A Brooks-type Theorem for the Generalized List T-Coloring

Jiří Fiala[a] Daniel Král'[a] Riste Škrekovski[a,b,*]

[a] Department of Applied Mathematics and
Institute for Theoretical Computer Science (ITI)[1]
Charles University,
Malostranské náměstí 25, 118 00, Prague, Czech Republic
{fiala,kral,skreko}@kam.mff.cuni.cz

[b] Department of Mathematics,
University of Ljubljana,
Jadranska 19, 1111 Ljubljana, Slovenia

Abstract

We study the notion of a generalized list T-coloring which is a common generalization of the channel assignment problem and the T-coloring. An instance of the generalized list T-coloring is described by a triple (G, Λ, t) where G is a graph, Λ is a mapping which assigns the vertices of G lists of numbers (colors) and t is a mapping which assigns each edge of G a set of forbidden differences. We require that $0 \in t(e)$ for each edge e of G. The goal is to find a labeling c of the vertices of G with $c(v) \in \Lambda(v)$ for each vertex v, and $|c(u) - c(v)| \notin t(uv)$ for each edge uv of G. An instance is balanced if the size of the list $\Lambda(v)$ for each vertex v is equal to the sum of the sizes of $t(e)$ for edges e incident with v.

We state and prove a Brooks-type theorem for the generalized list T-coloring problem. This generalizes and unifies the previously known Brooks-type theorems for the channel assignment problem and for the T-coloring. The theorem characterizes balanced instances of the generalized list T-coloring with a good labeling. As a consequence, if G is a connected graph different from a Gallai tree, then all balanced instances on G have good labelings.

*This research was partially done while the author was a postdoctoral fellow at Pacific Institute for the Mathematical Sciences (PIMS) at Simon Fraser University, Burnaby, B.C., Canada.

[1]Institute for Theoretical Computer Science is supported by Ministry of Education of Czech Republic as project LN00A056.

1 Introduction

In this paper, we study a common generalization of the graph coloring, the list coloring, the T-coloring and the (list) channel assignment problem. We call this coloring problem the *generalized list T-coloring*. Our approach unifies several previously known Brooks-type results, in particular [4, 10, 14, 22]. This coloring problem was suggested by Hale [8] under the name of "frequency constrained channel assignment problem" as a model for assigning frequencies to radio transmitters. The generalized list T-coloring is a more flexible model for problems from practice compared to the channel assignment problem because, in addition, you may prevent every two transmitters from assigning frequencies which have certain special differences. E.g., the choice $T = \{0, 7, 14, 15\}$ gives a model for interferences for the UHF standard television transmitters [13].

An instance of the *generalized list T-coloring* is described as a triple (G, Λ, t) where G is a graph, $\Lambda : V(G) \to 2^{\mathbb{N}}$ and $t : E(G) \to 2^{\mathbb{N}}$ where $\mathbb{N}$ is the set of all non-negative integers. We write $V(G)$ and $E(G)$ for the vertex set and the edge set of a graph G, respectively. Elements of the sets $\Lambda(v)$ are called *colors* and elements of the set $t(e)$ are called *forbidden differences* for the edge e. The function t must satisfy $0 \in t(e)$ for each edge $e \in E(G)$ (this condition is more essential than it might seem at the first sight, see our concluding remarks in Section 6, in particular, an example given in Proposition 1). An instance of the generalized list T-coloring is also called a *generalized list T-coloring problem*. The goal of the problem is to find a mapping $c : V(G) \to \mathbb{N}$ with $c(v) \in \Lambda(v)$ for each $v \in V(G)$ and $|c(x) - c(y)| \notin t(xy)$ for each edge $xy \in E(G)$. Such a mapping is called a *good labeling*.

The *t-degree* $\deg_t(v)$ of a vertex v is equal to the sum $\sum_{vw \in E(G)} |t(vw)|$. An instance (G, Λ, t) is called *balanced* if $|\Lambda(v)| = \deg_t(v)$ holds for each vertex $v \in V(G)$. The problem is called *overbalanced* if $|\Lambda(v)| \geq \deg_t(v)$ holds for each $v \in V(G)$ and at least at one vertex the inequality is strict. The main result of this paper can be summarized as follows: Each overbalanced instance of the generalized list T-coloring problem allows a good labeling (Theorem 1) and we completely describe all balanced instances with no good labelings (Theorem 5). In particular, we show that if G is a connected graph distinct from a Gallai tree, then all balanced instances (G, Λ, t) allow a good labeling.

We now explain how our results directly translate to the other coloring concepts mentioned above:

• **The graph coloring**

Instances of the generalized list T-coloring problem, where Λ is the function constantly equal to $\{1, \ldots, k\}$ for some integer k and t is the function constantly equal to $\{0\}$ at every edge, is just the usual graph k-coloring. Theorem 1 translates to the well-known inequality $\chi(G) \leq \Delta(G) + 1$ and Theorem 5 to Brooks' theorem [4]. An elegant short proof of Brooks' theorem was given by Lovász [14]. An extension of Brooks' theorem to hypergraphs can be found in [9].

• **The list coloring**

The list coloring is a variant of the graph coloring where each vertex has to be assigned a color from its list [12, 20]. In our setting, Λ is just the function assigning lists of colors to the vertices and t is again a constant mapping equal to $\{0\}$ at every edge. Theorem 1 translates to the claim that each graph G is $(\Delta+1)$-choosable and Theorem 5 coincides with Brooks-type theorems for choosability and the list coloring from [2, 3, 6, 21]. Another theorem for the list coloring in the spirit of Brooks' theorem can be found in [11].

• **The T-coloring and the list T-coloring**

In the T-coloring, the goal is to assign numbers (colors) to the vertices of a graph in such a way that the difference between the numbers assigned to two adjacent vertices does not belong to a certain fixed set of integers T (the set of forbidden differences), see [1, 13, 18, 23]. It is required that $0 \in T$. This condition assures that each T-coloring is also a coloring in the usual sense. In the list T-coloring, each vertex is in addition equipped with a list of available numbers (colors) and the assigned color must belong to the prescribed list. The generalized list T-coloring restricts to the T-coloring and the list T-coloring when the function t is a constant function equal to the set T. Our Theorem 5 for such a function t is just the Brooks-type theorem for the list T-coloring proved by Waller [22]: a 2-connected graph G is not $(|T| \cdot \Delta(G))$-T-choosable if and only if T is an arithmetic set and G is either a complete graph or an odd cycle. A set A of integers is called an *arithmetic set with a difference* d if $A = \{0, d, 2d, \ldots, (k-1)d\}$ for some integer k. Note that a set $\{0\}$ is arithmetic for all possible differences.

• **The (list) channel assignment problem**

Instance of the channel assignment problem are graphs with edges labeled by positive integers. The numbers assigned to adjacent vertices must differ by at least the weight of the edge joining them [15]. The notion of the channel assignment problem also includes a so-called $L(p, q)$-labeling problem in which numbers assigned to adjacent vertices must differ by at least p and numbers assigned to vertices at distance two by at least q, see [5, 7, 19]. Theorem 1 translates to a counterpart of the equality "$\chi \leq \Delta + 1$" for the channel assignment problem proven by McDiarmid [16] and Theorem 5 extends the Brooks-type theorem for the list channel assignment problem from [10].

2 Preliminaries

We write $A \uplus B$ for the union of disjoint sets A and B; this notation is used only to emphasize that the sets A and B are disjoint. Arithmetic sets are often considered in the paper, so we define $\mathrm{Ar}_d(k) = \{0, d, 2d, \ldots, d(k-1)\}$. For a set A of integer and an integer k_0, let $A+k_0$ denote the set $\{k+k_0 \mid k \in A\}$. If convenient, we use k_0+A instead of $A+k_0$. Similarly, $A-k_0$ denotes the set $\{k-k_0 \mid k \in A\}$ and k_0-A denotes the set $\{k_0-k \mid k \in A\}$.

Let (G, Λ, t) be a generalized list T-coloring problem, v a vertex of a graph G and α an element of $\Lambda(v)$. We say that the problem $(G', \Lambda', t') = (G, \Lambda, t)[v \to \alpha]$ is *obtained from the problem (G, Λ, t) by assigning the color α to the vertex v*. Formally, (G', Λ', t') is the following problem:

- $G' = G \setminus v$ is the subgraph of G induced by the vertex set $V(G) \setminus \{v\}$, i.e., $V(G') = V(G) \setminus \{v\}$ and $E(G') = \{ww' \mid ww' \in E(G) \ \& \ w, w' \in V(G')\}$.
- For each vertex w of G', the list $\Lambda'(w)$ is a subset of $\Lambda(w)$ consisting of the colors which do not conflict with the color assigning to the vertex v. Formally, $\Lambda'(w) = \{k \mid k \in \Lambda(w) \ \& \ |k-\alpha| \in t(vw)\}$.
- The function t' is the restriction of the function t to $E(G')$, i.e., $t'(e) = t(e)$ for all $e \in E(G')$.

Clearly, the problem $(G', \Lambda', t') = (G, \Lambda, t)[v \to \alpha]$ has a good labeling if and only if the original problem (G, Λ, t) has a good labeling c with $c(v) = \alpha$. For a generalized list T-coloring problem (G, Λ, t), let $\Lambda_{\min}$ and $\Lambda_{\max}$ denote the minimal and the maximal colors, respectively, contained in the union $\bigcup_{v \in V(G)} \Lambda(v)$ of all lists.

The following lemma illustrates the just introduced notation:

Lemma 1 *Let (G, Λ, t) be a balanced generalized list T-coloring problem, let α be either $\Lambda_{\min}$ or $\Lambda_{\max}$ and let v be an arbitrary vertex of G with $\alpha \in \Lambda(v)$. Then, the problem $(G, \Lambda, t)[v \to \alpha]$ is balanced or overbalanced. In particular, if there is a neighbor v' of v with that $\alpha \notin \Lambda(v')$, then $(G, \Lambda, t)[v \to \alpha]$ is overbalanced.*

Proof: By symmetry, it is enough to prove the lemma for $\alpha = \Lambda_{\min}$. The assignment of the color $\Lambda_{\min}$ to the vertex v reduces the size of the list $\Lambda(v')$ of each neighbor v' of the vertex v by at most $|t(vv')|$. Namely, only the elements of the set $t(vv') + \Lambda_{\min}$ can be removed. Observe that the t-degree of v' in $(G, \Lambda, t)[v \to \alpha]$ is $\deg_t(v') - |t(vv')|$. Thus, if (G, Λ, t) is balanced and $t(vv') + \Lambda_{\min} \subseteq \Lambda(v')$ for each neighbor v' of v, then the new problem is balanced, too. If the latter condition is not satisfied for some neighbor v' of v, then the size of the list of v' is decreased by at most $|t(vv')| - 1$ and thus the new problem is overbalanced. In particular, this happens if $\Lambda_{\min} \notin \Lambda(v')$.

∎

3 The counterpart of the inequality $\chi \leq \Delta + 1$

In this section, we prove the counterpart of the well-known graph inequality $\chi \leq \Delta + 1$:

Theorem 1 *An overbalanced generalized list T-coloring problem (G, Λ, t) has a good labeling whenever G is a connected graph.*

Proof: The proof is by induction on the number of vertices of G. If $|V(G)| = 1$, then $\Lambda(v) \neq \emptyset$ for the single vertex v of G, and hence (G, Λ, t) has a good labeling. Assume in the rest that $|V(G)| \geq 2$. Let $V_{\min}$ be the set of vertices v of G such that $\Lambda_{\min} \in \Lambda(v)$, and let v_0 be a vertex of G with $\deg_t(v_0) < |\Lambda(v_0)|$. In the proof, we distinguish three cases with respect to the vertex v_0 and the set $V_{\min}$.

If $V_{\min}$ contains a vertex v which is not a cut-vertex of G and $v \neq v_0$, then assign the color $\Lambda_{\min}$ to the vertex v in order to obtain an overbalanced problem $(G', \Lambda', t') = (G, \Lambda, t)[v \to \Lambda_{\min}]$. Since G' is connected, the problem (G', Λ', t') has a good labeling by the induction hypothesis. Hence, the problem (G, Λ, t) has a good labeling, too.

If $|V_{\min}| \geq 2$, it can be easily seen that $V_{\min}$ contains a cut vertex v, $v \neq v_0$, with the following property: If K is the component of $G \setminus v$ which contains the vertex v_0, then each component of $G \setminus v$ distinct from K contains no vertices of $V_{\min}$. Consider now the problem (G', Λ', t') obtained by assigning the color $\Lambda_{\min}$ to the vertex v. The problem (G', Λ', t') restricted to the component K is overbalanced because it contains the vertex v_0. The corresponding problems obtained by restricting to the other components are also overbalanced; each of the other components contains a neighbor of v whose list $\Lambda(v)$ does not contain the color $\Lambda_{\min}$. Each of these restricted problems is overbalanced with the underlying graph being connected. Hence they all have good labelings and thus the generalized list T-coloring problem (G, Λ, t) has also a good labeling.

The remaining case is $V_{\min} = \{v_0\}$. Consider now the problem (G', Λ', t') obtained by assigning the color $\Lambda_{\min}$ to v_0 and its restrictions to all the components of $G \setminus v_0$. Each of these restrictions is overbalanced because v_0 is the only vertex whose list contains the color $\Lambda_{\min}$. By the induction hypothesis, all of them have good labelings, and thus the problem (G, Λ, t) has a good labeling, too.

■

4 The case of 2-connected graphs

In this section, we characterize for 2-connected graphs G balanced generalized list T-coloring problems (G, Λ, t) which have no good labelings. These results are then used in Section 5 where a characterization of all balanced generalized list T-coloring problems with no good labeling is presented.

Lemma 2 *Let (G, Λ, t) be a balanced generalized list T-coloring problem such that G is 2-connected. If there is a vertex v such that $\Lambda_{\min} \notin \Lambda(v)$ or $\Lambda_{\max} \notin \Lambda(v)$, then the problem (G, Λ, t) has a good labeling.*

Proof: By symmetry, it is enough to prove the lemma for the case that $\Lambda_{\min}$ is not contained in all lists. In such case, since G is connected, there must be adjacent vertices v and w such that $\Lambda_{\min} \notin \Lambda(v)$ and $\Lambda_{\min} \in \Lambda(w)$. The problem $(G, \Lambda, t)[w \to \Lambda_{\min}]$ is overbalanced by Lemma 1. And since $G \setminus w$ is a connected graph, it follows from Theorem 1 that $(G, \Lambda, t)[w \to \Lambda_{\min}]$ has a good labeling. Thus, the original problem (G, Λ, t) must have a good labeling, too.

■

The following well-known lemma can be found in [17, Lemma 1.15]:

Lemma 3 *Every 2-connected graph G, which is neither a cycle nor a complete graph, contains three vertices x, y and z such that x and y are neighbors of z, the vertices x and y are non-adjacent, and $G \setminus \{x, y\}$ is a connected graph.*

Lemma 3 allows us to concentrate to the problems where G is either an odd cycle or a complete graph. The next lemma deals with balanced generalized list T-coloring problems whose underlying graphs are 2-connected but they are neither odd cycles nor complete graphs. The cases of odd cycles and complete graphs are later considered in separate subsections.

Lemma 4 *If a balanced generalized list T-coloring problem (G, Λ, t) does not have a good labeling and G is 2-connected, then G is either an odd cycle or a complete graph.*

Proof: Let us first consider the case that G is an even cycle. By Lemma 2, the color $\Lambda_{\min}$ is contained in the list $\Lambda(v)$ for every vertex $v \in V(G)$. Let $v_1, \ldots, v_n$ be the vertices of the cycle G enumerated in a cyclic order. Let k_i be a color of $\Lambda(v_i) \setminus ((t(v_{i-1}v_i) + \Lambda_{\min}) \cup (t(v_i v_{i+1}) + \Lambda_{\min}))$ for each $1 \leq i \leq n$. Since the problem is balanced (recall that $0 \in t(v_{i-1}v_i) \cap t(v_i v_{i+1})$), such a number k_i always exists. Then, we can define a good labeling c as follows:

$$c(v_i) = \begin{cases} \Lambda_{\min} & \text{if } i \text{ is odd,} \\ k_i & \text{otherwise.} \end{cases}$$

The remaining case is that the graph G is neither a complete graph nor a cycle. Let x, y and z be vertices of G with the properties as described in the statement of Lemma 3. Recall that the color $\Lambda_{\min}$ is contained in the list $\Lambda(v)$ of every vertex $v \in V(G)$. Consider now the problem (G', Λ', t') obtained from (G, Λ, t) by assigning the color $\Lambda_{\min}$ to the vertices x and y. By Lemma 1, the problem $(G, \Lambda, t)[x \to \Lambda_{\min}]$ is balanced and the color

$\Lambda_{\min}$ for $(G,\Lambda,t)[x\to\Lambda_{\min}]$ is not contained in the list of z. Note that z is a neighbor of y. Hence the problem $(G',\Lambda',t') = ((G,\Lambda,t)[x\to\Lambda_{\min}])[y\to\Lambda_{\min}]$ is overbalanced by Lemma 1. The problem (G',Λ',t') has a good labeling by Theorem 1, and thus the original problem (G,Λ,t) has a good labeling, too.

■

The following lemma is a corollary of the Brooks-type theorem for the T-coloring proved by Waller [22, Lemma 7]. We provide here its complete proof for the sake of completeness:

Lemma 5 *Let (K_2,Λ,t) be a balanced generalized list T-coloring problem with $K_2 = uv$. Then, (K_2,Λ,t) admits no good labeling if and only if $t(uv)$ is arithmetic and $\Lambda(u) = \Lambda(v) = \Lambda_{\min} + t(uv)$.*

Proof: By Lemma 2, it holds $\Lambda_{\min} \in L(u)$ and $\Lambda_{\min} \in L(v)$. If $\Lambda(u) \neq \Lambda_{\min} + t(uv)$, then there is a good labeling which assigns $\Lambda_{\min}$ to v and a color of $\Lambda(u) \setminus (\Lambda_{\min} + t(uv))$ to u. Hence, $\Lambda(u) = \Lambda_{\min} + t(uv)$ and similarly $\Lambda(v) = \Lambda_{\min} + t(uv)$.

If $t(uv)$ is not arithmetic, then K_2 has a good labeling from any pair of lists of size $|t(uv)|$: Let $0 = i_1 < i_2 < \cdots < i_k$ be the elements of $t(uv) = \Lambda(u) - \Lambda_{\min} = \Lambda(v) - \Lambda_{\min}$ and let k_0 be the largest index such that the set $\{i_1, \ldots, i_{k_0}\}$ is arithmetic. Since $t(uv)$ is not arithmetic, we have $2 \le k_0 < k$. Observe now that $i_{k_0+1} - i_2 \notin t(uv)$ by the choice of k_0. But, then the labeling c defined as $c(u) = \Lambda_{\min} + i_2$ and $c(v) = \Lambda_{\min} + i_{k_0+1}$ is good.

■

4.1 The case of odd cycles

Throughout this subsection, we consider cycles C_n of odd length n. The vertices of a cycle C_n are denoted by $v_1, \ldots, v_n$. In the next lemma, we study a possible structure of sets $t(e)$ in balanced generalized list T-coloring problems (C_n,Λ,t) with no good labelings.

Lemma 6 *Let (C_n,Λ,t) be a balanced generalized list T-coloring problem such that for some edge e of an odd cycle C_n, the set $t(e)$ is not arithmetic or $\Lambda_{\max} - \Lambda_{\min} \in t(e)$. Then, the problem (C_n,Λ,t) has a good labeling.*

Proof: We may assume that the colors $\Lambda_{\min}$ and $\Lambda_{\max}$ are contained in all the lists $\Lambda(v)$, $v \in V(C_n)$ by Lemma 2. Assume that the edge e from the statement of the lemma is the edge v_1v_2. We first define a sought good labeling c for vertices $v_3, \ldots, v_n$ as follows:

$$\begin{array}{lcll} c(v_i) & = & \Lambda_{\min} & \text{for } i = 3, 5, \ldots, n \text{ and} \\ c(v_i) & \in & \Lambda(v_i) \setminus ((\Lambda_{\min} + t(v_{i-1}v_i)) \cup (\Lambda_{\min} + t(v_{i+1}v_i))) & \text{for } i = 4, 6, \ldots, n-1. \end{array}$$

Note that the set $\Lambda(v_i) \setminus ((\Lambda_{\min} + t(v_{i-1}v_i)) \cup (\Lambda_{\min} + t(v_{i+1}v_i)))$ is non-empty for each $i = 4, 6, \ldots, n-1$ because the problem (C_n, Λ, t) is balanced and $0 \in t(v_{i-1}v_i) \cap t(v_{i+1}v_i)$.

Consider now the problem (G', Λ', t') obtained by assigning the color $c(v_i)$ to every vertex v_i for $i = 3, \ldots, n$. Note that G' is isomorphic to K_2 (it is just the edge v_1v_2) and the problem (G', Λ', t') is balanced (follow the proof of Lemma 2). If $t(e)$ is not arithmetic, then the problem (G', Λ', t') has a good labeling by Lemma 5. Otherwise, $\Lambda_{\max} - \Lambda_{\min} \in t(e) = t'(e)$ by the assumption of the lemma. Since the vertices v_3 and v_n are colored with $\Lambda_{\min}$, the colors contained in the list $\Lambda'(v_1)$ and $\Lambda'(v_2)$ are integers between $\Lambda_{\min} + 1$ and $\Lambda_{\max}$. Hence, the problem (G', Λ', t') has a good labeling by Lemma 5 in this case, too. Thus, the original problem (G, Λ, t) has a good labeling in both the cases.

■

The following lemma relates contents of lists $\Lambda(v)$ to sets $t(e)$ for balanced generalized list T-coloring problems (C_n, Λ, t) with no good labelings:

Lemma 7 *Let (C_n, Λ, t) be a balanced generalized list T-coloring problem with no good labeling. Then, the following equalities hold for each i, $1 \le i \le n$ (indices are taken modulo n):*

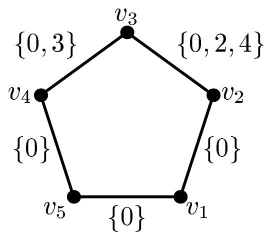
$$\begin{aligned} \Lambda(v_i) &= (\Lambda_{\min} + t(v_{i-1}v_i)) \uplus (\Lambda_{\max} - t(v_iv_{i+1})) && (1) \\ &= (\Lambda_{\min} + t(v_iv_{i+1})) \uplus (\Lambda_{\max} - t(v_{i-1}v_i)). && (2) \end{aligned}$$

Proof: By Lemma 2, each list $\Lambda(v_i)$ contains both the colors $\Lambda_{\min}$ and $\Lambda_{\max}$. In addition, by Lemma 6, there is no edge $e = v_iv_{i+1}$ with $\Lambda_{\max} - \Lambda_{\min} \in t(e)$. Suppose that there is a vertex v_i whose list $L(v_i)$ does not satisfy the equality of (1), i.e., the sets $\Lambda_{\min} + t(v_{i-1}v_i)$ and $\Lambda_{\max} - t(v_iv_{i+1})$ are not disjoint or $\Lambda(v_i) \neq (\Lambda_{\min} + t(v_{i-1}v_i)) \uplus (\Lambda_{\max} - t(v_iv_{i+1}))$. Since the problem is balanced, the size of the list $L(v_i)$ is $|t(v_{i-1}v_i)| + |t(v_iv_{i+1})|$ and there is a color k such that $k \in \Lambda(v_i) \setminus ((\Lambda_{\min} + t(v_{i-1}v_i)) \cup (\Lambda_{\max} - t(v_iv_{i+1})))$ in both the cases. Consider now the following labeling c (indices are taken modulo n):

$$c(v_j) = \begin{cases} k & \text{for } j = i, \\ \Lambda_{\min} & \text{for } j = i+2, i+4, \ldots, i-1, \\ \Lambda_{\max} & \text{for } j = i+1, i+3, \ldots, i-2. \end{cases}$$

The labeling c is good by the choice of the color k and the fact that $\Lambda_{\max} - \Lambda_{\min} \notin t(e)$ for all edges e of the cycle. The equality (2) can be proven analogously.

■



$$\Lambda(v_1) = \{2, 8\}$$
$$\Lambda(v_2) = \{2, 4, 6, 8\}$$
$$\Lambda(v_3) = \{2, 4, 5, 6, 8\}$$
$$\Lambda(v_4) = \{2, 5, 8\}$$
$$\Lambda(v_5) = \{2, 8\}$$

Figure 1: An example of a balanced generalized list T-coloring problem with an underlying graph being C_5. The sets of forbidden differences are at the middles of the corresponding edges and the lists of colors for the vertices are in the right part of the figure.

Before stating Theorem 2, we define three special types of vertices for the problems whose underlying graphs are cycles. Let (C_n, Λ, t) be a balanced generalized list T-coloring problem. We say that the vertex v_i is of the *first*, *second* or *third type*, if it satisfies the following condition 1, 2 or 3, respectively:

1. $t(v_{i-1}v_i) = t(v_iv_{i+1})$ is arithmetic and $\Lambda(v_i) = (\Lambda_{\text{min}} + t(v_{i-1}v_i)) \uplus (\Lambda_{\text{max}} - t(v_{i-1}v_i))$.

2. The sets $t(v_{i-1}v_i)$ and $t(v_iv_{i+1})$ are arithmetic with the same difference d but $t(v_{i-1}v_i) \neq t(v_iv_{i+1})$. The list $\Lambda(v_i)$ is $\Lambda_{\text{min}} + \text{Ar}_d(k)$ where $k = |t(v_{i-1}v_i)| + |t(v_iv_{i+1})|$. In particular, $\Lambda_{\text{max}} - \Lambda_{\text{min}} = d(k-1)$.

3. Both sets $t(v_{i-1}v_i)$ and $t(v_iv_{i+1})$ are arithmetic sets with at least two elements and their differences d and d' are distinct. Then $t(v_{i-1}v_i) = \text{Ar}_d(k)$ and $t(v_iv_{i+1}) = \text{Ar}_{d'}(k')$ where $kd = k'd' = \text{lcm}(d, d')$. In addition, $\Lambda_{\text{max}} - \Lambda_{\text{min}} = \text{lcm}(d, d')$ and

   $$\Lambda(v_i) = (\Lambda_{\text{min}} + \text{Ar}_d(k)) \uplus (\Lambda_{\text{max}} - \text{Ar}_{d'}(k')) = (\Lambda_{\text{min}} + \text{Ar}_{d'}(k')) \uplus (\Lambda_{\text{max}} - \text{Ar}_d(k)).$$

   Note that both unions in the above expression are disjoint because of the equality $kd = k'd' = \text{lcm}(d, d') = \Lambda_{\text{max}} - \Lambda_{\text{min}}$.

As an example, consider the generalized list T-coloring problem depicted in Figure 1. The vertices v_1 and v_5 are of the first type, the vertices v_2 and v_4 are of the second type and the vertex v_3 is of the third type. Note that the problem depicted in Figure 1 has no good labeling.

We finally characterize balanced generalized list T-coloring problems (C_n, Λ, t) with no good labelings:

Theorem 2 *A balanced generalized list T-coloring problem (C_n, Λ, t) where C_n is an odd cycle does not have a good labeling if and only if:*

- *the colors $\Lambda_{\min}$ and $\Lambda_{\max}$ are contained in all the lists $\Lambda(v)$, $v \in V(C_n)$,*
- *each vertex is one of the three types described above, in particular, all the sets $t(e)$, $e \in E(C_n)$, are arithmetic, and*
- *there is at least one vertex of the first or of the second type.*

Proof: We first prove that if a balanced problem (C_n, Λ, t) does not have a good labeling, then it is of the form described in the statement. The colors $\Lambda_{\min}$ and $\Lambda_{\max}$ are contained in all the lists by Lemma 2, and all the sets $t(e)$, $e \in E(C_n)$, are arithmetic by Lemma 6. Fix an arbitrary vertex v_i of C_n. We first show that the vertex v_i is one of the three types introduced before this theorem.

If $t(v_{i-1}v_i) = t(v_iv_{i+1})$, then $\Lambda(v_i) = (\Lambda_{\min} + t(v_{i-1}v_i)) \uplus (\Lambda_{\max} - t(v_{i-1}v_i))$ by Lemma 7. Hence the vertex v_i is of the first type.

We may now assume that $t(v_{i-1}v_i) \neq t(v_iv_{i+1})$. Let $t(v_{i-1}v_i) = \mathrm{Ar}_d(k)$ and $t(v_iv_{i+1}) = \mathrm{Ar}_{d'}(k')$. If $k = 1$ or $k' = 1$, i.e., the set assigned to the corresponding edge incident with v_i is $\{0\}$, then we may assume that the differences d and d' are equal. However, if $d = d'$, then, by Lemma 7:

$$\Lambda(v_i) = (\Lambda_{\min} + \mathrm{Ar}_d(k)) \uplus (\Lambda_{\max} - \mathrm{Ar}_d(k')) = (\Lambda_{\min} + \mathrm{Ar}_d(k')) \uplus (\Lambda_{\max} - \mathrm{Ar}_d(k)).$$

But this is possible only if $\Lambda_{\max} - \Lambda_{\min} = d(k + k' - 1)$. Hence the vertex v_i is of the second type.

The final case is that $d \neq d'$, say $d < d'$, and both k and k' are at least 2. By Lemma 7, we have:

$$\Lambda(v_i) = (\Lambda_{\min} + \mathrm{Ar}_d(k)) \uplus (\Lambda_{\max} - \mathrm{Ar}_{d'}(k')) = (\Lambda_{\min} + \mathrm{Ar}_{d'}(k')) \uplus (\Lambda_{\max} - \mathrm{Ar}_d(k)).$$

But this is possible only if $\Lambda_{\max} - \Lambda_{\min} = \mathrm{lcm}(d, d') = kd = k'd'$. Indeed, the set $\Lambda(v_i)$ contains the element $\Lambda_{\min} + d$ by the middle part of the above equality. Since $d < d'$, then $\Lambda_{\min} + d$ must be equal to $\Lambda_{\max} - (k-1)d$ by the right part of the equality. We now have $\Lambda_{\max} - \Lambda_{\min} = kd$ as desired. Since the unions of the equality are disjoint, we have also $\Lambda_{\max} - \Lambda_{\min} = k'd'$ and $\mathrm{lcm}(d, d') = kd = k'd'$. Hence, we have deduced that the vertex v_i is of the third type.

In order to complete the proof of the first implication of the theorem, it remains to exclude the case that all the vertices are of the third type. So, assume now that all the vertices are of the third type. Let d_i be the difference of the arithmetic set $t(v_iv_{i+1})$. Consider the labeling c defined as $c(v_i) = \Lambda_{\min} + d_i$ for each $i = 1, \ldots, n$.

Since $\Lambda_{\min} + d_i \in \Lambda(v_i)$, the labeling cannot be a good labeling only if there is an index i such that $|(\Lambda_{\min} + d_{i+1}) - (\Lambda_{\min} + d_i)| = |d_{i+1} - d_i| \in t(v_i v_{i+1})$. Then, $d_i|(d_{i+1} - d_i)$ and $d_i|d_{i+1}$. Hence, $\mathrm{lcm}(d_i, d_{i+1}) = d_{i+1}$ and $t(v_{i+1}v_{i+2}) = \mathrm{Ar}_{d_{i+1}}(1)$. But then, the vertex v_{i+1} is not of the third type.

We now prove the opposite implication of the theorem, namely, that a balanced generalized list T-coloring problem of the form described in the theorem does not have a good labeling. The proof proceeds by contradiction which is eventually obtained after several claims are established. Let c be a good labeling of such a problem (C_n, Λ, t) and let d_i be the difference of the arithmetic set $t(v_i v_{i+1})$. We construct another function $\mu : V(C_n) \to \mathbb{N} \cup \{\mathrm{Min}, \mathrm{Max}\}$ based on the labeling c:

$$\mu(v_i) = \begin{cases} \mathrm{Min} & \text{if } c(v_i) \in (\Lambda_{\min} + t(v_{i-1}v_i)) \cap (\Lambda_{\min} + t(v_i v_{i+1})), \\ \mathrm{Max} & \text{if } c(v_i) \in (\Lambda_{\max} - t(v_{i-1}v_i)) \cap (\Lambda_{\max} - t(v_i v_{i+1})), \\ d_{i-1} & \text{if } c(v_i) \in (\Lambda_{\min} + t(v_{i-1}v_i)) \setminus (\Lambda_{\min} + t(v_i v_{i+1})), \qquad (*) \\ d_i & \text{if } c(v_i) \in (\Lambda_{\min} + t(v_i v_{i+1})) \setminus (\Lambda_{\min} + t(v_{i-1}v_i)). \qquad (**) \end{cases}$$

Since all the vertices are of one of the three types, all the lists $\Lambda(v_i)$ satisfy the equalities (1) and (2) from Lemma 7. Hence, the function μ is well-defined. Observe that if v_i is of the first type (in which $t(v_{i-1}v_i) = t(v_i v_{i+1})$), then $\Lambda_{\min} + t(v_{i-1}v_i) = \Lambda_{\min} + t(v_i v_{i+1})$. Hence, $\mu(v_i)$ for such a vertex v_i is either Min or Max. In particular, we have:

Claim 1 *If $\mu(v_i) \notin \{\mathrm{Min}, \mathrm{Max}\}$, then v_i is of the second type or the third type.*

We now prove the following two claims:

Claim 2 *If c is a good labeling, then no two adjacent vertices are assigned by μ simultaneously both the label* Min *or both the label* Max.

If two adjacent vertices v_i and v_{i+1} are both mapped to Min, then $c(v_i) \in (\Lambda_{\min} + t(v_i v_{i+1}))$ and $c(v_{i+1}) \in (\Lambda_{\min} + t(v_i v_{i+1}))$ by the definition of μ. This immediately yields that $|c(v_i) - c(v_{i+1})| \in t(v_i v_{i+1})$ (recall that the set $t(v_i v_{i+1})$ is arithmetic). A similar argument excludes the case that both are mapped to Max.

Claim 3 *If the vertex v_i is assigned by μ the difference d_{i-1}, i.e., the condition in (*) is satisfied, then $d_{i-1}|\Lambda_{\max} - \Lambda_{\min}$ and the following holds:*

$$\{\Lambda_{\min}, \Lambda_{\min} + d_{i-1}, \Lambda_{\min} + 2d_{i-1}, \ldots, \Lambda_{\max}\} \subseteq (c(v_i) - t(v_{i-1}v_i)) \cup (c(v_i) + t(v_{i-1}v_i)). \quad (\triangle)$$

Since v_i is assigned by μ neither Min nor Max, the vertex v_i is of the second type or the third type by Claim 1. Hence, $d_{i-1}|\Lambda_{\max} - \Lambda_{\min}$. If v_i is of the third type, then $t(v_{i-1}v_i) = \mathrm{Ar}_{d_{i-1}}(k)$ where $k = (\Lambda_{\max} - \Lambda_{\min})/d_{i-1} - 1$. Since $\mu(v_i)$ is neither Min nor Max, the color $c(v_i)$ is neither $\Lambda_{\min}$ nor $\Lambda_{\max}$. We now infer from $c(v_i) \neq \Lambda_{\min}, \Lambda_{\max}$ that

$c(v_i) \in \{\Lambda_{\min}+d_{i-1}, \Lambda_{\min}+2d_{i-1}, \ldots, \Lambda_{\max}-d_{i-1}\}$ and hence the inclusion ($\triangle$) indeed holds. Next, we consider the case that v_i is of the second type. Let $k_{i-1} = |t(v_{i-1}v_i)|$ and $k_i = |t(v_i v_{i+1})|$. Note that $k_{i-1} > k_i$ by (*) and $\Lambda_{\max} - \Lambda_{\min} = (k_{i-1}+k_i-1)d_{i-1}$ because v_i is of the second type. By the condition from (*), the color $c(v_i)$ is one of the numbers $\Lambda_{\min}+k_i d_i, \Lambda_{\min}+(k_i+1)d_i, \ldots, \Lambda_{\min}+(k_{i-1}-1)d_i$ and thus the inclusion ($\triangle$) holds. This establishes the claim.

Similarly as Claim 3, we can prove the following claim (the details are left to the reader):

Claim 4 *If the vertex v_i is assigned by μ the difference d_i, i.e., the condition in (**) is satisfied, then $d_i|\Lambda_{\max}-\Lambda_{\min}$ and the following holds:*

$$\{\Lambda_{\min}, \Lambda_{\min}+d_i, \Lambda_{\min}+2d_i, \ldots, \Lambda_{\max}\} \subseteq (c(v_i)-t(v_i v_{i+1})) \cup (c(v_i)+t(v_i v_{i+1})). \quad (\triangle\triangle)$$

Now, some edges of the cycle are oriented in the following way: If v_i is labeled by μ with d_{i-1} according (*), then the edge $v_{i-1}v_i$ is oriented from v_i to v_{i-1}. If v_i is labeled by μ with d_i according (**), then the edge $v_i v_{i+1}$ is oriented from v_i to v_{i+1}. Since c is a good labeling, each edge is oriented in at most one direction. Indeed, assume for the sake of contradiction that both the vertex v_i satisfies (**) and the vertex v_{i+1} satisfies (*). We can now infer from ($\triangle\triangle$) that $d_i|(c(v_i)-\Lambda_{\min})$, in particular, $c(v_i) \in \{\Lambda_{\min}, \Lambda_{\min}+d_i, \Lambda_{\min}+2d_i, \ldots, \Lambda_{\max}\}$ holds. Since the vertex v_{i+1} satisfies ($\triangle$), we conclude that $|c(v_{i+1})-c(v_i)| \in t(v_i v_{i+1})$ — contradiction.

The proof of the second implication is completed by the following four claims:

Claim 5 *No edge can be oriented to a vertex which assigned by μ either* Min *or* Max.

Assume the opposite and say, e.g., that the edge $v_i v_{i+1}$ is oriented from v_i to v_{i+1} and $\mu(v_{i+1}) = \mathrm{Min}$. Then, $c(v_i) \in (\Lambda_{\min}+t(v_i v_{i+1})) \setminus (\Lambda_{\min}+t(v_{i-1}v_i))$ and the vertex v_i is assigned by μ the difference d_i. In particular, $\Lambda_{\max}-\Lambda_{\min}$ is divisible by d_i and $(c(v_i)+t(v_i v_{i+1})) \cup (c(v_i)-t(v_i v_{i+1})) \supseteq \Lambda_{\min}+\mathrm{Ar}_{d_i}(k+1)$ by ($\triangle\triangle$) where $k = (\Lambda_{\max}-\Lambda_{\min})/d_i$. Since the vertex v_{i+1} is assigned by μ the label Min, the difference $c(v_{i+1})-\Lambda_{\min}$ is divisible by d_i and thus $c(v_{i+1}) \in \Lambda_{\min}+\mathrm{Ar}_{d_i}(k+1)$. Then, $|c(v_i)-c(v_{i+1})| \in t(v_i v_{i+1})$ and the labeling c is not good — contradiction.

Claim 6 *All edges of the cycle are oriented.*

If all the vertices of the cycle are assigned by μ one of the labels Min or Max, then the vertices of the cycle should be assigned the labels Min and Max alternately. But this is impossible because the length of the cycle is odd. Hence, there is a vertex v_i assigned by μ neither Min nor Max. In particular, there is an edge leaving the vertex v_i and this edge must lead to a vertex which is again assigned by μ neither Min nor Max by Claim 5. There is also an edge leaving this vertex and it again leads to a vertex assigned by μ neither Min nor Max. In this way, we go around the whole cycle and show that all the edges are oriented.

Claim 7 *All the vertices of the cycle are of the second type or the third type.*

By Claim 6, all edges of the cycle are oriented. Since no edge can be oriented to a vertex which assigned by μ either Min or Max by Claim 5, all the vertices are of the second or the third type by Claim 1.

Claim 8 *All the vertices of the cycle are of the third type.*

Assume that the vertex v_i is of the second type. By symmetry, it can be assumed that $t(v_{i-1}v_i) \subseteq t(v_iv_{i+1})$. Since v_i is assigned by μ neither Min nor Max, the edge v_iv_{i+1} is oriented from v_i to v_{i+1} and $t(v_{i-1}v_i) \subset t(v_iv_{i+1})$. Let now k_{i-1} and k_i be such integers that $t(v_{i-1}v_i) = \mathrm{Ar}_{d_{i-1}}(k_{i-1})$ and $t(v_iv_{i+1}) = \mathrm{Ar}_{d_i}(k_i)$. Note that $k_{i-1} < k_i$ and $k_{i-1} + k_i = (\Lambda_{\max} - \Lambda_{\min})/d_{i-1} + 1$. In particular, $k_{i-1} < (\Lambda_{\max} - \Lambda_{\min})/d_{i-1}$. Since all the edges are oriented, the edge $v_{i-1}v_i$ is oriented from v_{i-1} to v_i. If the vertex v_{i-1} were of the third type, then it would hold that $k_{i-1} = (\Lambda_{\max} - \Lambda_{\min})/d_{i-1}$ (by the definition of the third vertex type). However, this does not hold. Hence, v_{i-1} is of the second type and $t(v_{i-2}v_{i-1}) = t(v_iv_{i+1}) = \mathrm{Ar}_{d_i}(k_i)$. But, then the edge $v_{i-1}v_i$ cannot be oriented from v_{i-1} to v_i because $t(v_{i-1}v_i) \subseteq t(v_{i-2}v_{i-1})$. This establishes the claim.

By Claim 8, all the vertices are of the third type, but then the balanced generalized list T-coloring problem is not as described in the statement of the theorem. This completes the proof of the second implication and so the proof of the whole theorem.

■

4.2 The case of complete graphs

We first formulate a lemma which is an immediate corollary of Theorem 2 but which will be useful in the analysis of the case of complete graphs:

Lemma 8 *Let (C_3, Λ, t) be a balanced generalized list T-coloring problem which does not have a good labeling and let $V(C_3) = \{x, y, z\}$. Then, the sets $t(xy)$, $t(xz)$ and $t(yz)$ are arithmetic. Moreover, if each of the sets $t(xy)$ and $t(xz)$ contains at least two elements and the differences of the arithmetic sets $t(xy)$ and $t(xz)$ are distinct, then $t(yz) = \{0\}$.*

Proof: Since the problem (C_3, Λ, t) does not have a good labeling, it is of the type described in the statement of Theorem 2. Therefore, the sets $t(xy)$, $t(xz)$ and $t(yz)$ are arithmetic. The vertex x must be of the third type because each of the arithmetic sets $t(xy)$ and $t(xz)$ contains at least two elements and their differences are distinct. At least one of the vertices y and z is of the first or the second type, again by Theorem 2. Assume that this vertex is y. Then, the difference of the arithmetic set $t(yz)$ and the difference of the arithmetic set $t(yx)$ are the same. Let this difference be denoted by d. Since x is of the third type, we have $t(yx) = \mathrm{Ar}_d((\Lambda_{\max} - \Lambda_{\min})/d)$ by the definition of the third type. By Lemma 7, the sets $\Lambda_{\min} + t(yx)$ and $\Lambda_{\max} - t(yz)$ are disjoint. But this is possible only if $t(yz) = \{0\}$ (recall that the difference of $t(yz)$ is d).

■

Next, we show that a balanced generalized list T-coloring problem with no good labeling can be reduced to a smaller one with the same property:

Lemma 9 *Let (K_n, Λ, t) be a balanced generalized list T-coloring problem with no good labeling. Let U be a subset of $V(K_n)$ of size $n' \geq 2$. Then, there is a balanced generalized list T-coloring problem $(K_{n'}, \Lambda', t')$ which does not have a good labeling, $V(K_{n'}) = U$ and $t'(uu') = t(uu')$ for $u, u' \in U$.*

Proof: The proof proceeds by induction on $n - n'$. If $n - n' = 0$, then the problems (K_n, Λ, t) and $(K_{n'}, \Lambda', t')$ are the same.

If $n - n' = 1$, then consider the problem $(K_n, \Lambda, t)[v \to \Lambda_{\min}]$ where v is the only vertex of K_n outside the set U. Since the problem (K_n, Λ, t) does not have a good labeling, it follows that the color $\Lambda_{\min}$ is contained in each list $\Lambda(v)$ by Lemma 2. Hence, the problem $(K_n, \Lambda, t)[v \to \Lambda_{\min}]$ is balanced and it does not have a good labeling.

If $n - n' \geq 2$, consider a set U' of the vertices of K_n such that $U \subset U' \subset V(K_n)$. By the induction hypothesis, for the set U', there is a balanced generalized list T-coloring problem which does not have a good labeling. Now, by induction applied to this new problem, there is a balanced generalized list T-coloring problem for the set U which does not have a good labeling.

■

As an immediate corollary of Lemma 9, we obtain that if a balanced generalized list T-coloring problem on a complete graph does not have a good labeling, then all sets $t(e)$ must be arithmetic:

Lemma 10 *Let (K_n, Λ, t) be a balanced generalized list T-coloring problem. If K_n has an edge e such that $t(e)$ is not arithmetic, then the problem (K_n, Λ, t) allows a good labeling.*

Proof: Let u and v be the end-vertices of the edge e such that $t(e)$ is not arithmetic. If the problem (K_n, Λ, t) does not have a good labeling, then apply Lemma 9 with $U = \{u, v\}$ to get a balanced generalized list T-coloring problem (K_2, Λ', t') such that $t'(e) = t(e)$ is not arithmetic. But this is impossible by Lemma 5.

■

We now focus on the relation between lists $\Lambda : V(G) \to 2^{\mathbb{N}}$ and forbidden sets $t : E(G) \to 2^{\mathbb{N}}$ in generalized list T-coloring problems on complete graphs with no good labelings:

Lemma 11 *Let (K_n, Λ, t) be a balanced generalized list T-coloring problem which does not have a good labeling. Let $v_1, \ldots, v_n$ be an arbitrary ordering of the vertices of K_n. Then, there exist numbers $k_1 < k_2 < \cdots < k_{n-1}$ such that*

$$\Lambda(v_n) = \biguplus_{1 \le i \le n-1} (t(v_i v_n) + k_i).$$

Moreover, for each k_i, $i = 1, \ldots, n-1$, and each $j = i, \ldots, n$:

$$k_i = \min \left(\Lambda(v_j) \setminus \bigcup_{1 \le i' < i} (t(v_{i'} v_j) + k_{i'}) \right).$$

In particular, $k_1 = \Lambda_{\min}$.

Proof: The proof proceeds by induction on n. The lemma vacuously holds for $n = 1$. For $n = 2$, it follows by Lemma 5.

Suppose now that $n \ge 3$ and set $k_1 = \Lambda_{\min}$. Let $(K_{n-1}, \Lambda', t') = (K_n, \Lambda, t)[v_1 \to \Lambda_{\min}]$. Note that the problem (K_{n-1}, Λ', t') is balanced since otherwise the problem (K_n, Λ, t) would have a good labeling. By the induction hypothesis, there are numbers $k_2 < \cdots < k_{n-1}$ such that:

$$\Lambda'(v_n) = \biguplus_{2 \le i \le n-1} (t'(v_i v_n) + k_i).$$

Moreover, for all $i \in \{2, \ldots, n-1\}$ and $j \in \{i, \ldots, n\}$, it holds:

$$k_i = \min \left(\Lambda'(v_j) \setminus \bigcup_{2 \le i' < i} (t'(v_{i'} v_j) + k_{i'}) \right).$$

Since $\Lambda'(v_j) = \Lambda(v_j) \setminus (t(v_1 v_j) + \Lambda_{\min})$ and $t(e) = t'(e)$ for each edge of K_{n-1}, we have:

$$\Lambda(v_n) = \Lambda'(v_n) \uplus (t(v_1 v_n) + k_1) = \biguplus_{1 \le i \le n-1} (t(v_i v_n) + k_i).$$

Similarly, we have for all i with $2 \leq i \leq n-1$, and all j with $i \leq j \leq n$:

$$k_i = \min \left(\Lambda(v_j) \setminus \bigcup_{1 \leq i' < i} (t(v_{i'} v_j) + k_{i'}) \right).$$

The final equality follows for all $1 \leq j \leq n$ from the choice of k_1:

$$k_1 = \min \left(\Lambda(v_j) \setminus \bigcup_{1 \leq i' < 1} (t(v_{i'} v_j) + k_{i'}) \right) = \min \Lambda(v_j) = \Lambda_{\min}.$$

■

Roughly speaking, if a balanced generalized list T-coloring problem (K_n, Λ, t) does not have a good labeling, all sets $t(e)$ incident with the same vertex must share the same difference as stated in the next lemma:

Lemma 12 *Let (K_n, Λ, t), $n \geq 4$, be a balanced generalized list T-coloring problem with no good labeling. Then, all sets $t(e)$ for $e \in E(K_n)$ are arithmetic, and all the sets $t(e)$ for all edges e incident with the same vertex v share the same difference.*

Proof: By Lemma 10, all the sets $t(e)$ are arithmetic. According to Lemma 9, it is enough to prove the claim for $n = 4$. Let us assume that $n = 4$ and v, x, y and z are the vertices of K_4 such that the sets $t(e)$ for the edges e incident with the vertex v do not share the same difference. Let d_x, d_y and d_z be the differences and k_x, k_y and k_z the sizes of the arithmetic sets $t(vx)$, $t(vy)$ and $t(vz)$, respectively. By our assumption, at most one of the three numbers k_x, k_y and k_z is equal to one. Hence, we may assume that $k_x \geq 1$, $k_y \geq 2$ and $k_z \geq 2$. The three differences d_x, d_y and d_z are not all the same by the choice of v. We distinguish three cases and eventually derive a contradiction in each of them:

- $k_x = 1$, $k_y \geq 2$, $k_z \geq 2$ *and* $d_y < d_z$ (the case $d_y > d_z$ is symmetric)
  Note that $t(vx) = \{0\}$. Consider the problem $(K_4, \Lambda, t)[x \to \Lambda_{\max}]$ obtained by assigning the color $\Lambda_{\max}$ to the vertex x. This is a balanced generalized list T-coloring problem which does not have a good labeling. Note that its underlying graph is a triangle. Recall that the sets of forbidden differences on its edges are $t(vy) = \mathrm{Ar}_{d_y}(k_y)$ and $t(vz) = \mathrm{Ar}_{d_z}(k_z)$. Hence, $t(yz) = \{0\}$ by Lemma 8. By Theorem 2, the vertex v must be of the third type since $d_y \neq d_z$. In addition, k_y and k_z satisfy $k_y d_y = k_z d_z = \mathrm{lcm}(d_y, d_z)$. In particular, d_z is not divisible by d_y (recall that $k_y \geq 2$ and $k_z \geq 2$). Since the vertex v is of the third type in

$(K_4,\Lambda,t)[x\to\Lambda_{\max}]$, its list in the new problem $(K_4,\Lambda,t)[x\to\Lambda_{\max}]$ is equal to the following set:

$$(\Lambda_{\min}+\mathrm{Ar}_{d_y}(k_y))\cup(\Lambda_{\min}+\mathrm{Ar}_{d_z}(k_z))\cup\{\Lambda_{\min}+\mathrm{lcm}(d_y,d_z)\}.$$

Hence, we infer that:

$$\Lambda(v)=(\Lambda_{\min}+\mathrm{Ar}_{d_y}(k_y))\cup(\Lambda_{\min}+\mathrm{Ar}_{d_z}(k_z))\cup\{\Lambda_{\min}+\mathrm{lcm}(d_y,d_z),\Lambda_{\max}\}. \quad (3)$$

Similarly, considering the problem $(K_4,\Lambda,t)[x\to\Lambda_{\min}]$ yields:

$$\Lambda(v)=(\Lambda_{\max}-\mathrm{Ar}_{d_y}(k_y))\cup(\Lambda_{\max}-\mathrm{Ar}_{d_z}(k_z))\cup\{\Lambda_{\max}-\mathrm{lcm}(d_y,d_z),\Lambda_{\min}\}. \quad (4)$$

The second largest element of $\Lambda(v)$ according to (3) is $\Lambda_{\min}+\mathrm{lcm}(d_y,d_z)$ and according to (4) is $\Lambda_{\max}-d_y$. Hence, $\Lambda_{\max}-\Lambda_{\min}=\mathrm{lcm}(d_y,d_z)+d_y$. On the other hand, the largest element of $\Lambda(v)$ which is not congruent to $\Lambda_{\max}$ modulo d_y is equal to $\Lambda_{\min}+\mathrm{lcm}(d_y,d_z)-d_z$ according to (3) and it is equal to $\Lambda_{\max}-d_z$ according to (4) (recall that $k_yd_y=k_zd_z=\mathrm{lcm}(d_y,d_z)$). Hence, we infer that $\Lambda_{\max}-\Lambda_{\min}=\mathrm{lcm}(d_y,d_z)$ — contradiction.

- *$k_x\geq 2$, $k_y\geq 2$, $k_z\geq 2$ and $d_x<d_y<d_z$*
  The problem obtained by assigning the color $\Lambda_{\max}$ to the vertex z does not have a good labeling. In this new problem, the vertex v must be of the third type described in Theorem 2 because differences on edges incident with it are different. We infer that $k_xd_x=k_yd_y=\mathrm{lcm}(d_x,d_y)$. Since $k_x\geq 2$ and $k_y\geq 2$, d_y is not divisible by d_x. And by Lemma 8, it must be $t(xy)=\{0\}$. Symmetric arguments yield $k_xd_x=k_zd_z=\mathrm{lcm}(d_x,d_z)$, $k_yd_y=k_zd_z=\mathrm{lcm}(d_y,d_z)$ and $t(xz)=t(yz)=\{0\}$. Let l be the following number:

  $$l=k_xd_x=k_yd_y=k_zd_z=\mathrm{lcm}(d_x,d_y)=\mathrm{lcm}(d_x,d_z)=\mathrm{lcm}(d_y,d_z). \quad (5)$$

  Consider again the problem $(K_4,\Lambda,t)[z\to\Lambda_{\max}]$ obtained by assigning the color $\Lambda_{\max}$ to the vertex z. Since $t(xz)=t(yz)=\{0\}$, the color $\Lambda_{\min}$ remains in the lists of the vertices of x and y. Then, the color $\Lambda_{\min}$ must remain also in the list of the vertex v by Lemma 2. Hence, the list of v in the obtained problem is equal to the following list:

  $$(\Lambda_{\min}+t(vx))\cup(\Lambda_{\min}+t(vy))\cup\{\Lambda_{\min}+l\}.$$

  The following inclusion now immediately follows:

  $$(\Lambda_{\min}+\mathrm{Ar}_{d_x}(k_x))\cup(\Lambda_{\min}+\mathrm{Ar}_{d_y}(k_y))\cup\{\Lambda_{\min}+l\}\subseteq\Lambda(v).$$

By symmetry, we have also the following:

$$(\Lambda_{\min} + \mathrm{Ar}_{d_x}(k_x)) \cup (\Lambda_{\min} + \mathrm{Ar}_{d_z}(k_z)) \cup \{\Lambda_{\min} + l\} \subseteq \Lambda(v).$$

The size of the following set is $k_x + k_y + k_z - 1$ by equation (5):

$$(\Lambda_{\min} + \mathrm{Ar}_{d_x}(k_x)) \cup (\Lambda_{\min} + \mathrm{Ar}_{d_y}(k_y)) \cup (\Lambda_{\min} + \mathrm{Ar}_{d_z}(k_z)) \cup \{\Lambda_{\min} + l\}. \tag{6}$$

Since the problem (K_4, Λ, t) is balanced, the size of $\Lambda(v)$ is $k_x + k_y + k_z$. We now have (observe that the missing color in (6) can be only $\Lambda_{\max}$):

$$\Lambda(v) = (\Lambda_{\min} + \mathrm{Ar}_{d_x}(k_x)) \cup (\Lambda_{\min} + \mathrm{Ar}_{d_y}(k_y)) \cup (\Lambda_{\min} + \mathrm{Ar}_{d_z}(k_z)) \cup \{\Lambda_{\min} + l, \Lambda_{\max}\}. \tag{7}$$

A symmetric argument based on the problems obtained by assigning the color $\Lambda_{\min}$ to some of the vertices gives the following equality:

$$\Lambda(v) = (\Lambda_{\max} - \mathrm{Ar}_{d_x}(k_x)) \cup (\Lambda_{\max} - \mathrm{Ar}_{d_y}(k_y)) \cup (\Lambda_{\max} - \mathrm{Ar}_{d_z}(k_z)) \cup \{\Lambda_{\max} - l, \Lambda_{\min}\}. \tag{8}$$

Now, the equalities (7) and (8) are compared: The second largest element of $\Lambda(v)$ according to (7) is $\Lambda_{\min} + l$ and according to (8) is $\Lambda_{\max} - d_x$. Hence, we can infer that $\Lambda_{\max} - \Lambda_{\min} = l + d_x$. The largest element of $\Lambda(v)$ which is not congruent to $\Lambda_{\max}$ modulo d_x is equal to $\Lambda_{\min} + l - d_y$ according to the equalities (5) and (7). But the largest element which is not congruent to $\Lambda_{\max}$ modulo d_x is equal to $\Lambda_{\max} - d_y$ according to the equalities (5) and (8). Hence, we have $\Lambda_{\max} - \Lambda_{\min} = l$ which is the desired contradiction.

- *$k_x \geq 2$, $k_y \geq 2$, $k_z \geq 2$ and $d_x = d_y \neq d_z$*
  As in the previous case, consider the problems $(K_4, \Lambda, t)[x \to \Lambda_{\max}]$ and $(K_4, \Lambda, t)[y \to \Lambda_{\max}]$ and conclude that $t(xz) = t(yz) = \{0\}$. In particular, it is possible to define $l = k_x d_x = k_y d_y = k_z d_z = \mathrm{lcm}(d_x, d_y)$ and $k_x = k_y$.

  Consider again the problem $(K_4, \Lambda, t)[y \to \Lambda_{\max}]$. Since $t(yz) = \{0\}$, the color $\Lambda_{\min}$ remains in the list of the vertex z. Then, the color $\Lambda_{\min}$ must remain also in the list of the vertex v by Lemma 2. Hence, the list of v in the new problem is equal to the following set:

  $$(\Lambda_{\min} + t(vx)) \cup (\Lambda_{\min} + t(vz)) \cup \{\Lambda_{\min} + l\}.$$

  The way in which the new problem was obtained immediately implies the following equality:

  $$\begin{aligned}\Lambda(v) \quad = \quad & ((\Lambda_{\min} + \mathrm{Ar}_{d_x}(k_x)) \cup (\Lambda_{\min} + \mathrm{Ar}_{d_z}(k_z)) \cup \{\Lambda_{\min} + l\}) \uplus \\ & (\Lambda_{\max} - \mathrm{Ar}_{d_y}(k_y)).\end{aligned} \tag{9}$$

A symmetric argument which is based on the problem obtained by assigning the color $\Lambda_{\min}$ to the vertex y gives the following:

$$\begin{aligned}\Lambda(v) &= ((\Lambda_{\max} - \mathrm{Ar}_{d_x}(k_x)) \cup (\Lambda_{\max} - \mathrm{Ar}_{d_z}(k_z)) \cup \{\Lambda_{\max} - l\}) \uplus \\ &\quad (\Lambda_{\min} + \mathrm{Ar}_{d_y}(k_y)). \end{aligned} \tag{10}$$

The equalities $k_x = k_y$ and $d_x = d_y$ implies that $\Lambda_{\min} + \mathrm{Ar}_{d_x}(k_x) = \Lambda_{\min} + \mathrm{Ar}_{d_y}(k_y)$ and $\Lambda_{\max} - \mathrm{Ar}_{d_x}(k_x) = \Lambda_{\max} - \mathrm{Ar}_{d_y}(k_y)$. This combined with the equalities (9) and (10) yields the following:

$$(\Lambda_{\min} + d_z + \mathrm{Ar}_{d_z}(k_z - 1)) \uplus \{\Lambda_{\min} + l\} = (\Lambda_{\max} - d_z - \mathrm{Ar}_{d_z}(k_z - 1)) \uplus \{\Lambda_{\max} - l\}. \tag{11}$$

Since $l = k_z d_z$, we can simplify (11) to the following equality:

$$\mathrm{Ar}_{d_z}(k_z) + \Lambda_{\min} + d_z = \Lambda_{\max} - d_z - \mathrm{Ar}_{d_z}(k_z).$$

Hence, we can infer (by considering the largest and the smallest element in the sets above) that $\Lambda_{\max} - \Lambda_{\min} = l + d_z$.

Let us consider now the problem $(K_3, \Lambda', t') = (K_4, \Lambda, t)[z \to \Lambda_{\max}]$. First, we have by the equality (10) (the union in the next equality is disjoint because the new problem must be balanced):

$$\Lambda'(v) = \Lambda(v) \setminus t(vz) = \Lambda(v) \setminus (\Lambda_{\max} - \mathrm{Ar}_{d_z}(k_z)) =$$

$$(\Lambda_{\min} + \mathrm{Ar}_{d_x}(k_x)) \uplus (\Lambda_{\max} - d_x - \mathrm{Ar}_{d_x}(k_x)).$$

Observe that, by Lemma 9, the problem (K_3, Λ', t') is a balanced generalized list T-coloring problem which does not have a good labeling. Let $\Lambda'_{\min}$ and $\Lambda'_{\max}$ be the smallest and the largest element contained in the lists Λ'. Since $\Lambda'_{\min} = \Lambda_{\min}$ and $\Lambda'_{\max} = \Lambda_{\max} - d_x = \Lambda_{\min} + l + d_z - d_x$ are not congruent modulo d_x, all the vertices v, x and y in the problem (K_3, Λ', t') must be of the first type described in the statement of Theorem 2. Hence, we infer that $t(xy) = t'(xy) = \mathrm{Ar}_{d_x}(k_x)$ and:

$$\Lambda'(x) = \Lambda'(y) = \Lambda'(v) = (\Lambda_{\min} + \mathrm{Ar}_{d_x}(k_x)) \uplus (\Lambda_{\max} - d_x - \mathrm{Ar}_{d_x}(k_x)).$$

In particular:

$$(\Lambda_{\min} + \mathrm{Ar}_{d_x}(k_x)) \uplus (\Lambda_{\max} - d_x - \mathrm{Ar}_{d_x}(k_x)) \subseteq \Lambda(x). \tag{12}$$

A symmetric argument based on the problem obtained by assigning the color $\Lambda_{\min}$ to the vertex z yields the following inclusion:

$$(\Lambda_{\max} - \mathrm{Ar}_{d_x}(k_x)) \uplus (\Lambda_{\min} + d_x + \mathrm{Ar}_{d_x}(k_x)) \subseteq \Lambda(x). \tag{13}$$

By comparing the inclusions (12) and (13), we get the following:

$$(\Lambda_{\min} + \mathrm{Ar}_{d_x}(k_x+1)) \uplus (\Lambda_{\max} - \mathrm{Ar}_{d_x}(k_x+1)) \subseteq \Lambda(x).$$

Thus the size of $\Lambda(x)$ must be at least $2k_x+2$. On the other hand, the t-degree of x in the problem (K_n,Λ,t) is $|t(xy)|+|t(xz)|+|t(xv)| = 2k_x+1$. This contradicts the fact that the problem (K_n,Λ,t) is balanced.

■

Now, we extend the argument from the previous lemma and show that all the sets $t(e)$ must share the same difference. Note that we cannot derive immediately this conclusion from Lemma 12 since there could exist edges e with $t(e)=\{0\}$.

Lemma 13 *Let (K_n,Λ,t) be a balanced generalized list T-coloring problem with no good labeling and $n \geq 4$. Then, all the sets $t(e)$ for $e \in E(K_n)$ are arithmetic sets with the same difference.*

Proof: By Lemma 12, all the sets $t(e)$ are arithmetic and the sets $t(e)$ for edges e incident with the same vertex have the same difference. If there are edges e and e' with $|t(e)|, |t(e')| \geq 2$ such that $t(e)$ and $t(e')$ do not have the same difference, then the edges e and e' cannot be incident. Let $e=vw$ and $e'=xy$. By Lemma 9, it is enough now to prove the statement for $n=4$, i.e., a balanced generalized list T-coloring problem whose underlying graph is the complete graph of order four comprised by the vertices v, w, x and y. By Lemma 12, we have $t(vx)=t(wx)=t(vy)=t(wy)=\{0\}$. Let k_{vw} and k_{xy} be the sizes of the sets $t(vw)$ and $t(xy)$, respectively. Similarly, let d_{vw} and d_{xy} be their differences. Recall that we have assumed that $d_{vw} \neq d_{xy}$.

Consider the problem $(G',\Lambda',t') = (K_n,\Lambda,t)[y \to \Lambda_{\max}]$. The problem (G',Λ',t') is balanced and it does not have a good labeling. Theorem 2 implies the following equalities:

$$\begin{aligned}\Lambda'(v) &= \Lambda'(w) = \Lambda_{\min} + \mathrm{Ar}_{d_{vw}}(k_{vw}+1)\\ \Lambda'(x) &= \{\Lambda_{\min}, \Lambda_{\min} + d_{vw}k_{vw}\}.\end{aligned} \tag{14}$$

Hence, $\Lambda_{\min} + \mathrm{Ar}_{d_{vw}}(k_{vw}+1) \subseteq \Lambda(v)$. Next, consider the problem $(K_n,\Lambda,t)[y \to \Lambda_{\min}]$. By a similar argument as before, we obtain that $\Lambda_{\max} - \mathrm{Ar}_{d_{vw}}(k_{vw}+1) \subseteq \Lambda(v)$. Since $|\Lambda(v)| = \deg_t(v) = k_{vw}+2$, we can infer that $\Lambda_{\max} - \Lambda_{\min} = d_{vw}(k_{vw}+1)$ and $\Lambda(v) = \Lambda_{\min} + \mathrm{Ar}_{d_{vw}}(k_{vw}+2)$. Similarly, we may determine that the lists of the vertices w, x and y are as follows:

$$\begin{aligned}\Lambda(v) &= \Lambda(w) = \Lambda_{\min} + \mathrm{Ar}_{d_{vw}}(k_{vw}+2) \quad \text{and}\\ \Lambda(x) &= \Lambda(y) = \Lambda_{\min} + \mathrm{Ar}_{d_{xy}}(k_{xy}+2).\end{aligned}$$

But we know that $\Lambda_{\min} + d_{vw}k_{vw} = \Lambda_{\max} - d_{vw} \in \Lambda(x)$ by (14). Since $\Lambda(x) = \Lambda_{\min} + \mathrm{Ar}_{d_{xy}}(k_{xy}+2)$, all the elements of $\Lambda(x)$ are congruent with $\Lambda_{\max}$ modulo d_{xy}. In particular, $\Lambda_{\max} - d_{vw}$ and $\Lambda_{\max}$ are congruent modulo d_{xy}. We can now infer that $d_{xy} \mid d_{vw}$. By symmetry, we also infer that $d_{vw} \mid d_{xy}$. So, we conclude $d_{xy} = d_{vw}$ — contradiction.

■

Finally, we extend our arguments to get some properties of the lists in balanced generalized list T-coloring problems (K_n, Λ, t) with no good labeling:

Lemma 14 *Let (K_n, Λ, t) be a balanced generalized list T-coloring problem with $n \geq 3$, which does not have a good labeling, and let v be a vertex of the graph K_n. Suppose that all the sets $t(e)$ for edges e incident with the vertex v are arithmetic with the same difference d, but there exist two edges $e, e' \in E(K_n)$ incident with v for which $t(e) \neq t(e')$. Then, all the elements of the list $\Lambda(v)$ are congruent modulo d.*

Proof: We prove by induction on n that if the elements of $\Lambda(v)$ are not congruent modulo d, then the problem (K_n, Λ, t) has a good labeling. This will establish the claim of the lemma. If $n = 3$, this is true by Theorem 2 because the vertex v must be of the second type.

Suppose now that $n \geq 4$. Let $k_{\min}$ and $k_{\max}$ be the minimum and the maximum size of the lists $t(e)$ for edges e incident with the vertex v. By the assumptions of the lemma, $k_{\min} < k_{\max}$. Let $v_{\min}$ and $v_{\max}$ be vertices of G such that $t(vv_{\min}) = \mathrm{Ar}_d(k_{\min})$ and $t(vv_{\max}) = \mathrm{Ar}_d(k_{\max})$. By Lemma 2, we can also assume that the colors $\Lambda_{\min}$ and $\Lambda_{\max}$ are contained in the lists of all the vertices. We consider three cases:

- If *the number of elements of $\Lambda(v)$ congruent with $\Lambda_{\min}$ modulo d is smaller than* $k_{\max}$, then $\Lambda_{\min} + t(vv_{\max}) = \Lambda_{\min} + \mathrm{Ar}_d(k_{\max}) \not\subseteq \Lambda(v)$. Hence, the problem $(K_n, \Lambda, t)[v_{\max} \to \Lambda_{\min}]$ is overbalanced. The problem (K_n, Λ, t) has then a good labeling by Lemma 1 and Theorem 1.

- If *the number of elements of $\Lambda(v)$ congruent with $\Lambda_{\min}$ modulo d is greater than* $k_{\max}$, we proceed as follows: Let u be a vertex distinct from v, $v_{\min}$ and $v_{\max}$. The problem $(K_n, \Lambda, t)[u \to \Lambda_{\min}]$ is overbalanced or the list of the vertex v contain two elements which are not congruent modulo d. In the former case, it has a good labeling by Theorem 1. In the latter case, it has a good labeling by induction. Hence, the problem (K_n, Λ, t) has a good labeling by Lemma 1.

- The final case is that *the number of elements of $\Lambda(v)$ congruent with $\Lambda_{\min}$ modulo d is exactly* $k_{\max}$. If there is a vertex $u \neq v_{\min}$ with $|t(vu)| < k_{\max}$, then $(K_n, \Lambda, t)[u \to \Lambda_{\min}]$ is overbalanced or the list of the vertex v contain two elements which are not congruent modulo d. Similarly, as in the previous case, we conclude that the problem (K_n, Λ, t) has a good labeling. The other possibility is that for each vertex $u \neq v_{\min}$, we have $t(vu) = t(vv_{\max}) = \mathrm{Ar}_d(k_{\max})$. Consider now the problem $(K_{n-1}, \Lambda', t') = (K_n, \Lambda, t)[v_{\min} \to \Lambda_{\min}]$. Since the problem (K_n, Λ, t) is assumed not to have a good labeling, the problem (K_{n-1}, Λ', t') should admit no good labeling as well. In particular, the problem (K_{n-1}, Λ', t') is balanced. By Lemma 11, the number of elements with the same remainder modulo d contained in the set $\Lambda'(v)$ is divisible by $k_{\max}$ because $t'(e) = \mathrm{Ar}_d(k_{\max})$ for every edge e incident with v. But the set $\Lambda'(v)$ contains exactly $k_{\max} - k_{\min} < k_{\max}$ elements congruent modulo d with $\Lambda_{\min}$ (of the original problem (K_n, Λ, t)).

■

We may now extend the arguments of Lemma 14 to show that all elements of the lists are congruent modulo d where d is the common difference of all the sets $t(e)$:

Lemma 15 *Let (K_n, Λ, t) be a balanced generalized list T-coloring problem which does not have a good labeling. Suppose that all $t(e)$ for $e \in E(K_n)$ are arithmetic sets with the same difference d and that there exist edges $e, e' \in E(K_n)$ such that $t(e) \neq t(e')$. Then, all the elements of the union $\bigcup_{v \in V(K_n)} \Lambda(v)$ are congruent modulo d.*

Proof: By the assumption of the lemma, there is a vertex w of K_n which satisfies the assumption of Lemma 14. Hence, the elements of the list $\Lambda(w)$ are congruent modulo d. Let w' be a vertex distinct from w. Order vertices of K_n in the sequence $v_1, v_2, \ldots, v_n$ in such a way that $v_{n-1} = w$ and $v_n = w'$. By Lemma 11, there exist numbers $k_1 < k_2 < \cdots < k_{n-1}$ such that

$$\Lambda(w') = \biguplus_{1 \leq i \leq n-1} (t(w'v_i) + k_i).$$

In addition, the following holds for each i, $1 \leq i \leq n-1$:

$$k_i = \min \left(\Lambda(w) \setminus \bigcup_{1 \leq i' < i} (t(wv_i) + k_i) \right).$$

In particular, $k_i \in \Lambda(w)$ and since all the sets $t(w'v_i)$ have the same difference d, all the elements of the list $\Lambda(w')$ are congruent with all the elements of $\Lambda(w)$ modulo d. Since the choice of w' was arbitrary, the proof is completed.

■

In the proof of the main theorem of this subsection, we use the Brooks-type theorem for the channel assignment problem on complete graphs from [10]. We formulate it in our notation:

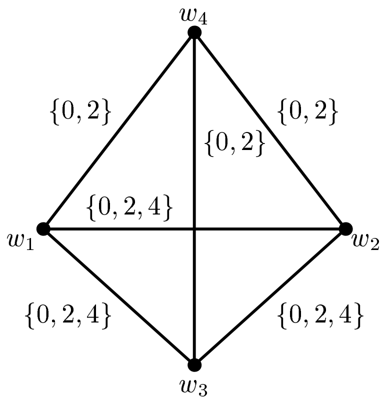
Theorem 3 *Let (K_n, Λ, t) be a balanced generalized list T-coloring problem such that each list $t(e)$ for $e \in E(K_n)$ is an arithmetic set with difference 1. Let $V(K_n) = \{v_1, \ldots, v_n\}$. Then, the problem (K_n, Λ, t) does not have a good labeling if and only if one of the following holds:*

- *There exist integers $1 \leq a$ and $1 \leq k_1 < \cdots < k_{n-1}$ such that:*
  - $k_i + a \leq k_{i+1}$ *for each* $i = 1, \ldots, n-2$,
  - $t(e) = \mathrm{Ar}_1(a)$ *for each edge* $e \in E(K_n)$, *and*
  - $\Lambda(v_i) = \bigcup_{1 \leq j \leq n-1}(k_j + \mathrm{Ar}_1(a))$ *for each vertex* v_i *of* K_n.
- *There exist integers $1 \leq a < b$ and $1 \leq k$ such that (possibly after an appropriate permutation of the vertices):*
  - $t(e) = \begin{cases} \mathrm{Ar}_1(a) & \text{if } e \text{ is incident with the vertex } v_n, \\ \mathrm{Ar}_1(b) & \text{otherwise.} \end{cases}$
  - $\Lambda(v_i) = \begin{cases} k + \mathrm{Ar}_1(a + b(n-2)) & \text{if } i \neq n, \\ \bigcup_{0 \leq j \leq n-2}(k + bj + \mathrm{Ar}_1(a)) & \text{otherwise.} \end{cases}$

We can now characterize in a similar way balanced generalized list T-coloring problems whose underlying graph is a complete graph and which do not have a good labeling (an example of such a balanced generalized list T-coloring problem with no good labeling is depicted in Figure 2):

Theorem 4 *Let (K_n, Λ, t) be a balanced generalized list T-coloring problem with $n \geq 4$. Let $V(K_n) = \{v_1, \ldots, v_n\}$. The problem (K_n, Λ, t) does not have a good labeling if and only if it is one of the following two types:*

- *There exist integers $1 \leq a, d$ and $k_1 < \cdots < k_{n-1}$ such that:*
  - $t(e) = \mathrm{Ar}_d(a)$ *for all* $e \in E(K_n)$ *and*
  - $\Lambda(v_i) = \biguplus_{1 \leq j \leq n-1}(k_j + \mathrm{Ar}_d(a))$ *for all* $1 \leq i \leq n$.
- *There exist integers $1 \leq a < b$, $1 \leq d$ and k such that (possibly after an appropriate permutation of the vertices):*
  - $t(e) = \begin{cases} \mathrm{Ar}_d(a) & \text{if } e \text{ is incident with the vertex } v_n, \\ \mathrm{Ar}_d(b) & \text{otherwise.} \end{cases}$
  - $\Lambda(v_i) = \begin{cases} k + \mathrm{Ar}_d(a + b(n-2)) & \text{if } i \neq n, \\ \bigcup_{0 \leq j \leq n-2}(k + bjd + \mathrm{Ar}_d(a)) & \text{otherwise.} \end{cases}$



$$\begin{aligned}\Lambda(w_1) &= \{1,3,5,7,9,11,13,15\}\\ \Lambda(w_2) &= \{1,3,5,7,9,11,13,15\}\\ \Lambda(w_3) &= \{1,3,5,7,9,11,13,15\}\\ \Lambda(w_4) &= \{1,3,7,9,13,15\}\end{aligned}$$

Figure 2: An example of a balanced generalized list T-coloring problem with an underlying graph being K_4. The sets of forbidden differences are at the middles of the corresponding edges and the lists of colors for the vertices are in the right side of the figure.

Proof: It is easy to check that if a problem (K_n, Λ, t) is of one of the above two types, then it is balanced. If it is of the first type described above, then it is easy to check that in any labeling from the lists, at most one vertex of K_n has a color from $k_j + \mathrm{Ar}_d(a)$. By the pigeon-hole principle, the problem (K_n, Λ, t) cannot have a good labeling. If the problem (K_n, Λ, t) is of the second type described above, we can assume without loss of generality that $d \mid k$. Observe that the problem (K_n, Λ, t) has a good labeling if and only if the problem (K_n, Λ', t') with the parameters $a' = a/d$, $b' = b/d$, $d' = 1$ and $k' = k/d$ has a good labeling. But this problem has no good labeling by Theorem 3.

We show that each balanced generalized list T-coloring problem (K_n, Λ, t) with no good labeling is of one of the two types described in the statement of the theorem. By Lemma 10, for each $e \in E(G)$, the set $t(e)$ is arithmetic. By Lemma 13, all the sets $t(e)$, $e \in E(G)$, have the same difference d. If there are edges e and e' such that $t(e) \neq t(e')$, then all the elements of all the lists $\Lambda(v)$, $v \in V(K_n)$, are congruent modulo d by Lemma 15. We may assume that $d \mid \Lambda_{\min}$, i.e., all the elements of all the lists $\Lambda(v)$ are divisible by d. Consider now the balanced problem (K_n, Λ', t') with $\Lambda'(v) = \{\frac{k}{d} \mid k \in \Lambda(v)\}$ and $t'(e) = \{\frac{k}{d} \mid k \in t(e)\}$. Observe that the problem (K_n, Λ, t) has a good labeling if and only if the problem (K_n, Λ', t') has a good labeling. Then, by the assumption, the problem (K_n, Λ', t') does not have a good labeling. Since the common difference of all the sets $t'(e)$ is one and there are edges e and e' such that $t'(e) \neq t'(e')$, it must be of the second type described in Theorem 3. Let a', b' and k' be the parameters from the statement of Theorem 3. We may conclude that the problem (K_n, Λ, t) is of the second type with the parameters $a = a'd$, $b = b'd$ and $k = k'd$.

The remaining case is that all the sets $t(e)$ for $e \in E(G)$ are the same. Suppose that they are equal to $\mathrm{Ar}_d(k)$. By Lemma 11, there exist integers $k_1 < \cdots < k_{n-1}$ such that for all the vertices v of K_n:

$$\Lambda(v) = \biguplus_{1 \le i \le n-1} (\mathrm{Ar}_d(k) + k_i).$$

Hence, the problem is of the first type described in the statement of this theorem. This completes the proof of the theorem.

■

5 The General Case

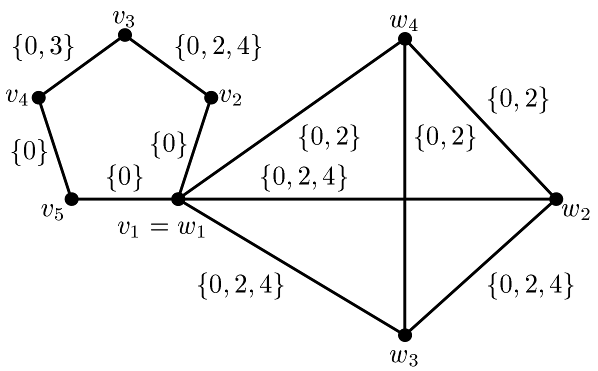
We now show that Lemma 5 and Theorems 2 and 4 can be combined to provide a full characterization of all balanced generalized list T-coloring problem which do not have a good labeling (an example of a balanced generalized list T-coloring problem with no good labeling whose underlying graph is not 2-connected can be found in Figure 3):

Theorem 5 *Let (G, Λ, t) be a balanced generalized list T-coloring problem where G is a connected graph and let $B_1, \ldots, B_l$ be the blocks of G. The problem (G, Λ, t) does not have a good labeling if and only if there exists $\Lambda_i : V(B_i) \to 2^{\mathbb{N}}$ and $t_i : E(B_i) \to 2^{\mathbb{N}}$, $1 \le i \le l$ such that:*

1. *For each $v \in V(G)$, it holds $\Lambda(v) = \biguplus_{\substack{1 \le i \le l \\ v \in V(B_i)}} \Lambda_i(v)$.*
2. *$t(e) = t_i(e)$ for the unique index i satisfying $e \in E(B_i)$.*
3. *Each generalized list T-coloring problem (B_i, Λ_i, t_i) is balanced and it does not have a good labeling.*

In particular, G is a Gallai tree and each (B_i, Λ_i, t_i) is as described in Lemma 5 and in Theorems 2 and 4.

Proof: We first prove that a balanced generalized list T-coloring problem (G, Λ, t) of the type described in the statement does not have a good labeling. The proof is by



$$\begin{aligned}
\Lambda(v_1) &= \{1,2,3,5,7,8,9,11,13,15\}\\
\Lambda(v_2) &= \{2,4,6,8\}\\
\Lambda(v_3) &= \{2,4,5,6,8\}\\
\Lambda(v_4) &= \{2,5,8\}\\
\Lambda(v_5) &= \{2,8\}\\
\Lambda(w_2) &= \{1,3,5,7,9,11,13,15\}\\
\Lambda(w_3) &= \{1,3,5,7,9,11,13,15\}\\
\Lambda(w_4) &= \{1,3,7,9,13,15\}
\end{aligned}$$

Figure 3: An example of a balanced generalized list T-coloring problem with no good labeling such that the underlying graph of the problem is not 2-connected. The problem is obtained by gluing the problems depicted in Figures 1 and 2.

induction on the number l of the blocks. If $l = 1$, the statement straightforwardly follows from Lemma 5 and Theorems 2 and 4. Otherwise, let B_l be an end-block of the graph G. Let v be the cut-vertex contained in B_l. Assume for the sake of contradiction that there is a good labeling c for the problem (G, Λ, t). If $c(v) \in \Lambda_l(v)$, then c restricted to B_l is a good labeling for the problem (B_l, Λ_l, t_l) which is impossible. If $c(v) \notin \Lambda_l(v)$, then c is a good labeling for the balanced problem (G', Λ', t'):

$$\begin{aligned}
V(G') &= \bigcup_{1\le i<l} V(B_i)\\
E(G') &= \bigcup_{1\le i<l} E(B_i)\\
\Lambda'(v) &= \bigcup_{1\le i<l, v\in V(B_i)} \Lambda_i(v)\\
t'(e) &= t_i(e) \text{ for the unique } i \text{ such that } e \in E(B_i).
\end{aligned}$$

But this is impossible by the assumption of the induction.

We now prove that if a problem (G, Λ, t) does not have a good labeling, then it is of the type described in the statement of the theorem. The proof again proceeds by induction on the number l of the blocks of G. If $l = 1$, the statement easily follows from Lemma 5, Theorems 2 and 4.

Assume that $l \geq 2$. Let B_1 be an end-block of the graph G and let v be the cut-vertex contained in B_1 and G' the graph comprised by the blocks $B_2, \ldots, B_l$. Let Λ_1 and Λ' be the function Λ restricted to $V(B_1)$ and $V(G')$, respectively. Similarly, let t_1 and t' be the function t restricted to $E(B_1)$ and $E(G')$, respectively. Let L_1 be the set of all the colors $k \in \Lambda(v) = \Lambda_1(v)$ such that there is not a good labeling c for the problem (B_1, Λ_1, t_1) with $c(v) = k$. By Theorem 1, $|L_1| \leq \deg_{t_1}(v)$. Let L' be the set of all the colors $k \in \Lambda(v) = \Lambda'(v)$ such that there is not a good labeling c for the problem (G', Λ', t') with $c(v) = k$. By Theorem 1, $|L'| \leq \deg_{t'}(v)$. If $|L_1 \cup L'| < \deg_{t_1}(v) + \deg_{t'}(v) = \deg_t(v)$, then there is a good labeling c for the problem (G, Λ, t) such that $c(v) = k$ where $k \in \Lambda(v) \setminus (L_1 \cup L')$. Otherwise, $|L_1| = \deg_{t_1}(v)$, $|L'| = \deg_{t'}(v)$ and so $\Lambda(v) = L_1 \uplus L'$. Reset $\Lambda_1(v) = L_1$ and $\Lambda'(v) = L'$. By the induction hypothesis, both problems (B_1, Λ_1, t_1) and (G', Λ', t') are of the type described in the statement of the theorem. Hence, it easily follows that the problem (G, Λ, t) is also of the desired type.

■

It is straightforward to check that all the proofs in this paper are algorithmic and hence we may conclude:

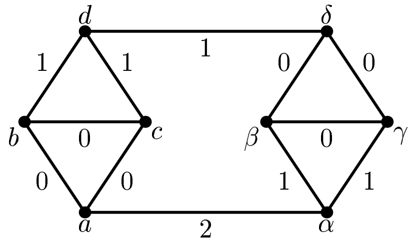
Corollary 1 *There is a polynomial-time algorithm which for each overbalanced generalized list T-coloring problem finds a good labeling. And, there is also a polynomial-time algorithm which for each balanced generalized list T-coloring problem decides whether the problem has a good labeling and if so, then the algorithm finds such a labeling.*

6 Conclusion

Throughout the paper, all considered generalized list T-coloring problems (G, Λ, t) satisfy that $0 \in t(e)$ for all sets of forbidden differences (as a part of the definition of the generalized list T-coloring). A natural question to ask is what happens, if we dismiss this requirement. In particular, the following problem naturally arises:

Problem 1 *Which (over)balanced generalized list T-coloring problems (G, Λ, t) do not have a good labeling when we do not require that $0 \in t(e)$ for all $e \in E(G)$?*

Surprisingly, it is **not** true that each such overbalanced generalized list T-coloring problem (G, Λ, t), where G is a connected graph, has a good labeling (this contrasts with the statement of Theorem 1 for overbalanced generalized list T-coloring problems with the requirement $0 \in t(e)$ for each set of forbidden differences). The example in Figure 4, which was obtained in discussions of the second author and Jiří Sgall, shows that such a statement is not true. Moreover, this example has some interesting properties such as



$$\Lambda(a) = \Lambda(b) = \Lambda(c) = \Lambda(d) = \{1,2,3\}$$
$$\Lambda(\alpha) = \{1,2,3,4\}$$
$$\Lambda(\beta) = \Lambda(\gamma) = \Lambda(\delta) = \{1,2,3\}$$

Figure 4: An example of an overbalanced generalized list T-coloring problem (G,Λ,t) which does not have a good labeling if we dismiss the condition $0 \in t(e)$. Each edge has a single forbidden difference which is represented by the number at the middle of the edge. The lists $\Lambda(v)$, $v \in V(G)$, are described in the right part of the figure.

its underlying graph is 2-connected but neither a cycle nor a complete graph, each set of forbidden differences is of size one, all the lists of vertices are the same except for a single vertex, etc.

Proposition 1 *If we dismiss a requirement that $0 \in t(e)$, then there exists an overbalanced generalized list T-coloring problem (G,Λ,t) which does not have a good labeling and which, in addition, satisfies:*

- *G is a 2-connected cubic graph.*
- *Each $\Lambda(v)$, $v \in V(G)$, is equal to $\{1,2,3\}$ except for a single vertex whose list is $\{1,2,3,4\}$.*
- *Each $t(e)$, $e \in E(G)$, is either $\{0\}$, $\{1\}$ or $\{2\}$. In particular, $|t(e)| = 1$ for every edge $e \in E(G)$.*

Proof: Consider the problem (G,Λ,t) depicted in Figure 4. It is easy to see that the problem has the properties from the statement of the proposition except that it does not have a good labeling. We now show that the problem (G,Λ,t) does not have a good labeling.

Assume for the sake of contradiction that the problem (G,Λ,t) has a good labeling λ. Let us consider first the case that $\lambda(b) = 2$. Then, $\lambda(d)$ cannot be 1 or 3 because of the edge bd. Since $\lambda(c)$ is either 1 or 3 (the edge bc), $\lambda(d)$ cannot be 2 (the edge cd), too. But then, the labeling λ cannot be proper. Hence, we can conclude that $\lambda(b) \neq 2$. Let $\lambda(c) \neq 2$ without loss of generality. We can now infer that $\lambda(a) = 2$ (consider the triangle

abc) and $\lambda(b), \lambda(c) \in \{1,3\}$. By symmetry, it can actually be assumed that $\lambda(b) = 1$ and $\lambda(c) = 3$. Finally, we derive that $\lambda(d)$ is 1 or 3 (consider the edges bd and cd).

Since $\lambda(d) \in \{1,3\}$, the vertex δ cannot be assigned by the labeling λ the number 2, i.e., $\lambda(\delta) \neq 2$. By symmetry, we can assume that $\lambda(\beta) = 2$ (consider the triangle $\beta\gamma\delta$). Thus, $\lambda(\gamma)$ is equal to 1 or 3. If $\lambda(\gamma) = 3$, then $\lambda(\alpha)$ cannot be 1 or 3 because of the edge $\alpha\beta$ and it cannot be 2 or 4 because of the edge $\gamma\delta$. Thus, $\lambda(\gamma) = 1$ and $\lambda(\alpha) = 4$. We eventually obtained the contradiction since $\lambda(a) = 2$, $\lambda(\alpha) = 4$ and $t(a\alpha) = \{2\}$.

■

We remark that it is not hard to show that the decision problem whether an overbalanced generalized list T-coloring problem has a good labeling is NP-complete when we dismiss the requirement $0 \in t(e)$ for all edges e. This contrasts the fact that the corresponding problem for overbalanced generalized list T-coloring problems with this requirement is trivial (the answer is simply always "yes" if the underlying graph is connected) and even the corresponding problem for balanced generalized list T-coloring problems can be solved in polynomial time (Corollary 1).

Acknowledgement

The research was partly conducted while the second author was visiting the third author at Pacific Institute for the Mathematical Sciences (PIMS) at Simon Fraser University (SFU) in November 2002. The authors would like to thank professor Pavol Hell for making their stay at PIMS and SFU pleasant and fruitful. The authors would also like to thank Jiří Sgall for his comments on the generalized list T-coloring without the requirement $0 \in t(e)$.

References

[1] N. Alon, A. Zaks: *T-choosability in graphs*, Discrete Appl. Math. **82** (1998), 1–13.

[2] O. V. Borodin: *Criterion of chromaticity of a degree prescription*, in: Abstracts of IV All-Union Conf. on Theoretical Cybernetics, Novosibirsk, 1977, 127–128 (in Russian).

[3] O. V. Borodin: *Problems of colouring and of covering the vertex set of a graph by induced subgraphs*, Ph. D. Thesis, Novosibirsk State University, Novosibirsk, 1979 (in Russian).

[4] R. L. Brooks: *On colouring the nodes of a network*, Proc. Cambridge Phil. Soc. **37** (1941), 194–197.

[5] G. J. Chang, D. Kuo: *The $L(2,1)$-labeling problem on graphs*, SIAM J. Discrete Math. **9(2)** (1996), 309–316.

[6] P. Erdös, A. L. Rubin, H. Taylor: *Choosability in graphs*, Proc. West Coast Conf. on Combinatorics, Graph Theory and Computing, Congr. Numer. XXVI (1979), 125–157.

[7] J. R. Griggs, R. K. Yeh: *Labeling graphs with a condition at distance 2*, SIAM J. Discrete Math. **5** (1992), 586–595.

[8] W. K. Hale: *Frequency assignment: Theory and applications*, Proceedings of the IEEE **68** (1980), 1497–1514.

[9] A. V. Kostochka, M. Stiebitz, B. Wirth: *The colour theorems of Brooks and Gallai extended*, Discrete Math. **162** (1996), 299–303.

[10] D. Král', R. Škrekovski: *A theorem about channel assignment problem*, SIAM J. Discrete Math. **16(3)** (2003), 426–437.

[11] J. Kratochvíl, Z. Tuza, M. Voigt: *Brooks-type theorems for choosability with separation*, J. Graph Theory **27** (1998), 43–49.

[12] J. Kratochvíl, Z. Tuza, M. Voigt: *New trends in the theory of graph colorings: Choosability and list coloring*, in: Contemporary Trends in Discrete Mathematics, DIMACS Series in Discrete Mathematics and Theoretical Computer Science, vol. 49, AMS, Providence, RI, 1999, 183–197.

[13] D. D.-F. Liu: *T-colorings of graphs*, Discrete Math. **101** (1992), 203–212.

[14] L. Lovasz: *Three short proofs in graph theory*, J. Combin. Theory Ser. B **19** (1975), 269–271.

[15] C. McDiarmid: *Discrete mathematics and radio channel assignment*, in: Recent advances in theoretical and applied discrete mathematics, Eds. C. Linhares-Salas and B. Reed, Springer-Verlag, 2001.

[16] C. McDiarmid: *On the span in channel assignment problems: bounds, computing and counting*, Discrete Math. **266** (2003), 387–397.

[17] M. Molloy, B. Reed: *Graph colouring and the Probabilistic Method*, Algorithms and Combinatorics, Vol. 23, Springer-Verlag, 2001.

[18] F. Roberts: *T-colorings of graphs: recent results and open problems*, Discrete Math. **93** (1991), 229–245.

[19] D. Sakai: *Labeling chordal graphs: distance two condition*, SIAM J. Discrete Math. **7** (1994), 133–140.

[20] Z. Tuza: *Graph colorings with local constraints — a survey*, Discuss. Math., Graph Theory **17(2)** (1997), 161–228.

[21] V. G. Vizing: *Colouring the vertices of a graph with prescribed colours* (in Russian), Metody Diskretnogo Analiza Teorii Kodov i Skhem **29** (1976), 3–10.

[22] A. Waller: *An upper bound for list T-colourings*, Bull. London Math. Soc. **28** (1996), 337–342.

[23] A. Waller: *Some results on list T-colourings*, Discrete Math. **174** (1997), 357–363.