fix an orientation, an edge-labeling and a list-assignment of the ℓ-literal gadget. If the vertex IN is precolored, then there exists a consistent coloring of the gadget.*

Proof: By Proposition 6, the first transfer gadget can be colored and this gives a color to the vertex β_1. Color now the vertex α_2 by a color from its list not conflicting with the color of the vertex β_1. By Proposition 6, we can color the second transfer gadget together with the vertex β_2. Then, assign color to the vertex α_3, color the third transfer gadget and the vertex β_3, etc. In this way, a consistent coloring of the inner vertices not conflicting with the color of the vertex IN is obtained. Finally, color the contact vertices $\mathrm{OUT}_1, \ldots, \mathrm{OUT}_\ell$ by colors from their lists not conflicting with the colors of their neighbors $\beta_1, \ldots, \beta_\ell$.

■

3.7 The construction

Finally, we can present the construction of the desired graph G:

Theorem 15 *Let Ψ be a given Π_2-3CNF-formula and $k \geq 3$ an integer. There exists a graph G with a list-size-assigning function ψ with the following property: If Ψ is true, then G is A-ψ-choosable for each Abelian group A of order k. If Ψ is false, then G is A-ψ-choosable for no Abelian group A of order k. Moreover, the graph G and the list-size-assigning function ψ can be constructed in time polynomial in k and in the size of the formula Ψ.*

Proof: Let $x_1, \ldots, x_m$ be the variables of Ψ which are universally quantified and $y_1, \ldots, y_n$ the variables which are existentially quantified. Let us consider a variable x_i contained in ℓ_T positive and ℓ_F negative literals in Ψ. The graph G contains a copy of the universal gadget, the ℓ_T-literal gadget and the ℓ_F-variable gadget where the contact vertex T of the universal gadget is identified with the contact vertex IN of the ℓ_T-literal gadget and the contact vertex F of the universal gadget is identified with the contact vertex IN of the ℓ_F-literal gadget. Each of the contact vertices OUT_j of the ℓ_T-literal gadget correspond to one of positive literals containing the variable x_i and each of the contact vertices OUT_j of the ℓ_F-literal gadget correspond to one of negative literals containing the variable x_i. Similarly, for a variable y_i contained in ℓ_T positive and ℓ_F negative literals, the graph G contains a copy of the existential gadget, the ℓ_T-literal gadget and the ℓ_F-literal gadget where the contact vertex T of the existential gadget is identified with the contact vertex IN of the ℓ_T-literal gadget and the contact vertex F of the existential gadget is identified with the contact vertex IN of the ℓ_F-literal gadget. Again, each of the contact vertices OUT_j of the ℓ_T-literal gadget correspond to one of positive literals containing the variable y_i and each of the contact vertices OUT_j of the ℓ_F-literal gadget correspond to one of negative literals containing the variable y_i. Finally, for each clause C of Ψ, add a new vertex w_C to the graph G and join it to three vertices corresponding to literals contained in the clause C. This finishes the construction of the graph G. The list-size-assigning function ψ is the same as the list-size-assigning function for the gadgets used in our construction extended to the clause vertices by assigning all of them the number three.

Let A be an Abelian group of odd order $k \geq 3$. First, we show that if Ψ is true, then G is A-ψ-choosable. Fix an orientation, the edge-labeling and the list-ψ-assignment of the graph G. By Proposition 10, for each universal gadget, it is possible to precolor the vertex T of it so the vertex F is free or it is possible to precolor the vertex F so the vertex T is free. In the first case, precolor the vertex T and set the corresponding variable x_i to be false. In the latter case, precolor the vertex F and set the corresponding variable x_i to be true. Since Ψ is true, there exists a truth-assignment for variables y_i such that each clause of Ψ is satisfied. Fix such a truth assignment for variables y_i. If y_i is true, precolor the vertex F of the existential gadget corresponding to the variable y_i so that its vertex T is free. If y_i is false, precolor the vertex T so that the vertex F is free. This is always possible by Proposition 8.

If the vertex IN of the literal gadget is now precolored, then extend the coloring to the whole literal gadget and color all its vertices OUT_i. This is possible by Proposition 14. In this way, contact vertices of exactly half of the literal gadgets (those corresponding to false literals) are precolored. We color now the clause vertices. Let us consider a clause C of the formula Ψ. The size of the list of the vertex w_C is three and it has at most two precolored neighbors (otherwise all the three literals contained in the clause C are false). Hence, it is possible to color the vertex w_C by a color from its list. Once, the vertex w_C is colored, color also its neighbors corresponding to true literals by colors from their lists (this is possible since each of them has a list of size two and w_C is its only colored neighbor). In this way, all the contact vertices of literal gadgets corresponding to true literals are precolored. By Proposition 13, the vertices IN of these variable gadgets are free and since they are also free in their universal/existential gadgets and their lists have sizes three, they can be colored so that the coloring can be extended to inner vertices of the incident gadgets. In this way, we have obtained a proper coloring of the whole graph G.

Next, we show that if Ψ is false, then G is not A-ψ-choosable. By our assumption, there exists a truth assignment for $x_1, \ldots, x_m$ such that for any truth assignment for $y_1, \ldots, y_n$, there is a clause C of Ψ which is not satisfied. Consider the orientation, the edge-labeling and the list-assignment for existential gadgets, universal gadgets and literal gadgets contained in the graph G as in Propositions 9, 11 and 12. In the case of universal gadgets, choose the orientation, the edge-labeling and the list-assignment so the color of the vertex T, F, is forced to be 0 if the corresponding variable x_i is false, true, respectively. At this moment, the only non-oriented edges and the edges without a label are the edges incident with the clause vertices. Let σ_1, σ_2 and σ_3 be any three distinct elements of A. Orient all these edges in the direction to the clause vertices and extend the edge-labeling ψ so that each clause is incident with exactly one edge labelled with σ_0, one edge with σ_1 and one edge with σ_2. Finally, assign all the clause vertices the same list $\{\sigma_1, \sigma_2, \sigma_3\}$. We show that G cannot be colored with respect to this orientation, this edge-labeling and this list-assignment.

Assume that G can be properly colored and fix such a coloring. By Proposition 9, at least one of the vertices F and T of each existential gadget is colored with 0. Set y_i to be true if the vertex F of the corresponding existential gadget is colored with 0 and set y_i to be false otherwise. Since the formula Ψ is false, there is a clause C for the just constructed truth-assignment which is not satisfied. Observe that all the three vertices OUT_j from literal gadgets which are joined to w_C must be colored with 0 by Proposition 12 since the vertices IN of those three literal gadgets are colored with 0 (this is immediately implied by the choice of the truth assignment and the fact that C is not satisfied). But then, the vertex w_C cannot be colored from its list because all the three elements σ_1, σ_2 and σ_3 appear on edges incident with it.

■

4 Group coloring is hard

In this section, we prove our main results about complexity of group coloring. First, we deal with the case of 3-group colorability to demonstrate our technique. Later we extend our results to groups of larger order:

Theorem 16 *Let Ψ be a given Π_2-3CNF-formula. There exists a graph G_0 which is Z_3-colorable if and only if the formula Ψ is true. The graph G_0 can be constructed in time polynomial in the size of the formula Ψ. Moreover, the graph G_0 is 3-degenerate.*

Proof: Let G be the graph and ψ the list-size-assigning function from Theorem 15 for $k = 3$ and the formula Ψ. Observe that $\psi(v) \in \{2, 3\}$ for each vertex v of G. Add a new vertex w_0 to the graph G and join w_0 to each vertex v of G with $\psi(v) = 2$. Let G_0 be the resulting graph. We claim that G_0 is Z_3-colorable if and only if the formula Ψ is true.

Assume that Ψ is true. Fix an orientation and an edge-labeling of the graph G_0. Color the vertex w_0 with the color 0 and consider now the following list-assignment for the graph G: If $\psi(v) = 3$ for $v \in V(G)$, then

$L(v) = \{0, 1, 2\}$. If $\psi(v) = 2$, then $L(v)$ consists of the two colors which do not conflict with the color of w_0. Observe that L is a list-ψ-assignment. Consider now a restriction of the orientation and the edge-labeling of G_0 to the graph G together with the list-assignment L. The graph G has a proper coloring with respect to this orientation, this edge-labeling and the list-assignment L because it is Z_3-ψ-choosable by Theorem 15. This proper coloring extended to w_0 by assigning the color 0 to it is a proper coloring of the whole graph G_0.

Assume now that Ψ is false. Fix an orientation, an edge-labeling and a list-ψ-assignment L of G for which G cannot be properly colored (they exist by Theorem 15). Extend the orientation of G to G_0 so that each edge incident with w_0 is oriented from the vertex w_0. The edge-labeling is extended so that the label of the edge w_0v is σ where σ is the element of Z_3 missing in the list $L(v)$ (note that the sizes of lists of vertices v incident with w_0 in G_0 are two in G and hence there exists a unique element of Z_3 with this property). Assume for the sake of contradiction that G_0 can be colored with respect to this orientation and this edge-labeling and fix such a coloring c. We may assume that $c(w_0) = 0$ by Lemma 1. Then, the coloring c restricted to the graph G is a proper coloring of G such that $c(v) \in L(v)$ because at each vertex v with $L(v) \neq Z_3$ the color conflicting with the color of the vertex w_0 is exactly the color missing in the list $L(v)$. But there is no proper coloring of G from the lists L — a contradiction.

In order to complete the proof, we have to show that G_0 is 3-degenerate. Observe first that degrees of all inner vertices (the vertices β_1, β_2, γ_1, γ_2 and δ in Figure 2) of transfer gadgets in G_0 are three. Hence, remove all these vertices. Now, all the remaining vertices except for w_0 has degree at most three. Thus, G_0 is indeed 3-degenerate.

■

Before considering groups of larger order, we prove a simple lemma about coloring of a certain special graph. This graph is then used in Theorem 18 in the construction of the sought graph G_0. The graph K_n^l for $n \geq 2$ and $l \geq 1$ is the graph obtained in the following way: Consider l copies of the complete graph K_n of order n and choose in each copy of the complete graph a single special vertex. Add to the graph a new vertex, called the *center vertex* of K_n^l in the rest, and join it to all the vertices of the complete graphs except for the special ones. This finishes the construction of K_n^l. The graph K_4^5 can be found in Figure 6.

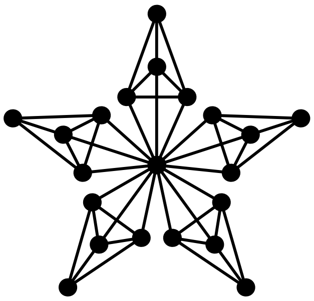

Figure 6: The graph K_4^5.

Lemma 17 *Let $n \geq 2$ and $l \geq 1$ be fixed integers. The group chromatic number of a graph K_n^l is n. If A is an Abelian group of order n, there exist an orientation and an edge-labeling such that in each proper coloring, the center vertex and all the special vertices get the same color.*

Proof: Since the graph K_n^l contains a clique of order n, we have $\chi_g(K_n^l) \geq n$. On the other hand, the graph K_n^l is $(n-1)$-degenerate and hence $\chi_g(K_n^l) \leq n$. In order to prove the second part of the statement of the lemma, consider any orientation of K_n^l and assign all the edges 0. Clearly, in each proper A-coloring of K_n^l, the center vertex and all the special vertices get the same color.

■

Theorem 18 *Let $k \geq 3$ be a fixed integer and Ψ a Π_2-3CNF-formula. There exists a k-degenerate graph G_0 with the following property: If Ψ is true, then G_0 is A-colorable for each Abelian group of order k and if Ψ is false, then G_0 is A-colorable for no Abelian group of order k. In addition, the graph G_0 can be constructed in time polynomial in k and in the size of the formula Ψ.*

Proof: Let G be the graph and ψ the list-size-assigning function from Theorem 15 for k and the formula Ψ. Note that $\psi(v) \in \{2, \ldots, \max\{3, k-1\}\}$ for each vertex v of G. Add a copy of the graph K_k^{k-2} to G and let $w_1, \ldots, w_{k-2}$ be the $k-2$ special vertices of K_k^{k-2}. If $\psi(v) = 2$ for $v \in V(G)$, then join the vertex v to the vertices $w_1, \ldots, w_{k-2}$, if $\psi(v) = 3$, then join the vertex v to the vertices $w_1, \ldots, w_{k-3}$, etc. Let G_0 be the resulting graph.

We claim that G_0 has all the properties from the statement of the lemma. In particular, if Ψ is true, then G_0 is A-colorable for each Abelian group of order k and if Ψ is false, then G_0 is A-colorable for no Abelian group of order k. Fix an Abelian group A of order k for the rest of the proof.

Assume first that Ψ is true. Fix an orientation and an edge-labeling of the graph G_0. Since the group chromatic of K_k^{k-2} is k, it is possible to color all vertices of K_k^{k-2}. Consider now the following list-assignment for the graph G: The list $L(v) \subseteq A$ of a vertex v of G consists of those elements of A which conflict with a color of no neighbor of v among the special vertices $w_1, \ldots, w_{k-2}$. Observe that $|L(v)| \geq \psi(v)$ for every $v \in V(G)$. Consider now a restriction of the orientation and the edge-labeling of G_0 to the graph G together with the list-assignment L. The graph G has a proper coloring with respect to this orientation, this edge-labeling and the list-assignment L because it is A-ψ-choosable by Theorem 15. This proper coloring and the proper coloring of the copy of K_k^{k-2} form a proper coloring of the graph G_0.

Assume now that Ψ is false. Fix an orientation, an edge-labeling and a list-ψ-assignment L of G for which G cannot be properly colored (they exist by Theorem 15). Next, fix an orientation and an edge-labeling of K_k^{k-2} such that in any proper coloring all the special vertices get the same color (such an orientation and an edge-labeling exist by Lemma 17). Extend the orientation of G to G_0 so that each edge incident with a special vertex is oriented from the special vertex to its neighbor in G. The edge-labeling is extended so that the labels of the edges $w_1 v, \ldots, w_{k-|L(v)|} v$ are precisely those elements of A missing in the list $L(v)$. Assume for the sake of contradiction that G_0 can be colored with respect to this orientation and this edge-labeling and fix such a coloring c. We may assume that $c(w_1) = 0$ by Lemma 1. Hence, we have $c(w_1) = \ldots = c(w_{k-2}) = 0$ by the choice of orientation inside K_k^{k-2}. Then, the coloring c restricted to the graph G is a proper coloring of G such that $c(v) \in L(v)$ because at each vertex v the colors conflicting with the colors of the vertices $w_1, \ldots, w_{k-2}$ are exactly the colors missing in the list $L(v)$. But there is no proper coloring of G from the lists L — a contradiction.

In order to complete the proof, we have to show that G_0 is k-degenerate. Observe first that degrees of all inner vertices of transfer gadgets in G_0 are k. Remove all these vertices. Now, all the remaining vertices except for the vertices of the copy K_k^{k-2} has degree at most k. Thus, we can remove these vertices. A copy of the graph K_k^{k-2} remains. Since the graph K_k^{k-2}

is k-degenerate, we can conclude that the graph G_0 is k-degenerate.

■

We are now ready to prove the two main theorems of this paper:

Theorem 19 *The decision problem whether the group chromatic number of a given graph G is at most k is Π_2^P-complete for each integer $k \geq 3$.*

Proof: Fix an integer $k \geq 3$. Let Ψ be a given Π_2-3CNF-formula. Consider the graph G_0 from Theorem 16 if $k = 3$ and from Theorem 18 if $k \geq 4$. Note that $\chi_g(G_0) \leq k + 1$ because G_0 is k-degenerate. Observe that the graph $\chi_g(G_0) \leq k$ if and only if Ψ is true by Theorems 16 and 18, respectively. Since G_0 can be constructed in time polynomial in the size of Ψ, the decision problem whether the group chromatic number of a given graph G is at most k is Π_2^P-complete.

■

Theorem 20 *The decision problem whether a given graph G is A-colorable for a fixed Abelian group A of order $k \geq 3$ is Π_2^P-complete.*

Proof: Fix an Abelian group A of order $k \geq 3$. Consider the graph G_0 from Theorem 16 if $k = 3$ and from Theorem 18 if $k \geq 4$. By Theorems 16 and 18, respectively, the graph G_0 is A-colorable if and only if Ψ is true. Since G_0 can be constructed in time polynomial in the size of Ψ, the decision problem whether a given graph G is A-colorable is Π_2^P-complete.

■

5 Conclusion

We have investigated the complexity of group coloring. Some of our results can be extended to A-ℓ-choosability. The case of A-2-choosability is easy since the following holds: A graph G is A-2-choosable for a group of even order iff G is forest. A graph G is A-2-choosable for a group of odd order iff the core of G (defined as in [3]) is a union of vertex-disjoint even cycles. Hence, it is possible to decide in polynomial time whether a given graph G is A-2-choosable for each Abelian group A. It can be shown that graph G_0 from Theorem 16 has the following two properties:

- If Ψ is true, then the graph G_0 is A-3-choosable for every group A of odd order.
- If Ψ is false, then the graph G_0 is A-3-choosable for no group A of odd order.

Hence, the problem to decide whether a given graph is A-ℓ-choosable for an Abelian group A of odd order $k \geq 3$ and $3 \leq \ell \leq k$ is Π_2^P-complete. The constructed list-size-assigning function ψ for groups of order k in Theorem 15 has values $2, \dots, \max\{3, k-1\}$. A proof similar to the proof of Theorem 18 yields that the problem to decide whether a given graph is A-ℓ-choosable is Π_2^P-complete for every Abelian group A of order $k \geq 4$ with $\ell = k-1, k$. We do not have the proof of Π_2^P-completeness of A-ℓ-choosability for groups A of even order $k \geq 6$ and for ℓ, $3 \leq \ell \leq k-2$. We conjecture that the problem is Π_2^P-complete even in these cases:

Conjecture 21 *For each Abelian group A of order $k \geq 3$ and each integer ℓ, $3 \leq \ell \leq k$, the problem to decide whether a given graph is A-k-choosable is Π_2^P-complete.*

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