# Group coloring is $\Pi_2^P$-complete

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**Abstract**

The group chromatic number of a graph $G$ is the smallest integer $k$ such that for each Abelian group $A$ of order at least $k$, each orientation of $G$ and each edge-labeling $\varphi : E(G) \to A$, there exists a vertex-coloring $c : V(G) \to A$ with $c(v) - c(u) \neq \varphi(uv)$ for all oriented edges $uv$ of $G$. We show that the decision problem whether a given graph has group chromatic number at most $k$ is $\Pi_2^P$-complete for each integer $k \geq 3$.

## 1 Introduction

Group colorings of graphs have been introduced by Jaeger, Linial, Payan and Tarsi [4]. For plane graphs, this concept is dual (in the usual sense) to group connectivity which generalizes the intensively studied concept of nowhere-zero flows in graphs. For an Abelian group $A$, a graph $G$ is said to be $A$-*colorable* if for every orientation of $G$ and for every *edge-labeling* $\varphi : E(G) \to A$, there is a vertex coloring $c : V(G) \to A$ such that $c(w) - c(v) \neq \varphi(vw)$ for each oriented edge $vw \in E(G)$. Note that the choice of an orientation of edges of $G$ is not essential since reversing the orientation of the edge $e$ can be replaced by changing the forbidden difference $\varphi(e)$ to $-\varphi(e)$. A plane graph is $A$-colorable iff its dual is $A$-connected [4]. We remark that it is unknown whether the property of being $A$-colorable ($A$-connected) depends on the structure or only on the order of the group $A$ (like in the case of nowhere-zero flows [12]). The least number $\chi_g(G)$ such that $G$ is $A$-colorable for each

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Abelian group $A$ of order at least $\chi_g(G)$ is the *group chromatic number* of $G$.

Group colorings have attracted a lot of attention from the combinatorial point of view. Some theorems for ordinary colorings, e.g., Brooks' theorem, can be translated to group colorings [6], some cannot. However, the concept of group coloring is closer to the concept of list-coloring [5]: The group chromatic number of a graph $G$ with the average degree $d$ is at least $\Omega(d/\log d)$. Hence, as in the concept of list coloring, the group chromatic number, the choice number, respectively, of graphs with large average degree is large. Lai and Zhang [7] proved that the group chromatic number of planar graphs is bounded by five and the author with Pangrác and Voss [5] constructed a planar graph with the group chromatic number five. This is the same bound as in the case of list-coloring [11]. Planar graphs without 3-cycles and 4-cycles have the group chromatic number at most three and planar graphs without 4-cycles at most four [5] matching the same bounds for choosability [8, 10]. On the other hand, the choice number of planar bipartite graphs is at most three [1] but there is a planar bipartite graph with the group chromatic number equal to four [5].

In this paper, we address the complexity of group coloring. We show that for each Abelian group $A$ of order at least three it is $\Pi_2^P$-complete to decide whether a given graph $G$ is $A$-colorable. The problem to decide whether the group chromatic number of a graph $G$ is at most $k$ is also $\Pi_2^P$-complete for each integer $k \geq 3$.

## 2 Definitions and notation

All groups considered in this paper are finite and Abelian. The neutral element is always denoted by 0. Elements of the group are sometimes also referred to as colors. If an orientation and edge-assignment $\varphi$ of a graph $G$ is given, then a vertex coloring $c$ is *proper* if $c(v) - c(u) \neq \varphi(uv)$ for every oriented edge $uv$ of $G$. Hence, a graph $G$ is $A$-colorable if there is a proper coloring for each orientation of it and each edge-assignment $\varphi : E(G) \to A$. An edge $uv$ with $c(v) - c(u) \neq \varphi(uv)$ is said to be *properly* colored. On the other hand, if $c(v) - c(u) = \varphi(uv)$, the colors of $u$ and $v$ *conflict* with each other.

Let us state a simple lemma about group coloring which is used later in this paper:

**Lemma 1** *Let $G$ be a graph and $v$ a fixed vertex of it. If a graph $G$ is $A$-colorable, then for each orientation and each edge-labeling, there exists a proper coloring $c$ of $G$ with $c(v) = 0$.*

We first show that a variant of group coloring where sizes of lists are not the same for all vertices of a graph is $\Pi_2^P$-complete and then we derive the $\Pi_2^P$-completeness of the original problem. A *list-size-assigning* function is a mapping $\psi : V(G) \to \{1, \ldots, |A|\}$. An *$A$-list-$\psi$-assignment* for a graph $G$ and an Abelian group $A$ is a mapping $L : V(G) \to 2^A$ with $|L(v)| = \psi(v)$ for every vertex $v \in V(G)$. If the group $A$ is clear from the context, the list assignment $L$ is just said to be a *list-$\psi$-assignment*. The graph $G$ is said to be *$A$-$\psi$-choosable* if for each orientation of $G$, each edge-labeling $\varphi : E(G) \to A$ and each list-$\psi$-assignment $L$, there is a vertex coloring $c : V(G) \to A$ such that $c(v) \in L(v)$ for every vertex $v \in L(v)$ and $c(w) - c(v) \neq \varphi(vw)$ for every oriented edge $vw \in E(G)$. We also say that $G$ can be colored from lists $L$ or that it is *$A$-$L$-colorable* in such a situation. If $\psi$ is a constant function equal to $\ell$, then the list assignment $L$ is said to be *list-$\ell$-assignment* and the graph $G$ is said to be *$A$-$\ell$-choosable*. Note that a graph is $A$-$|A|$-choosable iff it is $A$-colorable. Group choosability has sometimes a somewhat strange behavior, e.g., a cycle of even length is $A$-2-choosable for every group $A$ of odd order and it is not $A$-2-choosable for any group $A$ of even order.

The complexity class $\Pi_2^P$ is defined to be the class of the complements of problems which can be solved by non-deterministic algorithms running in polynomial time which have access to the oracle solving problems from the class NP [2, 9]. Only very few natural combinatorial problems are known to be $\Pi_2^P$-complete, e.g., the problem to decide whether a graph is 3-choosable is $\Pi_2^P$-complete [3], to compute the generalized Ramsey number [2]. Similarly as ordinary satisfiability of formulas is the basic NP-complete problem, the decision problem whether a given $\Pi_2$-formula is true is the basic $\Pi_2^P$-complete problem. A formula $\Psi$ is a *$\Pi_2$-3CNF-formula* if it is of the following form:

$$\Psi = \forall x_1 \cdots \forall x_m \exists y_1 \cdots \exists y_n \Psi_0(x_1, \ldots, x_m, y_1, \ldots, y_n)$$

where $\Psi_0$ is a 3CNF-formula with variables $x_1, \ldots, x_m, y_1, \ldots, y_n$, i.e., a formula $\Psi_0$ is in the conjunctive normal form (CNF) and each clause of it has size exactly three. Formulas $\Psi$ where $\Psi_0$ is of this restricted form are called *$\Pi_2$-3CNF-formulas*. The *size* of a $\Pi_2$-3CNF-formula is the number of its clauses. We remind that occurrences of variables in clauses of $\Psi_0$ are called *literals*. A literal is *positive* if it is of the form $x_i$, $i = 1, \ldots, m$, or $y_i$, $i = 1, \ldots, n$. Similarly, a literal is *negative* if it is of the form $\neg x_i$, $i = 1, \ldots, m$, or $\neg y_i$, $i = 1, \ldots, n$.

# 3 Group choosability is hard

In this section, we construct, for a given $\Pi_2$-3CNF-formula $\Psi$ and an integer $k$, a graph $G$ which is $A$-$\psi$-choosable for a certain list-size-assigning function $\psi$ for each Abelian group $A$ of order $k$ if $\Psi$ is true and for no Abelian group $A$ of order $k$ if $\Psi$ is false. This shows that $A$-choosability is $\Pi_2^P$-complete if sizes of lists for vertices form a part of the input. In the next section, these results are used to deduce that the decision problem whether a given graph is $A$-colorable is $\Pi_2^P$-complete.

## 3.1 Preliminary results on group choosability

First, let us prove a simple lemma on group choosability which later helps us to simplify some of our arguments:

**Lemma 2** *Let $G$ be a directed graph, $T$ a spanning tree of $G$, $A$ a fixed Abelian group and $\psi$ a list-size-assigning function. Then, the graph $G$ is $A$-$\psi$-choosable if and only if $G$ can be colored from every $A$-$\psi$-lists for every edge-labeling $\varphi : E(G) \to A$ with $\varphi(uv) = 0$ for all $uv \in E(T)$.*

**Proof:** Clearly, it is enough to prove that if $G$ can be colored from any $A$-$\psi$-lists for each edge-labeling $\varphi : E(G) \to A$ with $\varphi(uv) = 0$ for all $uv \in E(T)$, then $G$ is $A$-$\psi$-choosable. Fix an orientation, an edge-labeling $\varphi : E(G) \to A$ and an $A$-$\psi$-list assignment $L$ of $G$. We can assume that $T$ is a branching, i.e., $T$ has a root $r$ and all edges of $T$ are oriented in the direction from the root. We modify the edge-labeling $\varphi$ so that the resulting edge-labeling is equal to 0 on the edges of the tree $T$. In order to do this, we process one edge after another in the order determined by their distance from the vertex $r$. The edges incident with the root $r$ are processed first.

Let us consider an edge $uv$ of the tree $T$ and modify the edge-labeling $\varphi$ and the list assignment $L$ as follows:

- Decrease the label of each edge $e$ leading to the vertex $v$ by $\varphi(uv)$ and increase the label of each edge $e$ going out from the vertex $v$ by $\varphi(uv)$. Observe that the label of the edge $uv$ after the modification is equal to 0.

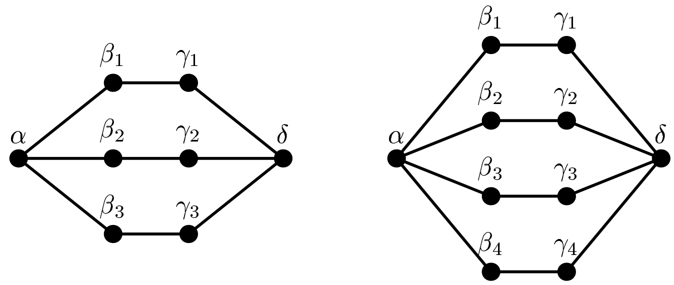


Figure 1: The graphs $P_4^3$ and $P_4^4$.

- Decrease each element in the list $L(v)$ of the vertex $v$ by $\varphi(uv)$.

It is easy to see that there is a proper vertex-coloring $c : V(G) \to A$ with respect to the original edge-labeling and list assignment if and only if there is a proper vertex-coloring with respect to the modified ones. The modification does not change edge-labels of edges closer to the root $r$ in the tree $T$ than the edge $uv$ and hence after all edges are processed, the edge-labeling $\varphi$ has been modified so that $\varphi(uv) = 0$ for all edges $uv \in E(T)$.

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The core of our gadgets is the following graph denoted as $P_4^k$: Consider $k$ copies of the four-vertex path $P_4$. Let $\alpha_i\beta_i\gamma_i\delta_i$, $i = 1, \ldots, k$, be these paths. We identify all the vertices $\alpha_1, \ldots, \alpha_k$ to a single vertex $\alpha$ and the vertices $\delta_1, \ldots, \delta_k$ to a single vertex $\delta$. Next, define the list-size-assigning function $\psi$ as follows: $\psi(\alpha) = \psi(\beta_1) = \ldots = \psi(\beta_k) = \psi(\gamma_1) = \ldots = \psi(\gamma_k) = 2$ and $\psi(\delta) = k$. The graphs $P_4^3$ and $P_4^4$ are depicted in Figure 1. A property of the graph $P_4^k$ which is crucial for our construction is stated in the next proposition:

**Proposition 3** *Let $k \geq 2$ be an integer and $A$ an Abelian group of order $k$. The graph $P_4^{k-1}$ is $A$-$\psi$-choosable.*

**Proof:** Fix an orientation, an edge-labeling $\varphi$ and a list-$\psi$-assignment $L$. In particular, the size of the list $|L(\delta)|$ of the vertex $\delta$ is $k-1$. We may assume that the edges are oriented so that the edges $\alpha\beta_i$ are oriented from $\alpha$ to $\beta_i$, the edges $\beta_i\gamma_i$ from $\beta_i$ to $\gamma_i$ and $\gamma_i\delta$ from $\gamma_i$ to $\delta$ for $i = 1, \ldots, k-1$. By Lemma 2, we can also assume that $\varphi(\alpha\beta_i) = 0$ and $\varphi(\beta_i\gamma_i) = 0$ for every $i = 1, \ldots, k-1$.

In the proof, we distinguish two cases. The first case is that there is an edge $\alpha\beta_i$ with $L(\alpha) \neq L(\beta_i)$ or an edge $\beta_i\gamma_i$ with $L(\beta_i) \neq L(\gamma_i)$ for some $i = 1, \ldots, k-1$. Assume that there is such an edge $\alpha\beta_i$. Color the vertex $\alpha$ by a color from its list which is not contained in the list $L(\beta_i)$. Color now all the vertices $\beta_{i'}$ for $i' \neq i$ by colors from their lists which do not conflict with the color of the vertex $\alpha$ and then color all the vertices $\gamma_{i'}$ for $i' \neq i$ by colors from their lists which do not conflict with the color of the vertex $\beta_{i'}$. The vertex $\delta$ has now $k-2$ colored neighbors and thus it can be colored by a color from its list which conflicts with the color of no vertex $\gamma_{i'}$ with $i' \neq i$. Color now the vertex $\gamma_i$ by a color from its list which does not conflict with the color of the vertex $\delta$ and then the vertex $\beta_i$ by a color from its list which does not conflict with the color of the vertex $\gamma_i$. By the choice of the color of the vertex $\alpha$, the edge $\alpha\beta_i$ is also properly colored. An analogous argument applies if there is an edge $\beta_i\gamma_i$ with $L(\beta_i) \neq L(\gamma_i)$.

The second (and last) case to deal with is that the lists of all the vertices $\alpha, \beta_1, \ldots, \beta_{k-1}, \gamma_1, \ldots, \gamma_{k-1}$ are the same. Let $\sigma_1$ and $\sigma_2$ be the two elements of $A$ contained in these lists. We now consider the following two colorings $c_1$ and $c_2$ of all the vertices of $P_4^{k-1}$ except for the vertex $\delta$: $c_\ell(\alpha) = \sigma_\ell$, $c_\ell(\beta_i) = \sigma_{3-\ell}$ and $c_\ell(\gamma_i) = \sigma_\ell$ for $\ell = 1, 2$ and $i = 1, \ldots, k-1$. Define now a set $B_\ell$ for $\ell = 1, 2$ to be the set containing all the elements $\sigma_\ell + \varphi(\gamma_i\delta)$ for $i = 1, \ldots, k-1$. If the graph $G$ does not have a proper coloring from the list-assignment $L$ with respect to the orientation and the edge-labeling $\varphi$, both the sets $B_1$ and $B_2$ are equal to $L(\delta)$. In particular, all the labels $\varphi(\gamma_1\delta), \ldots, \varphi(\gamma_{k-1}\delta)$ are mutually distinct. Let $\tau$ be the only element of $A$ missing in the list $L(\delta)$. By the definition of $B_1$, the equality implies $B_1 = L(\delta)$ that none of the labels $\varphi(\gamma_1\delta), \ldots, \varphi(\gamma_{k-1}\delta)$ is equal to $\tau - \sigma_1$. Similarly, none of the labels $\varphi(\gamma_1\delta), \ldots, \varphi(\gamma_{k-1}\delta)$ is equal to $\tau - \sigma_2$. Since the labels $\varphi(\gamma_1\delta), \ldots, \varphi(\gamma_{k-1}\delta)$ are mutually distinct, it must hold that $\tau - \sigma_1 = \tau - \sigma_2$. This implies $\sigma_1 = \sigma_2$ which is impossible. Hence, $B_1 \neq L(\delta)$ or $B_2 \neq L(\delta)$ and the graph $G$ can be properly colored from the lists $L$ with respect to the orientation and the edge-labeling $\varphi$.

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## 3.2 Overview of the construction

We aim to construct a graph $G$ and a list-size-assigning function $\psi$ for a given $\Pi_2$-3CNF-formula $\Psi$ and a given integer $k \geq 3$ such that $G$ is $A$-$\psi$-choosable for each group $A$ of order $k$ if $\Psi$ is true and $G$ is $A$-$\psi$-choosable for no group $A$ of order $k$ if $\Psi$ is false. The structure of the graph $G$ depends on the order of the group $A$ but not on its structure. The graph $G$ will be glued from several types of gadgets. The *gadget* is a graph with several (usually two) distinguished *contact vertices* together with a list-size-assigning function defined for all vertices including the contact ones. Contact vertices are always non-adjacent and vertices of the gadget which are not contact are called *inner vertices* of the gadget.

Once the Abelian group $A$, the orientation of the edges of the gadget, the edge-labeling and the list-assignment for inner vertices are chosen, there exist one or more proper vertex-colorings of the gadget. Such a proper vertex-coloring is said to be *consistent* with the given orientation, the edge-labeling and the list-assignment. If the orientation, the edge-labeling and the list-assignment are clear from the context, the vertex-coloring is just said to be consistent. A contact vertex is *precolored* if it is colored with a color from its list. If all but one contact vertices are precolored, the remaining contact vertex is said to be *rigid* if only a single color from its list can be used to color it, i.e., every consistent coloring which extends the precoloring assigns this vertex the same color. The contact vertex is said to be *free* if there are at least two colors in its list with which it can be colored. Finally, the contact vertex is said to be *colorable* if it is rigid or free. In our construction, free vertices correspond to true literals and rigid vertices to false ones.

## 3.3 Transfer gadget

First, we construct a so-called *transfer gadget* which is used to transfer the truth value of a literal throughout the constructed graph. The gadget has two contact vertices IN and OUT. For an Abelian group $A$ of order $k$, the transfer gadget is obtained from the graph $P_4^{k-1}$ by adding a new vertex OUT and joining it to the vertices $\beta_1$ and $\gamma_1$ of $P_4^{k-1}$. The other contact vertex IN is the vertex $\alpha$ of the graph $P_4^{k-1}$. The list-size-assigning function $\psi$ is the same as for the graph $P_4^{k-1}$ except for the vertices $\alpha = \mathrm{IN}$, $\beta_1$ and $\gamma_1$. For these three vertices and the vertex OUT, the function $\psi$ is set to be three. The transfer gadgets for $k = 3$ and $k = 4$ can be found in Figure 2. In the rest the transfer gadget is denoted by a bold double arrow joining the vertex IN to the vertex OUT (as in Figure 3). We describe some properties of the transfer gadget essential for our construction in the following propositions:

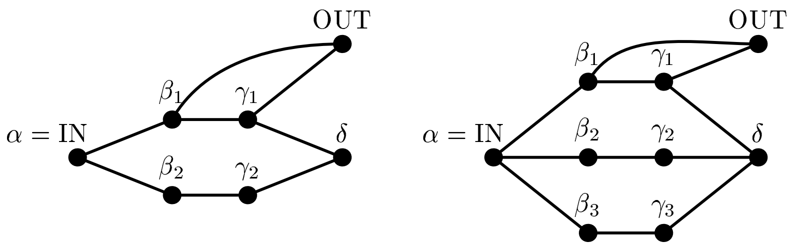


Figure 2: The transfer gadgets for $k = 3$ and $k = 4$.

**Proposition 4** *Let $A$ be an Abelian group of order $k \geq 3$, $\rho \in A$ and $B \subseteq A$ with $|B| = 3$ and $0 \in B$. There exist an orientation, an edge-labeling and a list-assignment $L$ of the transfer gadget with $L(\text{OUT}) = B$ such that if the contact vertex IN is precolored with $\rho$, then the contact vertex OUT is rigid and it must be colored with $0$.*

**Proof:** Let $B = \{0, \rho_1, \rho_2\}$ and let $\rho'$ be an element of $A$ different from both $\rho$ and $\rho_2 - \rho_1$. Let $R$ be a set of $k - 2$ elements of $A$ such that $0 \notin R$. Orient the edges $\alpha\beta_i$ from $\alpha$ to $\beta_i$, $\beta_i\gamma_i$ from $\beta_i$ to $\gamma_i$ and $\gamma_i\delta$ from $\gamma_i$ to $\delta$ for $i = 1, \ldots, k-1$, the edge $\beta_1\text{OUT}$ from $\beta_1$ to OUT and the edge $\gamma_1\text{OUT}$ from $\gamma_1$ to OUT. Next, we define the edge-labeling $\varphi$: The edges $\alpha\beta_2$, $\ldots$, $\alpha\beta_{k-1}$ are assigned $-\rho$ by $\varphi$ and the edges $\beta_1\gamma_1$, $\beta_2\gamma_2$, $\ldots$, $\beta_{k-1}\gamma_{k-1}$ are assigned 0 by $\varphi$. The edges $\gamma_2\delta$, $\ldots$, $\gamma_{k-1}\delta$ are labeled by mutually distinct elements from the set $R$. The labels of the remaining six edges are defined as follows: $\varphi(\alpha\beta_1) = \rho'$, $\varphi(\gamma_1\delta) = -\rho'$ and $\varphi(\beta_1\text{OUT}) = \varphi(\gamma_1\text{OUT}) = \rho_1$. Finally, we define the list-assignment $L$. The lists of all the vertices $\beta_2, \ldots, \beta_{k-1}$ and $\gamma_2, \ldots, \gamma_{k-1}$ are the same and they are equal to $\{0, \rho_1\}$. The lists of the remaining four vertices are the following: $L(\delta) = R \cup \{0\}$, $L(\beta_1) = L(\gamma_1) = \{0, \rho_2 - \rho_1, \rho'\}$ and $L(\text{OUT}) = B = \{0, \rho_1, \rho_2\}$. If the vertex IN$= \alpha$ is precolored with $\rho$, all the vertices $\beta_2, \ldots, \beta_{k-1}$ must be colored with the color $\rho_1$ and the vertices $\gamma_2, \ldots, \gamma_{k-1}$ with the color 0. This forces the vertex $\delta$ to be colored with the color 0 because of the choice of the edge-labeling $\varphi$ for the edges $\gamma_2\delta, \ldots, \gamma_{k-1}\delta$. Then, the colors of the vertices $\beta_1$ and $\gamma_1$ must be 0 and $\rho_2 - \rho_1$ (the color $\rho'$ conflicts with the colors of the vertices $\alpha$ and $\delta$, respectively). The color of the vertex OUT is forced to be neither $\rho_1$ nor $\rho_2$ as desired.

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**Proposition 5** *Let $A$ be an Abelian group of order $k \geq 3$. The following holds for each orientation, each edge-labeling and each list-assignment of the transfer gadget: If the contact vertex OUT is precolored, then the vertex IN is free.*

**Proof:** Assume for the sake of contradiction that the statement is false. Fix an orientation, an edge-labeling $\phi$, a list-assignment $L$ of the transfer gadget and a precoloring of the OUT vertex. Assume that there exist two colors $\rho_1$ and $\rho_2$ in the list $L(\text{IN})$ conflicting with any consistent coloring. We now create a list-assignment $L'$ for the vertices of the copy $P_4^{k-1}$ based on a list-assignment $L$ and the color of the vertex OUT. The lists $L'(\beta_1)$ and $L'(\gamma_1)$ are the lists $L(\beta_1)$ and $L(\gamma_1)$ without colors conflicting with the color of the vertex OUT. The list $L'(v)$ for the remaining vertices $v$ except for $\alpha$ is the same as the list $L'(v)$. The list $L'(\alpha)$ is $\{\rho_1, \rho_2\}$. By our assumption, the graph $P_4^{k-1}$ cannot be colored from the lists $L'$ for the restriction of the orientation and the edge-labeling $\phi$ to it. But this is impossible by Proposition 3.

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**Proposition 6** *Let $A$ be an Abelian group of order $k \geq 3$. The following holds for each orientation, each edge-labeling and each list-assignment of the transfer gadget: If the contact vertex IN is precolored, the vertex OUT is colorable.*

**Proof:** Fix an orientation, an edge-labeling, a list-assignment of the transfer gadget and a precoloring of the vertex IN. Color the remaining vertices in the order $\alpha_2$, $\alpha_3$, $\ldots$, $\alpha_{k-1}$, $\beta_2$, $\beta_3$, $\ldots$, $\beta_{k-1}$, $\delta$, $\alpha_1$, $\beta_1$ and OUT. Observe that each time a vertex of the gadget is to be colored, it always has at least one available color in its list and hence it can be colored.

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**Proposition 7** *Let $A$ be an Abelian group of order $k \geq 3$. Fix an orientation, an edge-labeling and a list-assignment of the transfer gadget. There is at most one color $\rho$ in the list of the vertex IN such that if the vertex IN is precolored with $\rho$, the vertex OUT is rigid. In particular, for any other color $\rho' \neq \rho$ from the list of the vertex IN, if IN is precolored with $\rho'$, the vertex OUT is free.*

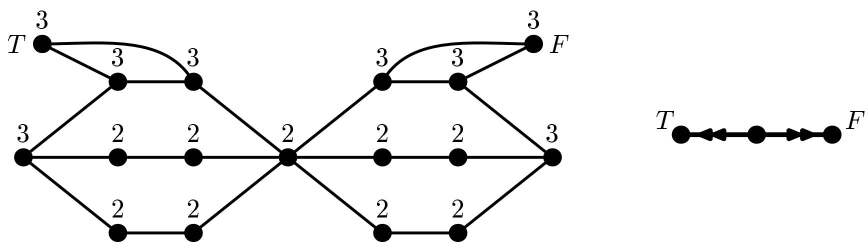


Figure 3: The existential gadget for $k = 4$. The numbers represent sizes of lists and the letters their names. The gadget is in the left part of the figure and its simplified diagram is in the right part.

**Proof:** If the vertex IN is precolored, then the vertex OUT is always colorable by Proposition 6. Assume for the sake of contradiction that there are two different colors $\rho_1$ and $\rho_2$ in the list of the vertex IN such that if the vertex IN is precolored with $\rho_1$ or $\rho_2$, then the vertex OUT is rigid. Let $\sigma_1$, $\sigma_2$, be the colors forced to the vertex OUT if the vertex IN is precolored with the color $\rho_1$, $\rho_2$, respectively, and let $\sigma_3$ be the remaining element of the list of the vertex OUT. If the vertex OUT is precolored with $\sigma_3$, the vertex IN is rigid since it can be colored neither with $\rho_1$ nor with $\rho_2$. But this is impossible by Proposition 5.

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### 3.4 Existential gadget

The *existential gadget* is obtained from two transfer gadgets by identifying their IN vertices. The vertex obtained by identifying these two vertices is called the *middle vertex* of the existential gadget. The list-size-assigning function of the gadgets is modified by assigning the middle vertex two instead of three (the remaining vertices preserve their original sizes of lists). The contact vertices of the existential gadgets are the two OUT vertices of the transfer gadgets and they are denoted by $T$ and $F$. The existential gadget for groups of order $k = 4$ can be found in Figure 3. The following two propositions summarize properties of the existential gadget which will be needed later:

**Proposition 8** *Let $A$ be an Abelian group of order $k \geq 3$. The following holds for each orientation, each edge-labeling, each list-assignment $L$ of the* *existential gadget: There are colors $\alpha_F \in L(F)$ and $\alpha_T \in L(T)$ such that if $F$ is precolored with $\alpha_F$, then the vertex $T$ is free and, on the other hand, if $T$ is precolored with $\alpha_T$, then the vertex $F$ is free.*

**Proof:** Fix an orientation, an edge-labeling and a list-assignment of the existential gadget. Let $\rho_1$ and $\rho_2$ be the colors contained in the list of the middle vertex. Let us first consider the following case: If the middle vertex is precolored with the color $\rho_1$, then both the vertices $T$ and $F$ are free. Let $t_0$ and $f_0$ be colors with which the vertices $T$ and $F$, respectively, can be colored. Set $\alpha_T = t_0$ and $\alpha_F = f_0$. If $T$ is precolored with $\alpha_T = t_0$, then precolor the middle vertex with the color $\rho_1$ and consider a consistent coloring of the transfer gadget containing $T$ which conflicts neither with the color of the middle vertex nor with the color of $T$. The vertex $F$ is now free. A symmetric argument applies when $F$ is precolored with the color $\alpha_F = f_0$.

The next case to consider is the following: If the middle vertex is precolored with the color $\rho_2$, then both the vertices $T$ and $F$ are free with respect to their transfer gadgets. However, this case is analogous to the previous case.

The final case is the following: If the middle vertex is precolored with either of the colors $\rho_1$ and $\rho_2$, then one of the vertices $T$ and $F$ is rigid. By Proposition 7, we may assume that if the middle vertex is precolored with the color $\rho_1$, then the vertex $T$ is rigid and if the middle vertex is precolored with the color $\rho_2$, then the vertex $F$ is rigid. Let $\sigma_1 \in L(T)$, $\sigma_2 \in L(F)$, be the color which the vertex $T$, $F$, is forced to get, if the middle vertex is precolored with $\rho_1$, $\rho_2$, respectively. Set now $\alpha_T = \sigma_1$ and $\alpha_F = \sigma_2$. If $T$ is precolored with $\alpha_T = \sigma_1$, then precolor the middle vertex with the color $\rho_1$ and consider a consistent coloring of the transfer gadget containing $T$ which conflicts neither with the color of the middle vertex nor with the color of $T$. The vertex $F$ is now free. A symmetric argument applies when $F$ is precolored with the color $\alpha_F = \sigma_2$.

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**Proposition 9** *Let $A$ be an Abelian group of order $k \geq 3$. There exist an orientation, an edge-labeling and a list-assignment of the existential gadget such that in every consistent coloring one of the vertices $T$ and $F$ is rigid and forced to be colored with $0$.*

**Proof:** Let $\rho_0$ be any element of $A$ different from 0. Set the list of the middle vertex of the existential gadget to be the set $\{0, \rho_0\}$ and choose

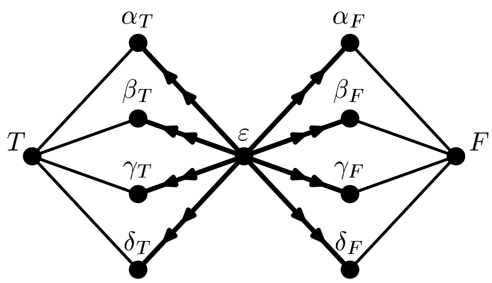


Figure 4: The universal gadget. The letters represent the names of the vertices.

the orientation, the edge-labeling and the list-assignment of each of the two transfer gadget as in Proposition 4 with $\rho = 0$ for the transfer gadget containing the vertex $F$ and with $\rho = \rho_0$ for the transfer gadget containing the vertex $T$. The middle vertex must clearly be colored either with the color 0 or with the color $\rho_0$. In the first case, the vertex $F$ is rigid and it must be colored with 0. In the latter case, the vertex $T$ is rigid and it must be colored with 0.

■

### 3.5 Universal gadget

We construct a so-called *universal gadget* in this subsection. The gadget is obtained by identifying the vertices IN of eight copies of the transfer gadget to a single vertex $\varepsilon$. Let $\alpha_T$, $\beta_T$, $\gamma_T$, $\delta_T$, $\alpha_F$, $\beta_F$, $\gamma_F$ and $\delta_F$ be the contact vertices OUT of the transfer gadgets. Introduce two new vertices $T$ and $F$. Join the vertex $T$ by an edge to each of the vertices $\alpha_T$, $\beta_T$, $\gamma_T$ and $\delta_T$ and the vertex $F$ to each of the vertices $\alpha_F$, $\beta_F$, $\gamma_F$ and $\delta_F$. The contact vertices of the universal gadget are the two vertices $T$ and $F$. Both $T$ and $F$ are assigned three by the list-size-assigning function while the other vertices preserve their original values. The universal gadget is depicted in Figure 4. We now state and prove two properties of the universal gadget needed in the rest:

**Proposition 10** *Let $A$ be an Abelian group of order $k \geq 3$. Fix an orientation, an edge-labeling and a list-assignment of the universal gadget. It is* *possible to precolor the vertex $T$ by a color from its list so that the vertex $F$ is free or it is possible to precolor the vertex $F$ by a color from its list so that the vertex $T$ is free.*

**Proof:** Let $\{\sigma_1, \sigma_2, \sigma_3\}$ be the list of the vertex $\varepsilon$. Precolor the vertex $\varepsilon$ by $\sigma_1$ and let $t_1$ be the number of vertices of $\alpha_T$, $\beta_T$, $\gamma_T$ and $\delta_T$ which are rigid. Similarly, let $f_1$ be the number of vertices of $\alpha_F$, $\beta_F$, $\gamma_F$ and $\delta_F$ which are rigid. Analogously, let $t_2$ and $f_2$ be the number of such rigid vertices if the vertex $\varepsilon$ is precolored with $\sigma_2$ and $t_3$ and $f_3$ if it is precolored with $\sigma_3$. By Proposition 7, we have $t_1 + t_2 + t_3 \leq 4$ and $f_1 + f_2 + f_3 \leq 4$. Since $t_1 + t_2 + t_3 + f_1 + f_2 + f_3 \leq 8$, there exists $i \in \{1, 2, 3\}$ such that $t_i + f_i \leq 2$. Assume that $t_i \leq 2$ and $f_i \leq 1$. The case that $t_i \leq 1$ and $f_i \leq 2$ is symmetric.

Precolor the vertex $\varepsilon$ by $i$. At most two of the vertices $\alpha_T$, $\beta_T$, $\gamma_T$ and $\delta_T$ are rigid with respect to the transfer gadgets, i.e., at most two of them are forced to get a certain color. We may assume that these vertices are $\alpha_T$ are $\beta_T$. Precolor now the vertex $T$ so its color conflicts neither with the color of $\alpha_T$ nor with the color of $\beta_T$. Since the vertices $\gamma_T$ and $\delta_T$ are free by Proposition 7 and the choice of $i$, we can color each them by a color which does not conflict with the color of $T$ and then we can extend this coloring to inner vertices of their transfer gadgets.

We show that vertex $F$ is free (note that the vertex $T$ is now precolored). By the choice of $i$, at most one of the vertices $\alpha_F$, $\beta_F$, $\gamma_F$ and $\delta_F$ is rigid. We may assume that this vertex is the vertex $\alpha_F$. Color the vertex $F$ by any of the two colors which do not conflict with the color of the vertex $\alpha_F$. Since the vertices $\beta_F$, $\gamma_F$ and $\delta_F$ are free by Proposition 7 and the choice of $i$, we can color each of them by a color which does not conflict with the color of $F$ and then we can extend this coloring to inner vertices of their transfer gadgets. Since the vertex $F$ can be colored by (at least two) colors from its list which do not conflict with the color of the vertex $\alpha_F$, the vertex $F$ is free as desired.

■

**Proposition 11** *Let $A$ be an Abelian group of order $k \geq 3$. There exist an orientation, an edge-labeling and a list-assignment of the universal gadget such that the vertex $T$ is rigid and forced to be colored with 0. Similarly, there exist an orientation, an edge-labeling and a list-assignment of the universal gadget such that the vertex $F$ is rigid and forced to be colored with 0.*

**Proof:** We construct an orientation, an edge-labeling and a list-assignment which forces the vertex $T$ to get the color 0. The case of the vertex $F$ is completely symmetric. Let us fix two elements $\sigma_1 \neq \sigma_2$ of $A$ different from 0. First we fix orientations, edge-labelings and list-assignments for transfer gadgets as in Proposition 4 with $\rho = 0$ for the transfer gadgets containing $\alpha_F$, $\beta_F$, $\gamma_F$ and $\delta_F$, with $\rho = \sigma_1$ for the gadgets containing $\alpha_T$ and $\beta_T$ and with $\rho = \sigma_2$ for the gadgets containing $\gamma_T$ and $\delta_T$. The edges not contained in transfer gadgets are oriented to the vertices $T$ and $F$ and the edge-labeling $\varphi$ is extended to these edges as follows:

$$\begin{array}{cccc} \varphi(\alpha_F F) = 0 & \varphi(\beta_F F) = \sigma_1 & \varphi(\gamma_F F) = \sigma_2 & \varphi(\delta_F F) = 0 \\ \varphi(\alpha_T T) = \sigma_1 & \varphi(\beta_T T) = \sigma_2 & \varphi(\gamma_T T) = \sigma_1 & \varphi(\delta_T T) = \sigma_2 \end{array}$$

Finally, the lists of the vertices $\varepsilon$, $T$ and $F$ are set to be $\{0, \sigma_1, \sigma_2\}$.

We now show that the vertex $T$ is forced to get the color 0. If the vertex $\varepsilon$ is colored with 0, then the vertices $\alpha_F$, $\beta_F$ and $\gamma_F$ must be colored with 0 by the choice of the orientations, the edge-labelings and the list-assignments in the corresponding transfer gadgets. But then the vertex $F$ cannot be colored because of the edges $\alpha_F F$, $\beta_F F$ and $\gamma_F F$. Hence, the vertex $\varepsilon$ must be colored with $\sigma_1$ or $\sigma_2$. If the vertex $\varepsilon$ is colored with $\sigma_1$, then the vertices $\alpha_T$ and $\beta_T$ must get the color 0 by the choice the orientations, the edge-labelings and the list-assignments in the corresponding transfer gadgets. Thus, the edges $\alpha_T T$ and $\beta_T T$ force the vertex $T$ to be colored with the color 0. Similarly, if the vertex $\varepsilon$ is colored with $\sigma_2$, then the vertices $\gamma_T$ and $\delta_T$ must get the color 0 and the edges $\gamma_T T$ and $\delta_T T$ force the vertex $T$ to be colored with the color 0. In either of the cases, the vertex $T$ must be colored with the color 0 as desired.

■

## 3.6 Literal gadget

The *$\ell$-literal gadget* is formed by $\ell$ copies of the transfer gadget and $\ell$ new vertices denoted by $\mathrm{OUT}_1, \ldots, \mathrm{OUT}_\ell$. Let $\alpha_i$ and $\beta_i$ be the vertices IN and OUT of the $i$-th copy of the transfer gadget. The $\ell$-literal gadget has $\ell + 1$ contact vertices which are the vertices $\mathrm{OUT}_1, \ldots, \mathrm{OUT}_\ell$ and the vertex $\alpha_1$ denoted further as the vertex IN. In addition to the edges of the transfer gadget, the $\ell$-literal gadget contains edges $\beta_i \alpha_{i+1}$ for $i = 1, \ldots, \ell - 1$ and edges $\beta_i \mathrm{OUT}_i$ for $i = 1, \ldots, \ell$. The list-size-assigning function $\psi$ is modified so that $\psi(\alpha_2) = \ldots = \psi(\alpha_\ell) = 2$ and extended to the new vertices $\mathrm{OUT}_1$,

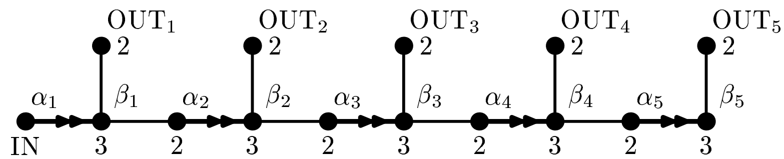


Figure 5: The 5-literal gadget.

$\ldots$, $\mathrm{OUT}_\ell$ so that $\psi(\mathrm{OUT}_1) = \ldots = \psi(\mathrm{OUT}_\ell) = 2$. An example of the $\ell$-literal gadget with list sizes for $\ell = 5$ can be found in Figure 5. Some properties of the $\ell$-literal gadget which are essential for our construction are stated in the following three propositions:

**Proposition 12** *Let $A$ be an Abelian group of order $k \geq 3$ and $\ell$ a positive integer. There exist an orientation, an edge-labeling and a list-assignment of the $\ell$-literal gadget such that if the contact vertex IN is precolored with* 0*, then all contact vertices $OUT_1$, $\ldots$, $OUT_\ell$ must be colored with* 0*.*

**Proof:** Fix orientations, edge-labelings and list-assignments in each of the $\ell$ copies of the transfer gadget as in Proposition 4 for $\rho = 0$. Let $\varphi$ be the resulting partial edge-labeling and $L$ the list-assignment. Next, orient the remaining edges so that there are leading from the vertices $\beta_i$ to $\mathrm{OUT}_i$ for $i = 1, \ldots, \ell$. Define $\varphi(\beta_i \mathrm{OUT}_i) = \sigma$ for $i = 1, \ldots, \ell$ where $\sigma \neq 0$ is an element of $A$. In addition, set $\varphi(\beta_i \alpha_{i+1}) = \sigma$ for $i = 1, \ldots, \ell - 1$. The edge-labeling $\varphi$ is now defined for all edges of the $\ell$-literal gadget. Next, define $L(\alpha_i) = \{0, \sigma\}$ for $i = 2, \ldots, \ell$ and $L(\mathrm{OUT}_i) = \{0, \sigma\}$ for $i = 1, \ldots, \ell$. The list assignment $L$ is now defined for all vertices of the $\ell$-literal gadget, too. In the next paragraph, we show that if the vertex IN is precolored with 0, then all the contact vertices $\mathrm{OUT}_1$, $\ldots$, $\mathrm{OUT}_\ell$ must be colored with 0.

If the vertex IN $= \alpha_1$ is precolored with 0, then the vertex $\beta_1$ must also get the color 0 by the choice the orientation, the edge-labeling and the list-assignment inside the first transfer gadget. Because $\varphi(\beta_1 \alpha_2) = \sigma$ and $L(\alpha_2) = \{0, \sigma\}$, the vertex $\alpha_2$ must get the color 0, too. Again, the choice of the orientation, the edge-labeling and the list-assignment inside the transfer gadget forces the vertex $\beta_2$ to get the color 0 and consequently the vertex $\alpha_3$ to get the color 0. In this way, we derive that all the vertices $\alpha_2, \ldots, \alpha_\ell$ and $\beta_1, \ldots, \beta_\ell$ must get the color 0. Then, none of the contact vertices $\mathrm{OUT}_1$, $\ldots$, $\mathrm{OUT}_\ell$ can be colored with the color $\sigma$ and hence they all must get the color 0.

■

**Proposition 13** *Let $A$ be an Abelian group of order $k \geq 3$ and $\ell$ a positive integer. Fix an orientation, an edge-labeling and a list-assignment of the $\ell$-literal gadget. If the vertices $OUT_1$, …, $OUT_\ell$ are precolored, then the vertex IN is free.*

**Proof:** It is enough to show that it is possible to color all the inner vertices of the $\ell$-literal gadget except for the inner vertices of the first transfer gadget so that there is no conflict. Once we have such a coloring, then the vertex OUT (which is the vertex $\beta_1$) of the first transfer gadget is precolored and by Proposition 6, the vertex IN is free.

Color first the vertex $\beta_\ell$ by a color from its list which does not conflict with the color of the vertex $\mathrm{OUT}_\ell$. If the vertex $\beta_\ell$ is precolored, then the vertex $\alpha_\ell$ would free by Proposition 6 if its list had size three. Since its list has size two, it can be colored with a color from its list so that there is a consistent coloring not conflicting with the colors assigned to the vertices $\alpha_\ell$ and $\beta_\ell$. The vertices $\mathrm{OUT}_{\ell-1}$ and $\alpha_\ell$ forbid at most two colors at the vertex $\beta_{\ell-1}$ and hence it is possible to color the vertex $\beta_{\ell-1}$ with a color from its list. The same argument as applied to the $\ell$-th transfer gadget yields that the vertex $\alpha_{\ell-1}$ can be colored with a color from its list. If we proceed in this way, all the inner vertices of the $\ell$-literal gadget except for the inner vertices of the first transfer gadget are precolored so that there is no conflict.

■

**Proposition 14** *Let $A$ be an Abelian group of order $k \geq 3$ and $\ell$ a positive integer. Fix an orientation, an edge-labeling and a list-assignment of the $\ell$-literal gadget. If the vertex IN is precolored, then there exists a consistent coloring of the gadget.*

**Proof:** By Proposition 6, the first transfer gadget can be colored and this gives a color to the vertex $\beta_1$. Color now the vertex $\alpha_2$ by a color from its list not conflicting with the color of the vertex $\beta_1$. By Proposition 6, we can color the second transfer gadget together with the vertex $\beta_2$. Then, assign color to the vertex $\alpha_3$, color the third transfer gadget and the vertex $\beta_3$, etc. In this way, a consistent coloring of the inner vertices not conflicting with the color of the vertex IN is obtained. Finally, color the contact vertices $\mathrm{OUT}_1$, …, $\mathrm{OUT}_\ell$ by colors from their lists not conflicting with the colors of their neighbors $\beta_1, \ldots, \beta_\ell$.

■

## 3.7 The construction

Finally, we can present the construction of the desired graph $G$:

**Theorem 15** *Let $\Psi$ be a given $\Pi_2$-3CNF-formula and $k \geq 3$ an integer. There exists a graph $G$ with a list-size-assigning function $\psi$ with the following property: If $\Psi$ is true, then $G$ is $A$-$\psi$-choosable for each Abelian group $A$ of order $k$. If $\Psi$ is false, then $G$ is $A$-$\psi$-choosable for no Abelian group $A$ of order $k$. Moreover, the graph $G$ and the list-size-assigning function $\psi$ can be constructed in time polynomial in $k$ and in the size of the formula $\Psi$.*

**Proof:** Let $x_1, \ldots, x_m$ be the variables of $\Psi$ which are universally quantified and $y_1, \ldots, y_n$ the variables which are existentially quantified. Let us consider a variable $x_i$ contained in $\ell_T$ positive and $\ell_F$ negative literals in $\Psi$. The graph $G$ contains a copy of the universal gadget, the $\ell_T$-literal gadget and the $\ell_F$-variable gadget where the contact vertex $T$ of the universal gadget is identified with the contact vertex IN of the $\ell_T$-literal gadget and the contact vertex $F$ of the universal gadget is identified with the contact vertex IN of the $\ell_F$-literal gadget. Each of the contact vertices $\mathrm{OUT}_j$ of the $\ell_T$-literal gadget correspond to one of positive literals containing the variable $x_i$ and each of the contact vertices $\mathrm{OUT}_j$ of the $\ell_F$-literal gadget correspond to one of negative literals containing the variable $x_i$. Similarly, for a variable $y_i$ contained in $\ell_T$ positive and $\ell_F$ negative literals, the graph $G$ contains a copy of the existential gadget, the $\ell_T$-literal gadget and the $\ell_F$-literal gadget where the contact vertex $T$ of the existential gadget is identified with the contact vertex IN of the $\ell_T$-literal gadget and the contact vertex $F$ of the existential gadget is identified with the contact vertex IN of the $\ell_F$-literal gadget. Again, each of the contact vertices $\mathrm{OUT}_j$ of the $\ell_T$-literal gadget correspond to one of positive literals containing the variable $y_i$ and each of the contact vertices $\mathrm{OUT}_j$ of the $\ell_F$-literal gadget correspond to one of negative literals containing the variable $y_i$. Finally, for each clause $C$ of $\Psi$, add a new vertex $w_C$ to the graph $G$ and join it to three vertices corresponding to literals contained in the clause $C$. This finishes the construction of the graph $G$. The list-size-assigning function $\psi$ is the same as the list-size-assigning function for the gadgets used in our construction extended to the clause vertices by assigning all of them the number three.

Let $A$ be an Abelian group of odd order $k \geq 3$. First, we show that if $\Psi$ is true, then $G$ is $A$-$\psi$-choosable. Fix an orientation, the edge-labeling and the list-$\psi$-assignment of the graph $G$. By Proposition 10, for each universal gadget, it is possible to precolor the vertex $T$ of it so the vertex $F$ is free or it is possible to precolor the vertex $F$ so the vertex $T$ is free. In the first case, precolor the vertex $T$ and set the corresponding variable $x_i$ to be false. In the latter case, precolor the vertex $F$ and set the corresponding variable $x_i$ to be true. Since $\Psi$ is true, there exists a truth-assignment for variables $y_i$ such that each clause of $\Psi$ is satisfied. Fix such a truth assignment for variables $y_i$. If $y_i$ is true, precolor the vertex $F$ of the existential gadget corresponding to the variable $y_i$ so that its vertex $T$ is free. If $y_i$ is false, precolor the vertex $T$ so that the vertex $F$ is free. This is always possible by Proposition 8.

If the vertex IN of the literal gadget is now precolored, then extend the coloring to the whole literal gadget and color all its vertices $\mathrm{OUT}_i$. This is possible by Proposition 14. In this way, contact vertices of exactly half of the literal gadgets (those corresponding to false literals) are precolored. We color now the clause vertices. Let us consider a clause $C$ of the formula $\Psi$. The size of the list of the vertex $w_C$ is three and it has at most two precolored neighbors (otherwise all the three literals contained in the clause $C$ are false). Hence, it is possible to color the vertex $w_C$ by a color from its list. Once, the vertex $w_C$ is colored, color also its neighbors corresponding to true literals by colors from their lists (this is possible since each of them has a list of size two and $w_C$ is its only colored neighbor). In this way, all the contact vertices of literal gadgets corresponding to true literals are precolored. By Proposition 13, the vertices IN of these variable gadgets are free and since they are also free in their universal/existential gadgets and their lists have sizes three, they can be colored so that the coloring can be extended to inner vertices of the incident gadgets. In this way, we have obtained a proper coloring of the whole graph $G$.

Next, we show that if $\Psi$ is false, then $G$ is not $A$-$\psi$-choosable. By our assumption, there exists a truth assignment for $x_1, \ldots, x_m$ such that for any truth assignment for $y_1, \ldots, y_n$, there is a clause $C$ of $\Psi$ which is not satisfied. Consider the orientation, the edge-labeling and the list-assignment for existential gadgets, universal gadgets and literal gadgets contained in the graph $G$ as in Propositions 9, 11 and 12. In the case of universal gadgets, choose the orientation, the edge-labeling and the list-assignment so the color of the vertex $T$, $F$, is forced to be 0 if the corresponding variable $x_i$ is false, true, respectively. At this moment, the only non-oriented edges and the edges without a label are the edges incident with the clause vertices. Let $\sigma_1$, $\sigma_2$ and $\sigma_3$ be any three distinct elements of $A$. Orient all these edges in the direction to the clause vertices and extend the edge-labeling $\psi$ so that each clause is incident with exactly one edge labelled with $\sigma_0$, one edge with $\sigma_1$ and one edge with $\sigma_2$. Finally, assign all the clause vertices the same list $\{\sigma_1, \sigma_2, \sigma_3\}$. We show that $G$ cannot be colored with respect to this orientation, this edge-labeling and this list-assignment.

Assume that $G$ can be properly colored and fix such a coloring. By Proposition 9, at least one of the vertices $F$ and $T$ of each existential gadget is colored with 0. Set $y_i$ to be true if the vertex $F$ of the corresponding existential gadget is colored with 0 and set $y_i$ to be false otherwise. Since the formula $\Psi$ is false, there is a clause $C$ for the just constructed truth-assignment which is not satisfied. Observe that all the three vertices $\mathrm{OUT}_j$ from literal gadgets which are joined to $w_C$ must be colored with 0 by Proposition 12 since the vertices IN of those three literal gadgets are colored with 0 (this is immediately implied by the choice of the truth assignment and the fact that $C$ is not satisfied). But then, the vertex $w_C$ cannot be colored from its list because all the three elements $\sigma_1$, $\sigma_2$ and $\sigma_3$ appear on edges incident with it.

■

# 4 Group coloring is hard

In this section, we prove our main results about complexity of group coloring. First, we deal with the case of 3-group colorability to demonstrate our technique. Later we extend our results to groups of larger order:

**Theorem 16** *Let $\Psi$ be a given $\Pi_2$-3CNF-formula. There exists a graph $G_0$ which is $Z_3$-colorable if and only if the formula $\Psi$ is true. The graph $G_0$ can be constructed in time polynomial in the size of the formula $\Psi$. Moreover, the graph $G_0$ is 3-degenerate.*

**Proof:** Let $G$ be the graph and $\psi$ the list-size-assigning function from Theorem 15 for $k = 3$ and the formula $\Psi$. Observe that $\psi(v) \in \{2, 3\}$ for each vertex $v$ of $G$. Add a new vertex $w_0$ to the graph $G$ and join $w_0$ to each vertex $v$ of $G$ with $\psi(v) = 2$. Let $G_0$ be the resulting graph. We claim that $G_0$ is $Z_3$-colorable if and only if the formula $\Psi$ is true.

Assume that $\Psi$ is true. Fix an orientation and an edge-labeling of the graph $G_0$. Color the vertex $w_0$ with the color 0 and consider now the following list-assignment for the graph $G$: If $\psi(v) = 3$ for $v \in V(G)$, then

$L(v) = \{0, 1, 2\}$. If $\psi(v) = 2$, then $L(v)$ consists of the two colors which do not conflict with the color of $w_0$. Observe that $L$ is a list-$\psi$-assignment. Consider now a restriction of the orientation and the edge-labeling of $G_0$ to the graph $G$ together with the list-assignment $L$. The graph $G$ has a proper coloring with respect to this orientation, this edge-labeling and the list-assignment $L$ because it is $Z_3$-$\psi$-choosable by Theorem 15. This proper coloring extended to $w_0$ by assigning the color 0 to it is a proper coloring of the whole graph $G_0$.

Assume now that $\Psi$ is false. Fix an orientation, an edge-labeling and a list-$\psi$-assignment $L$ of $G$ for which $G$ cannot be properly colored (they exist by Theorem 15). Extend the orientation of $G$ to $G_0$ so that each edge incident with $w_0$ is oriented from the vertex $w_0$. The edge-labeling is extended so that the label of the edge $w_0 v$ is $\sigma$ where $\sigma$ is the element of $Z_3$ missing in the list $L(v)$ (note that the sizes of lists of vertices $v$ incident with $w_0$ in $G_0$ are two in $G$ and hence there exists a unique element of $Z_3$ with this property). Assume for the sake of contradiction that $G_0$ can be colored with respect to this orientation and this edge-labeling and fix such a coloring $c$. We may assume that $c(w_0) = 0$ by Lemma 1. Then, the coloring $c$ restricted to the graph $G$ is a proper coloring of $G$ such that $c(v) \in L(v)$ because at each vertex $v$ with $L(v) \neq Z_3$ the color conflicting with the color of the vertex $w_0$ is exactly the color missing in the list $L(v)$. But there is no proper coloring of $G$ from the lists $L$ — a contradiction.

In order to complete the proof, we have to show that $G_0$ is 3-degenerate. Observe first that degrees of all inner vertices (the vertices $\beta_1$, $\beta_2$, $\gamma_1$, $\gamma_2$ and $\delta$ in Figure 2) of transfer gadgets in $G_0$ are three. Hence, remove all these vertices. Now, all the remaining vertices except for $w_0$ has degree at most three. Thus, $G_0$ is indeed 3-degenerate.

■

Before considering groups of larger order, we prove a simple lemma about coloring of a certain special graph. This graph is then used in Theorem 18 in the construction of the sought graph $G_0$. The graph $K_n^l$ for $n \geq 2$ and $l \geq 1$ is the graph obtained in the following way: Consider $l$ copies of the complete graph $K_n$ of order $n$ and choose in each copy of the complete graph a single special vertex. Add to the graph a new vertex, called the *center vertex* of $K_n^l$ in the rest, and join it to all the vertices of the complete graphs except for the special ones. This finishes the construction of $K_n^l$. The graph $K_4^5$ can be found in Figure 6.

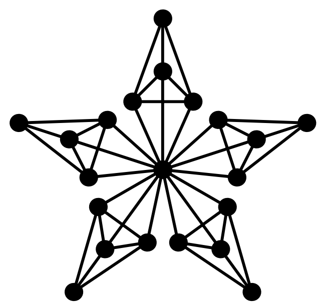

Figure 6: The graph $K_4^5$.

**Lemma 17** *Let $n \geq 2$ and $l \geq 1$ be fixed integers. The group chromatic number of a graph $K_n^l$ is $n$. If $A$ is an Abelian group of order $n$, there exist an orientation and an edge-labeling such that in each proper coloring, the center vertex and all the special vertices get the same color.*

**Proof:** Since the graph $K_n^l$ contains a clique of order $n$, we have $\chi_g(K_n^l) \geq n$. On the other hand, the graph $K_n^l$ is $(n-1)$-degenerate and hence $\chi_g(K_n^l) \leq n$. In order to prove the second part of the statement of the lemma, consider any orientation of $K_n^l$ and assign all the edges 0. Clearly, in each proper $A$-coloring of $K_n^l$, the center vertex and all the special vertices get the same color.

■

**Theorem 18** *Let $k \geq 3$ be a fixed integer and $\Psi$ a $\Pi_2$-3CNF-formula. There exists a $k$-degenerate graph $G_0$ with the following property: If $\Psi$ is true, then $G_0$ is $A$-colorable for each Abelian group of order $k$ and if $\Psi$ is false, then $G_0$ is $A$-colorable for no Abelian group of order $k$. In addition, the graph $G_0$ can be constructed in time polynomial in $k$ and in the size of the formula $\Psi$.*

**Proof:** Let $G$ be the graph and $\psi$ the list-size-assigning function from Theorem 15 for $k$ and the formula $\Psi$. Note that $\psi(v) \in \{2, \ldots, \max\{3, k-1\}\}$ for each vertex $v$ of $G$. Add a copy of the graph $K_k^{k-2}$ to $G$ and let $w_1, \ldots, w_{k-2}$ be the $k-2$ special vertices of $K_k^{k-2}$. If $\psi(v) = 2$ for $v \in V(G)$, then join the vertex $v$ to the vertices $w_1, \ldots, w_{k-2}$, if $\psi(v) = 3$, then join the vertex $v$ to the vertices $w_1, \ldots, w_{k-3}$, etc. Let $G_0$ be the resulting graph.

We claim that $G_0$ has all the properties from the statement of the lemma. In particular, if $\Psi$ is true, then $G_0$ is $A$-colorable for each Abelian group of order $k$ and if $\Psi$ is false, then $G_0$ is $A$-colorable for no Abelian group of order $k$. Fix an Abelian group $A$ of order $k$ for the rest of the proof.

Assume first that $\Psi$ is true. Fix an orientation and an edge-labeling of the graph $G_0$. Since the group chromatic of $K_k^{k-2}$ is $k$, it is possible to color all vertices of $K_k^{k-2}$. Consider now the following list-assignment for the graph $G$: The list $L(v) \subseteq A$ of a vertex $v$ of $G$ consists of those elements of $A$ which conflict with a color of no neighbor of $v$ among the special vertices $w_1, \ldots, w_{k-2}$. Observe that $|L(v)| \geq \psi(v)$ for every $v \in V(G)$. Consider now a restriction of the orientation and the edge-labeling of $G_0$ to the graph $G$ together with the list-assignment $L$. The graph $G$ has a proper coloring with respect to this orientation, this edge-labeling and the list-assignment $L$ because it is $A$-$\psi$-choosable by Theorem 15. This proper coloring and the proper coloring of the copy of $K_k^{k-2}$ form a proper coloring of the graph $G_0$.

Assume now that $\Psi$ is false. Fix an orientation, an edge-labeling and a list-$\psi$-assignment $L$ of $G$ for which $G$ cannot be properly colored (they exist by Theorem 15). Next, fix an orientation and an edge-labeling of $K_k^{k-2}$ such that in any proper coloring all the special vertices get the same color (such an orientation and an edge-labeling exist by Lemma 17). Extend the orientation of $G$ to $G_0$ so that each edge incident with a special vertex is oriented from the special vertex to its neighbor in $G$. The edge-labeling is extended so that the labels of the edges $w_1v, \ldots, w_{k-|L(v)|}v$ are precisely those elements of $A$ missing in the list $L(v)$. Assume for the sake of contradiction that $G_0$ can be colored with respect to this orientation and this edge-labeling and fix such a coloring $c$. We may assume that $c(w_1) = 0$ by Lemma 1. Hence, we have $c(w_1) = \ldots = c(w_{k-2}) = 0$ by the choice of orientation inside $K_k^{k-2}$. Then, the coloring $c$ restricted to the graph $G$ is a proper coloring of $G$ such that $c(v) \in L(v)$ because at each vertex $v$ the colors conflicting with the colors of the vertices $w_1, \ldots, w_{k-2}$ are exactly the colors missing in the list $L(v)$. But there is no proper coloring of $G$ from the lists $L$ — a contradiction.

In order to complete the proof, we have to show that $G_0$ is $k$-degenerate. Observe first that degrees of all inner vertices of transfer gadgets in $G_0$ are $k$. Remove all these vertices. Now, all the remaining vertices except for the vertices of the copy $K_k^{k-2}$ has degree at most $k$. Thus, we can remove these vertices. A copy of the graph $K_k^{k-2}$ remains. Since the graph $K_k^{k-2}$

is $k$-degenerate, we can conclude that the graph $G_0$ is $k$-degenerate.

■

We are now ready to prove the two main theorems of this paper:

**Theorem 19** *The decision problem whether the group chromatic number of a given graph $G$ is at most $k$ is $\Pi_2^P$-complete for each integer $k \geq 3$.*

**Proof:** Fix an integer $k \geq 3$. Let $\Psi$ be a given $\Pi_2$-3CNF-formula. Consider the graph $G_0$ from Theorem 16 if $k = 3$ and from Theorem 18 if $k \geq 4$. Note that $\chi_g(G_0) \leq k+1$ because $G_0$ is $k$-degenerate. Observe that the graph $\chi_g(G_0) \leq k$ if and only if $\Psi$ is true by Theorems 16 and 18, respectively. Since $G_0$ can be constructed in time polynomial in the size of $\Psi$, the decision problem whether the group chromatic number of a given graph $G$ is at most $k$ is $\Pi_2^P$-complete.

■

**Theorem 20** *The decision problem whether a given graph $G$ is $A$-colorable for a fixed Abelian group $A$ of order $k \geq 3$ is $\Pi_2^P$-complete.*

**Proof:** Fix an Abelian group $A$ of order $k \geq 3$. Consider the graph $G_0$ from Theorem 16 if $k = 3$ and from Theorem 18 if $k \geq 4$. By Theorems 16 and 18, respectively, the graph $G_0$ is $A$-colorable if and only if $\Psi$ is true. Since $G_0$ can be constructed in time polynomial in the size of $\Psi$, the decision problem whether a given graph $G$ is $A$-colorable is $\Pi_2^P$-complete.

■

# 5 Conclusion

We have investigated the complexity of group coloring. Some of our results can be extended to $A$-$\ell$-choosability. The case of $A$-2-choosability is easy since the following holds: A graph $G$ is $A$-2-choosable for a group of even order iff $G$ is forest. A graph $G$ is $A$-2-choosable for a group of odd order iff the core of $G$ (defined as in [3]) is a union of vertex-disjoint even cycles. Hence, it is possible to decide in polynomial time whether a given graph $G$ is $A$-2-choosable for each Abelian group $A$. It can be shown that graph $G_0$ from Theorem 16 has the following two properties:

- If $\Psi$ is true, then the graph $G_0$ is $A$-3-choosable for every group $A$ of odd order.
- If $\Psi$ is false, then the graph $G_0$ is $A$-3-choosable for no group $A$ of odd order.

Hence, the problem to decide whether a given graph is $A$-$\ell$-choosable for an Abelian group $A$ of odd order $k \geq 3$ and $3 \leq \ell \leq k$ is $\Pi_2^P$-complete. The constructed list-size-assigning function $\psi$ for groups of order $k$ in Theorem 15 has values $2, \ldots, \max\{3, k-1\}$. A proof similar to the proof of Theorem 18 yields that the problem to decide whether a given graph is $A$-$\ell$-choosable is $\Pi_2^P$-complete for every Abelian group $A$ of order $k \geq 4$ with $\ell = k-1, k$. We do not have the proof of $\Pi_2^P$-completeness of $A$-$\ell$-choosability for groups $A$ of even order $k \geq 6$ and for $\ell$, $3 \leq \ell \leq k-2$. We conjecture that the problem is $\Pi_2^P$-complete even in these cases:

**Conjecture 21** *For each Abelian group $A$ of order $k \geq 3$ and each integer $\ell$, $3 \leq \ell \leq k$, the problem to decide whether a given graph is $A$-$k$-choosable is $\Pi_2^P$-complete.*

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