Lightness, heaviness and gravity

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Abstract

The gravity $g(H, \mathcal{H})$ of a graph H in the family of graphs $\mathcal{H}$ is the greatest integer n with the property that for every integer m, there are infinitely many graphs $G \in \mathcal{H}$ such that each subgraph of G, which is isomorphic to H, contains at least n vertices of degree $\geq m$ in G. We study the basic properties of the gravity function for various families of plane graphs. We also introduce and study the almost-light graphs and the absolutely heavy graphs. The paper concludes with few open problems.

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1 Introduction

Let $\mathcal{H}$ be a family of graphs, and let H be a connected graph such that infinitely many members of $\mathcal{H}$ contain a subgraph isomorphic to H. Let $\varphi(H, \mathcal{H})$ be the smallest integer with the property that each graph $G \in \mathcal{H}$ which contains a subgraph isomorphic to H, contains also a subgraph $K \cong H$ such that, for every vertex $v \in K$,

$$\deg_G(v) \leq \varphi(H, \mathcal{H}).$$

If such a finite $\varphi(H, \mathcal{H})$ does not exist, we write $\varphi(H, \mathcal{H}) = +\infty$. We say that the graph H is *light* in the family $\mathcal{H}$ if $\varphi(H, \mathcal{H}) < +\infty$, otherwise we call it *heavy*. Thus, H is heavy in $\mathcal{H}$ if, for every integer m, there is a graph $G \in \mathcal{H}$ such that each isomorphic copy of H in G contains a vertex of degree $\geq m$ in G. The set of all light graphs in the family $\mathcal{H}$ is denoted $\mathcal{L}(\mathcal{H})$.

The above definition of the lightness was firstly introduced in [7], but the notion appears in [3] (see also particular results in [10, 9, 2, 4, 5]). The article [8] gives the survey of results for various families of plane graphs.

On the other hand, the heavy graphs were not studied. In this paper, based on the above definition of a heavy graph, we introduce the following measure of heaviness of graphs: The *gravity* $g(H, \mathcal{H})$ of a connected graph H in the family $\mathcal{H}$ of planar graphs is the greatest integer n with the property that for every integer m there are infinitely many graphs $G \in \mathcal{H}$ such that each isomorphic copy of H in G contains at least n vertices of degree $\geq m$ in G. Hence, a graph is light in a family of graphs if and only if its gravity is zero. A graph whose gravity in a family is equal to n is called *n-heavy*.

In general, one can determine the gravity of particular graphs also for families of nonplanar graphs; note, however, that if a family $\mathcal{H}$ contains complete graphs of arbitrarily big order, then the gravity of every graph G is trivially equal to the number of its vertices.

When considering the gravity $g(H, \mathcal{H})$, we always assume that H is contained in infinitely many graphs of $\mathcal{H}$. Under this assumption, a nice property of gravity is the following observation:

(O) *If $\mathcal{H}_2 \subseteq \mathcal{H}_1$, then $g(H, \mathcal{H}_2) \leq g(H, \mathcal{H}_1)$.*

By (O), gravity is monotone with respect to inclusion of the families of graphs. On the other hand, it is not monotone with respect to taking subgraphs. This can be seen from the following example: when considering the family comprised of all polyhedral graphs plus all the stars, the 4-path is light (see [3]), but the 3-path is 1-heavy.

For proving the lightness of a graph in a given family of graphs, usually, the Discharging method is used; for proving the heaviness, a construction of particular plane graphs is used. To determine the gravity of a graph, both these techniques are involved.

Throughout the paper, we consider connected graphs without loops or multiple edges. By P_k we denote the path on k vertices, i.e. the k-path and by C_k and S_k the k-cycle and the k-star $K_{1,k}$, respectively. By $\mathcal{P}_d$ we denote the family of planar graphs with the minimum vertex degree $\geq d$, and by $\mathcal{P}_d(w)$ the family of planar graphs with minimum vertex degree $\geq d$ and the minimum edge-weight (that is, the minimal sum of degrees of the endvertices of an edge in the graph) $\geq w$. The family of all 3-connected plane graphs is denoted by $\wp$. The minimum degree of a graph G is denoted $\delta(G)$. A vertex is called *big* if it is of degree $\geq m$, where m is a large enough integer. In the following two paragraphs, we describe some constructions which will be used to determine the gravity of some particular graphs.

Let G be a graph, and let v be a vertex of G. Take m vertex-disjoint copies of G and identify all the counterparts of v. The new graph is called an (m, G, v)-*star*. If G is a vertex-transitive graph, then we use to say an (m, G)-*star*. The vertex of identification is the *center* of the star. Thus, the (m, K_2)-star is just the m-star. Denote by $T_{m,h}$ the complete m-ary tree of height h. So, $T_{m,1}$ is the m-star. The vertices of degree 1 are called *leafs* and the highest vertex is the *root*. If we identify each leaf of $T_{m,h}$ with the vertex v of a copy of G, the resulting graph is denoted by $T_{m,h}(G, v)$. Moreover, if G is a vertex-transitive graph or G is an (m, G_0)-star for some graph G_0 with v being the center of that star, then we write $T_{m,h}(G)$ instead of $T_{m,h}(G, v)$. Notice that $T_{m,h+1} = T_{m,h}(K_{1,m})$.

Let G be a connected plane graph on at least 3 vertices and let u, v be two distinct vertices lying on the outerface of G. We use to say that a triple (G, u, v) is a *slice* and u, v are the *poles* of this slice. For the sake of the simplicity, we use to write for the slice (G, u, v) just G, when u, v are clear from the context. By a $(G, u, v; n)$-*melon* (or simply, *melon*) we denote the graph constructed in the following way: take n copies (slices) of G, identify all vertices corresponding to u into a new vertex and identify all vertices corresponding to v into another new vertex in all copies. In addition, if u and v are adjacent in G, then delete the multiple edges in the melon in order to obtain a simple graph. Two vertices resulted from this identification are called also the *poles* of the melon, the graph $G - u - v$ is the *pulp* of the slice (G, u, v). Observe that the melons are always planar graphs.

In each proof in this paper, based on the Discharging method, we consider a hypothetical counterexample G with vertex set $V(G)$ and face set $F(G)$. We assign the initial charge to every vertex $v \in V(G)$ and every face $f \in F(G)$ of the graph G in the following way:

$$c(v) = \alpha\, d(v) - 6 \quad \text{and} \quad c(f) = (3-\alpha)\, d(f) - 6, \tag{1}$$

where α is some prescribed number. If follows from the Euler's formula that the total sum of the charge of the vertices and the faces of G is equal to -12 according to the assignment by (1). We will redistribute the charge of the vertices and the faces of G by applying certain rules without changing the total sum of all charges. Denote by $c^*(x)$ the charge of a vertex or a face x after applying these rules (the *final charge* of x). In the proof of each claim, we will prove that each face and each vertex of G has a nonnegative final charge, which gives a contradiction.

2 Light stars and heavy paths

In this section, we study the gravity of the paths and the stars in the families of the planar graphs with bounded minimum degree. It is shown that the stars are either light or 1-heavy. And, for the paths, it is shown that their gravity is close to their length. We first prove the following lemma:

Lemma 2.1 *Let G be a planar graph with precisely b vertices of degree strictly greater than d, where $d \in \{1, \ldots, 5\}$. Then, $g(G, \mathcal{P}_d) \geq b$.*

Proof. Identify each vertex of G of degree $\geq d+1$ with the center of an (m, S)-star, where $S := K_2$ for $d = 1$, $S := C_3$ for $d = 2$, $S := K_4$ for $d = 3$, $S := O$ (octahedron) for $d = 4$, and $S := I$ (icosahedron) for $d = 5$. Denote the resulting graph by G^* and observe that it is a graph in $\mathcal{P}_d$. Notice that every copy of G in G^* contains all b big vertices of G. Now, the proof easily follows. $\square$

2.1 The gravity of stars in $\mathcal{P}_d$

Proposition 2.2 *Let $s_1 = s_2 = 0$, $s_3 = 1$, $s_4 = 2, s_5 = 4$ and $d \in \{1, \ldots, 5\}$. If $k \leq s_d$ then the star S_k is light in $\mathcal{P}_d$, and otherwise it is 1-heavy in $\mathcal{P}_d$.*

Proof. We will show first that the gravity of each star in $\mathcal{P}_d$ is at most 1. Suppose that, for fixed integer k and large enough integer m, there exists a graph $G \in \mathcal{P}_d$ which contains at least one k-star as a subgraph, and every such k-star contains at least two big vertices. Note that in that case G has at least two big vertices. Moreover, every big vertex of G has at least $m-k+1$ big neighbours; otherwise we encounter a k-star with only one big vertex. Now, consider the subgraph M induced by the big vertices of G. Then, $\delta(M) \geq m-k+1$. But, M is planar, so it contains a vertex of degree at most 5 and since m is large enough, it is a contradiction.

Now, we consider several cases regarding d. Since $\mathcal{L}(\mathcal{P}_1) = \mathcal{L}(\mathcal{P}_2) = \{P_1\}$, it follows that the 0-star S_0 $(= P_1)$ is the only light star in $\mathcal{P}_1$ and $\mathcal{P}_2$. Hence, $s_1 = s_2 = 0$. For $d = 3$, we have $\mathcal{L}(\mathcal{P}_3) = \{P_1, P_2\}$, and thus we have that $s_3 = 1$ is the right number. In [11] it is shown that $\mathcal{L}(\mathcal{P}_4) = \{P_1, P_2, P_3, P_4\}$. Thus, S_0, S_1, S_2 are the only light stars in $\mathcal{P}_4$. For $\mathcal{P}_5$, the set of light graphs $\mathcal{L}(\mathcal{P}_5)$ is not known, but in [6] it is proven that the k-star is heavy in $\mathcal{P}_5$ if and only if $k \geq 5$. Thus, $s_4 = 2$ and $s_5 = 4$.

□

2.2 The gravity of paths in $\mathcal{P}_1$

Theorem 2.3 $g(P_n, \mathcal{P}_1) = \begin{cases} n-2, & n = 3, 5; \\ n-1, & \textit{otherwise.} \end{cases}$

Proof. If $n = 3$ then the proof follows by Lemma 2.2. Suppose now that $n = 5$. By Lemma 2.1, $g(P_5, \mathcal{P}_1) \geq 3$. In order to prove the equality, suppose that G is a planar graph with at least one 5-path, and every such path contains at least four big vertices. Now, color each small vertex (i.e. each vertex of degree $< m$) by 0 and color each big vertex having at least two small neighbours by color 1. The remaining vertices of G color by 2.

If G contains a 3-path which vertices are consecutively colored by $1, 2, 1$ or $1, 1, 1$, then we can easily find a 5-path in G with both endvertices being small. Otherwise, there is no vertex colored by 2 or every vertex colored by 2 has at most one neighbour of color 1. In the first case, G must contain a 3-path with all vertices colored 1, which is a contradiction. In the second case consider the subgraph of G induced by the vertices colored by 2. Note that this graph has minimum degree $m-2$. This implies that G is not planar for $m \geq 8$, a contradiction.

Finally, suppose that $n \neq 3, 5$. For n even, consider the complete m-ary tree of height $n/2$. In this graph, every n-path P_n contains $n-1$ big vertices. If n is odd then consider the graph constructed from a 4-path by identifying each endvertex with the root of a copy of $T_{m,\lfloor n/2 \rfloor -1}$ and each of the two inner vertices of the 4-path identify with the center of a copy of the m-star. Observe that in this graph each n-path contains $n-1$ vertices of degree $\geq m$. Thus, $g(P_n, \mathcal{P}_1) \geq n-1$.

To show equality $g(P_n, \mathcal{P}_1) = n-1$, assume that for each large enough integer m there exists a connected graph G_m in which every n-path consists only of big vertices. Moreover, we assume that G_m has at least one n-path, say P. Then, P can be easily extended to a path P^* of length $\geq m$, consisting only of big vertices. If G_m contains a vertex x of degree $< m$, then, by connectedness of G_m, there exists an x–y-path with $y \in P^*$. Now, this x–y-path can be extended (using the part of P^*, if necessary) to an n-path with less than n big vertices, a contradiction. Hence, each vertex of G_m has to be big, which contradicts the planarity of G_m. □

2.3 The gravity of paths in $\mathcal{P}_2$

Theorem 2.4 *For each $n \geq 4$, the gravity of the n-path P_n in the family $\mathcal{P}_2$ is at most $n-2$.*

Proof. Suppose that the theorem is false, and there exists an $n \geq 4$ that for large enough integer m, there exists a graph G with at least one n-path, and each its n-path has at least $n-1$ big vertices. For the sake of simplicity, an n-path with at least two small vertices is *good*. Thus, the assumption is that G has no good n-path.

Claim 1. *G contains a path $P^* = y_1 y_2 \dots y_{2n+1}$ such that y_{n+1} is a small vertex.*

By the assumptions, G contains a path on n vertices, say $P = x_{-s} \dots x_{-1} x_0 x_1 x_2 \dots x_{n-s-1}$. Since $\mathcal{P}_2 \subseteq \mathcal{P}_1$, by Theorem 2.3, we can assume that P has a small vertex, say x_0. We may assume that all other vertices of P are big.

Consider first the case that $s \neq 0$ and $n-s-1 \neq 0$. In this case both endvertices of P are big. In what follows, we will extend P in both directions to obtain the required path P^*. First, set $i := n-s$. Next, repeat the following procedure until $i > n$: Choose a vertex which is a neighbour of x_{i-1} and which does not belong to P. This is possible since x_{i-1} is a big vertex and so it is adjacent to a vertex, which does not belong to P. Denote this vertex by x_i, and afterwards extend P by this vertex, i.e. set $P := Px_i$. Note that x_i is a big vertex, otherwise we obtain an n-subpath of P with two small vertices x_0 and x_i. Finally, set $i := i + 1$.

We apply above procedure also in the other direction in order to obtain the required path $P^* = x_{-n}x_{-n+1} \dots x_{-1}x_0x_1x_2 \dots x_n$.

Suppose now that $s = 0$. Then, $P = x_0x_1x_2 \dots x_{n-1}$. If x_0 has a neighbour which is not on P, then it must be a big vertex. In this case, denote it by x_{-1} and consider the path $x_{-1}x_0x_1 \dots x_{n-2}$ as in the previous case in order to construct P^*. Now, we may assume that all neighbours of x_0 belong on P. Let x_l $(l > 1)$ be a neighbour of x_0. In this case, set $P := x_{l-1} \dots x_1x_0x_lx_{l+1} \dots x_{n-1}$, and afterwards argue as in the first case of this claim. This establishes Claim 1.

Notice that, besides y_{n+1}, possible small vertices of P^* are y_1 and y_{2n+1}. And, all other vertices of P^* are big.

Claim 2. *G has no two adjacent small vertices.*

Suppose that the claim is false and suppose that u, v are two such vertices. Since the graph G is connected, it has a path Q with one endvertex in $\{u, v\}$, say u, and the other endvertex in $V(P^*)$, say y_p, where $p \in \{0, \dots, 2n\}$. One remark here is that we do not exclude the possibility that one of the vertices u and v (or both) belong to P^*. We may assume that Q does not contain the vertex v, and it does not contain any other vertex of P^*. Now, it is easy to see that one of two subgraphs $vQy_{p+1} \dots y_{2n}$ and $vQy_{p-1} \dots y_0$ contains a good n-path as a subgraph, a contradiction.

Claim 3. *G has no big vertex adjacent to two small vertices.*

Suppose that the claim is false and that a big vertex w is adjacent to two small vertices x_1 and x_2. By the previous claim, x_1 and x_2 are non-adjacent. If there is a path from x_1 to some vertex of P^*, which does not contain w and x_2, then we can easily find a path of length n which contains both x_1 and w. Similarly, we find a good n-path, if there exists a path from x_2 to a vertex of P^*, which does not contain x_1 and w. (Again, we do not exclude the possibility that some of x_1, x_2, w may belong to P^*.) Otherwise, each path with one endvertex in $\{x_1, x_2\}$ and other one in P^* must contain w. So, we can conclude that w is a cut-vertex of G, and it departs P^* from x_1 and x_2. In this case, let Q^* be a shortest path from w to P^*.

Suppose first that x_1 and x_2 belong to a same block of G. Let R be a shortest path between x_1, x_2 which does not contain w. Since x_1, x_2 are from the same block, the path R exists. If R is of length $\geq n-2$, then the cycle wR contains a good n-path. And, if R is of length smaller than $n-2$, then using a subpath of $Q^* \cup P^*$, the path R can be extended to a good n-path.

Now, we may assume that x_1 and x_2 belong to different blocks of G. Here, we argue similarly as above. Let R be a longest path in $G-w$, which contains x_2 as an endvertex. Denote by x_2^* the other endvertex of R. If R is of length $\geq n-3$, then the path $x_1 wR$ contains a good n-path. Otherwise, R is of length $< n-3$. By the choice of R it follows that all neighbours of x_2^* in $G-u$ belong to R. Hence, x_2^* is of degree $\leq n-3$ in $G-w$, and so it is of degree $\leq n-2$ in G. So, x_2^* is a small vertex. Hence, R can be extended to a good n-path with the vertices from $Q^* \cup P^*$. This proves the claim.

From the last claim, it follows that the planar graph, constructed from G by removing the small vertices, has a minimum degree at least $m-1$. But it is a contradiction and the end of the proof. □

We conclude this section with the following table for the gravity of the paths in $\mathcal{P}_2$. For $n = 1, 2, 3$, the values follow by Proposition 2.2. So, assume that $n \geq 4$. Theorem 2.4 gives us that the gravity of the n-path is at most the value in the corresponding entry of the table. For $n = 3, 4, 5$ consider the graph $K_{2,m}$. For $n = 6, 7, 8$ consider $T_{m,1}(K_{2,m}, v)$ where v is one of the two m-vertices in $K_{2,m}$. Let S be a $T(m, C_3)$-star. For $n \geq 10$, construct the following graph: identify each endvertex of P_{n-6} with the root of a copy of $T_{m,2}(S)$ and identify each inner vertex of P_{n-6} with the center of a copy of S. Finally, for $n = 9$ we use a similar construction: identify each endvertex of P_4 with the root of a copy of $T_{m,3}(S)$ and identify each inner vertex of P_4 with the center of a copy of S. In so constructed graph, each n-path contains $n-2$ big vertices.

n	1	2	3	4	5	6	7	8	9	≥ 10
$g(P_n, \mathcal{P}_2)$	0	1	1	2	2 or 3	4	4 or 5	5 or 6	6 or 7	$n-2$

Table 1: The gravity of n-paths in $\mathcal{P}_2$

The table leaves for the n-path P_n with $n \in \{5, 7, 8, 9\}$, the question whether its gravity is $n-3$ or $n-2$. The authors of this paper inspect that the right bound is $n-3$.

2.3.1 The gravity of paths in the subfamily of 2-connected graphs of $\mathcal{P}_2$

Considering the lower bound of the gravity in Table 1, all presented graphs have the property that each big vertex is a cut-vertex, and so those graphs have many blocks. However, the next result shows that restriction to the family of 2-connected graphs of $\mathcal{P}_2$, call it $\mathcal{P}_2^*$, the gravity of infinitely many n-paths P_n is asymptotically close to n. Perhaps, the claim remains valid if we consider the realm of all paths and not only the infinite subfamily.

Proposition 2.5 *For infinitely many k, the k-path P_k has the gravity of order $k-o(k)$ in $\mathcal{P}_2^*$.*

Proof. Let G_i be an $(S_i, u_i, v_i; m)$-melon, where $S_1 := P_3$ with poles being the vertices of degree 1, and S_i is obtained from G_{i-1} by joining the poles of G_{i-1} with two new vertices u_i, v_i.

Firstly, we will determine the length of the longest path L_i in G_i. Observe that L_i is contained in precisely three slices of G_i. Let $S_i^{(1)}$ be the slice where L_i starts, $S_i^{(2)}$ be the one that L_i passes through and $S_i^{(3)}$ be the one where L_i ends. Then L_i contains two poles of G_i, $2i-1$ vertices of the pulp of $S_i^{(2)}$, and $2i^2$ vertices in pulps of $S_i^{(1)}$ and $S_i^{(3)}$ (the first and last parts of L_i in $S_i^{(1)}$ and $S_i^{(3)}$ together lie in four slices $S_{i-1}^{(1)}, S_{i-1}^{(2)}, S_{i-1}^{(2)}$ and $S_{i-1}^{(3)}$ of two copies of G_{i-1}; then the result follows by induction). Thus, L_i has $2i(i+1)+1$ vertices.

Next, we will determine how many vertices of L_i are big. In $S_i^{(2)}$, L_i passes through exactly one 2-vertex; in each of $S_i^{(1)}$ and $S_i^{(3)}$, L_i passes through i 2-vertices. All other vertices of L_i are big. Thus, L_i contains $2i(i+1)+1-1-2i = 2i^2$ big vertices. Counting the ratio of big and all vertices of L_i, we obtain that it is $\geq \frac{2i^2}{2i(i+1)+1} = 1 - \frac{2i+1}{2i^2+2i+1}$.

Now, let l be a large integer, $2(i-1)i+1 < l < 2i(i+1)+1$. Consider a path L of length l in the graph G_i. Then L is contained in two or three slices of G_i which separate it to two or three subpaths. Each of these subpaths contains at most i 2-vertices. Hence, roughly counting, L consists of at least $l-3i$ big vertices and the ratio of big and all vertices of L is $\geq \frac{l-3i}{l} = 1 - \frac{3i}{l} > 1 - \frac{3i}{2(i-1)i+1} = 1 - \frac{3}{i} = 1 - o(1)$.

From the estimations above, the claim follows. □

By similar constructions, one can obtain similar asymptotical results also for the families of all 2-connected plane graphs of minimum degree 3, 4 and 5.

3 Almost-light graphs

We say that a graph G is *almost-light* in the family $\mathcal{H}$ if it is 1-heavy and every its proper connected subgraph is light. Unfortunately, the converse is not true, just consider the odd cycles and the family $\wp$ (see Theorem 4.1).

Let $\mathcal{AL}(\mathcal{G})$ be the set of all almost-light graphs in the family $\mathcal{G}$. Given a set X of graphs of a family $\mathcal{G}$, let $\overline{X}$ denote the set of graphs of $\mathcal{G}$ such that for each graph $G \in \overline{X}$, every proper connected subgraph of G belongs to X. Now, we immediately obtain that $\mathcal{AL}(\mathcal{G}) \subseteq \overline{\mathcal{L}(\mathcal{G})}$.

For $\mathcal{L}(\mathcal{G})$ finite, the set of heavy graphs in $\mathcal{G}$ may be infinite, as seen from Lemma 2.2. On the other hand, in this case only the finite number of graphs has to be examined for specifying the set $\mathcal{AL}(\mathcal{G})$. Note that for the families $\mathcal{P}_1$ and $\mathcal{P}_2$, the 2-path P_2 is the only almost-light graph. The next theorem describes the set of almost-light graphs in another three families of plane graphs, where the characterization of light graphs is complete (see [11]).

Theorem 3.1 *For the families of graphs $\mathcal{P}_3$, $\mathcal{P}_3(7)$, $\mathcal{P}_4$, it holds*

(a) $\mathcal{AL}(\mathcal{P}_3) = \{P_3\}$,

(b) $\mathcal{AL}(\mathcal{P}_3(7)) = \{P_4, S_3\}$,

(c) $\mathcal{AL}(\mathcal{P}_4) = \{C_3, S_3, P_5\}$.

Proof. **(a)** We have $\mathcal{L}(\mathcal{P}_3) = \{P_1, P_2\}$ and $\overline{\mathcal{L}(\mathcal{P}_3)} = \{P_1, P_2, P_3\}$. Then, the result follows immediately by Lemma 2.2.

(b) We have $\mathcal{L}(\mathcal{P}_3(7)) = \{P_1, P_2, P_3\}$ and $\overline{\mathcal{L}(\mathcal{P}_3(7))} = \{P_1, P_2, P_3, P_4, C_3, S_3\}$. By Lemma 2.2, gravity of S_3 in $\mathcal{P}_3$ and $\mathcal{P}_4$ is 1. Since $\mathcal{P}_4 \subseteq \mathcal{P}_3(7) \subseteq \mathcal{P}_3$, by (O) it follows that gravity of S_3 in $\mathcal{P}_3$ is 1.

Next, we will show that $C_3 \notin \mathcal{AL}(\mathcal{P}_3(7))$. Let S be the plane graph comprised of two 8-cycles $B = a_1b_1a_2b_2a_3b_3a_4b_4$, $C = a_1c_1a_2c_2a_3c_3a_4c_4$ and two vertices b, c such that $N(b) = \{b_1, \ldots, b_4\}$, $N(c) = \{c_1, \ldots, c_4\}$; further, let u, v be two adjacent vertices of the outer face $a_4b_4a_1c_4$ of S.

Let m be an integer. Consider a 3-cycle $C_3 = x_1x_2x_3$ and, for each $i = 1, 2, 3$ construct the $(S, u, v; m)$-melon G^i. Then identify endvertices of each edge x_ix_{i+1} (index is taken modulo 3) with poles u^i, v^i (the counterparts of u, v in G^i). The resulting graph $\widetilde{G}$ belongs to $\mathcal{P}_3(7)$ and contains only one 3-cycle, all vertices of which are big. Thus $g(C_3, \mathcal{P}_3(7)) = 3$, which establishes the claim.

It remains to show that the gravity of P_4 in $\mathcal{P}_3(7)$ is 1. If it is not true, then for every integer m, there exists a graph $G_m \in \mathcal{P}_3(7)$ in which every 4-path P_4 contains at least two big vertices. Consider the initial charge assignment by (1) with $\alpha = \frac{3}{2}$ for $G := G_m$, and the following discharging rules:

Rule R1: Each big vertex sends $\frac{3}{4}$ to each adjacent 3-vertex.

Rule R2: Each big vertex sends $\frac{3}{2k}$ to each incident triangular face f where $k \in \{1, 2, 3\}$ is the number of big vertices incident with f.

It is enough to deal only with 3-vertices, 3-faces, and big vertices. Each 3-vertex is adjacent with at least two big vertices (otherwise, it is adjacent with at least two vertices of degree $< m$ and, together, they can be extended to a 4-path containing at most one big vertex, a contradiction). Hence, by R1, its final charge is nonnegative. Similarly, each triangular face is incident with at least one big vertex and, by R2, it has also nonnegative final charge.

Consider a big vertex x of degree $d \geq m$. If x is not incident with a triangular face, then $c^*(x) \geq \frac{3}{2}d - 6 - d \cdot \frac{3}{4} \geq 0$ for large m. If x is incident with a triangular face α having two remaining vertices of degree $< m$, then all remaining $d - 2$ neighbours of x are big (otherwise, the vertices of α can be extended to a 4-path contradicting the assumptions on G). Thus, $c^*(v) \geq \frac{3}{2}d - 6 - 2 \cdot \frac{3}{4} - 2 \cdot \frac{3}{4} - \frac{3}{6} \cdot (d - 2) \geq 0$ for m large enough. So we can assume that every triangular face incident with x contains at least two big vertices. Denote by $x_1, \dots, x_d$ the neighbours of x as they appear around x in a cyclic order, and by f_i the face which contains the subwalk x_ixx_{i+1}. Considering, for each $i \in \{1, \dots, d\}$ the three consecutive vertices x_i, x_{i+1}, x_{i+2} (indices are taken modulo d), the total charge sent from x to these vertices and the faces f_i, f_{i+1}, f_{i+2} is always ≤ 3. This implies that x sends at most d to all its adjacent vertices and incident faces. Thus, $c^*(x) \geq \frac{3}{2}d - 6 - \frac{15}{16} \cdot d \geq 0$ for m being large enough. This completes the proof.

(c) We have $\mathcal{L}(\mathcal{P}_4) = \{P_1, P_2, P_3, P_4\}$ and $\overline{\mathcal{L}(\mathcal{P}_4)} = \{P_1, P_2, P_3, P_4, P_5, C_3, C_4, S_3\}$.

First, we show that $g(C_4, \mathcal{P}_4) \geq 2$. Fix m integer and take the graph S of the Archimedean polytope of type $(3,5,3,5)$. This graph can be obtained from dodecahedron by cutting its vertices in such a way that the resulting graph is 4-regular. Let u, v be two nonadjacent vertices of S lying on the outer pentagonal face of S and let $\widetilde{G}$ be the $(S, u, v; m)$-melon. Now, take C_4 and identify endvertices of each its edge with poles of a copy of $\widetilde{G}$. It is easy to see that, in the resulting graph, every 4-cycle contains at least two vertices of degree $\geq m$.

Since C_3 is heavy in $\mathcal{P}_4$, the gravity $g(C_3, \mathcal{P}_4) \geq 1$. Assume $g(C_3, \mathcal{P}_4) \geq 2$. Then for each m there is a graph $G_m \in \mathcal{P}_4$ such that every its triangle contains at least two big vertices. Consider the initial charge assignment by (1) with $\alpha = \frac{3}{2}$ and the following discharging rule:

Rule R3: Each big vertex sends $\frac{3}{4}$ to each incident triangular face.

It is enough to consider only the triangular faces and the big vertices. If f is a triangular face, it contains at least two big vertices, thus $c^*(f) \geq -\frac{3}{2} + 2 \cdot \frac{3}{4} = 0$. And, if v is a big vertex of degree d, then $c^*(v) \geq \frac{3}{2}d - 6 - \frac{3}{4}d \geq 0$. Thus, the proof is complete.

In what follows we show that the gravity of P_5 is 1 in $\mathcal{P}_4$. So suppose that it is false. Then, for every integer m there exists a graph $G_m \in \mathcal{P}_4$ such that every 5-path in G contains at least two big vertices. We say that two adjacent vertices u and v are *j-adjacent* ($j \in \{0,1,2\}$), if the edge uv is incident with precisely j triangular faces. Consider the initial charge assignment by (1) with $\alpha = 1$ and the following discharging rules:

Rule R4: Each face f of degree ≥ 4 sends $\frac{1}{2}$ to each incident vertex of degree 4 or 5.

Rule R5: Each big vertex sends $\frac{1}{3}$ to each adjacent 5-vertex and $\frac{2+j}{4}$ to each j-adjacent 4-vertex.

It is easy to see that each face of G has even positive final charge. Consider a 5-vertex x. If x is incident with at least two faces of degree ≥ 4, then $c^*(x) \geq -1 + 2 \cdot \frac{1}{2} = 0$; otherwise, at least three of the neighbours of x are big and $c^*(x) \geq -1 + 3 \cdot \frac{1}{3} = 0$.

Now, let x be a 4-vertex. If x is not incident with a triangular face, then $c^*(x) \geq -2 + 4 \cdot \frac{1}{2} = 0$. Otherwise, x is adjacent with at least two big vertices and, as it can be checked in a routine manner, they send in total at least $\frac{l}{2}$ to x, where l is the number of triangular faces incident with x. Thus, the final charge of x is nonnegative.

Finally, let x be a big vertex of degree $d \geq m$ and $x_1, \ldots, x_d$ be its neighbours. Observe that if there are three consecutive 4-vertices x_{i-1}, x_i, x_{i+1} each of them receiving 1 from x, then all remaining neighbours of x must be necessarily big, thus $c^*(x) \geq 0$. So, by rough estimation, we obtain that each three consecutive neighbours of x receive $< 1 + 1 + \frac{3}{4}$, which again ensures $c^*(x) \geq 0$ for m being large enough. This completes the proof. $\square$

4 Absolutely heavy graphs

According to its definition, the gravity of a graph H in a given family $\mathcal{H}$ is bounded above by the number of its vertices. In the case of equality, we will say that H is *absolutely heavy* in $\mathcal{H}$. The following theorem shows that there are many absolutely heavy graphs in the families of graphs which are subject of our consideration.

Theorem 4.1 *(a) Every planar graph of minimum degree greater than* d *is absolutely heavy in* $\mathcal{P}_d$*, where* $d \in \{1, 2, 3, 4\}$.

(b) Every planar graph which is not a tree is absolutely heavy in $\mathcal{P}_1$.

(c) Infinitely many trees are absolutely heavy in $\mathcal{P}_1$.

(d) Every cycle is absolutely heavy in $\mathcal{P}_3$.

(e) Each of the cycles C_4, C_6, C_8, C_{10}, *and* C_n *with* n *odd is absolutely heavy in* $\wp$.

Proof. For the purpose of the proof, let Q^-, D^-, O^- and K_4^- be the cube, the dodecahedron, the octahedron, and K_4 minus one edge, respectively. If not stated otherwise, the endvertices of the deleted edge will be referred as u and v in this proof.

(a) For given d and m, we construct a graph $\widetilde{G} \in \mathcal{P}_d$ in which every isomorphic copy of G consists only of big vertices.

If $d = 1$, $\widetilde{G}$ is constructed by identifying each vertex of G with the central vertex of a copy of the m-star. If $d > 1$, we form the basic slice S firstly. For $d = 2$, let S be a 3-path with endvertices u, v. For $d = 3$, let S be K_4^- and for $d = 4$ let S be O^-.

Let M be the $(S, u, v; m)$-melon. Identify the endvertices of each edge of G with poles of a copy of M. It is easy to see that, in the resulting graph

$\widetilde{G}$, the only subgraph isomorphic to G is the original one and all its vertices are big.

(b) Let G' be the maximal subgraph of G, whose all vertices are of degree > 1, i.e. G' is a maximal (induced) 2-degenerated subgraph of G. Then, the set $E(G) \setminus E(G')$ induces a set of trees. Each of these trees is considered as a rooted tree with the root being the unique common vertex with G'. Let h be the maximum of heights of these rooted trees. Construct the graph $\widetilde{G}$ in the following way: identify each vertex of G with the root of a copy of $T_{m,h+1}$. In $\widetilde{G}$, every vertex of each subgraph isomorphic to G is big.

(c) To show that there are trees which are absolutely heavy in $\mathcal{P}_1$, consider the k-star S_k with $k \geq 3$ with edges $e_1, \dots, e_k$. Now, subdivide each edge e_i with 10^x new vertices; denote by T' the obtained tree. Now, identify each vertex of T' with the central vertex of a copy of the m-star. In the resulting graph, every isomorphic copy of T' contains the center of the original S_k, and every its vertex is big. Notice that this construction does not work for case $k = 2$, since paths are not absolutely heavy in $\mathcal{P}_1$ (see Theorem 2.3).

(d) For each n and m, we construct a graph $G \in \mathcal{P}_3$ in which every n-cycle consists only of big vertices. Firstly we form the basic slice S. For n odd, let S be Q^-. For n even, $n \geq 8$, let S be K_4^-. For $n = 4$ or 6, let S be D^-.

Let M be the $(S, u, v; m)$-melon. Then G is obtained from C_n by identifying endvertices of each its edge with the poles of a copy of M. It is easy to see that, in the resulting graph, the only n-cycle is the original one and all its vertices are big.

(e) Recall that the dual of the m-antiprism can be constructed from a $2m$-cycle $a_1 b_1 a_2 b_2 \dots a_m b_m$ and two vertices a and b where a is adjacent to $a_1, \dots, a_m$ and b is adjacent to $b_1, \dots, b_m$. Let S be the graph obtained from the dual of m-antiprism by deleting the 3-vertex a_1. Notice that b_1 and b_m are only 2-vertices in S.

Take an n-cycle $C_n = x_1 x_2 \dots x_n$ with $n \geq 3$ odd, and for each x_i take a new copy S^i of S. Next, identify x_i with a vertex b^i (the counterpart vertex in S^i of b). In addition, connect the vertex b_m^i with b_1^{i+1}, where these two vertices are counterparts of b_m and b_1 in S^i and S^{i+1}, respectively. The resulting graph $\widetilde{G}$ is a 3-connected planar graph with only two faces of odd size, one of them is a $5n$-cycle and the other one is the original n-cycle C_n. Moreover, all other faces of $\widetilde{G}$ are of size 4. Using the basis of the cycle space of $\widetilde{G}$ consisting of all its facial cycles, one can show that each odd cycle of $\widetilde{G}$ contains $x_i x_{i+1}$ or $b^i_m b^{i+1}_1$ for each $i = 1, \ldots, n$. This easily implies that the only n-cycle in $\widetilde{G}$ is $x_1 x_2 \ldots x_n$. Since each X_i is a big vertex in $\widetilde{G}$, we conclude that C_n with n odd is absolutely heavy in $\wp$.

For cycles of length 4 or 6, we will use the following construction: Take two paths $P_a = a_1 a_2 \ldots a_m$, $P_b = b_1 b_2 \ldots b_{m-1}$ and two additional vertices a and b. Add edges aa_i for $i = 1, \ldots, m$ and bb_j for $j = 1, \ldots, m-1$. Next, subdivide each edge $a_i a_{i+1}$, $i = 1, \ldots, m-1$ by two new vertices u_i and v_i such that $a_i u_i, u_i v_i$ and $v_i a_{i+1}$ are edges. Similarly, subdivide each edge $b_j b_{j+1}$, $j = 1, \ldots, m-2$ by two new vertices w_j, z_j such that $b_j w_j, w_j z_j, z_j b_{j+1}$ are edges. Add new edges $v_j w_j$ and $z_j u_{j+1}$ for $j = 1, \ldots, m-2$. Then, add two new vertices k, l and edges kb_1, ku_1, lb_{m-1} and lv_{m-1}. In this way, we obtain a graph S with exactly one 10-face $aa_m v_{m-1} l b_{m-1} b b_1 k u_1 a_1$; all other faces of S are 5-faces.

Now, take an n-cycle $C_n = x_1 x_2 \ldots x_n$ with $n = 4$ or 6, and for each x_i take a new copy S^i of S. Next, identify x_i with the vertex b^i (the counterpart vertex in S^i of b). In addition, connect the vertex a^i_m with a^{i+1}_1 and identify the vertex l^i with the vertex k^{i+1}, where these four vertices are counterparts of a_m, a_1, k and l in S^i and S^{i+1}, respectively (see Figure 1 for the case $n = 4$).

The resulting graph $\widetilde{G}$ is a 3-connected planar graph with only one face of size $3n$, only one face of size n (the original n-cycle C_n) and all remaining faces being pentagonal. By a routine check, one can easily check that no graph S^i, $i = 1, \ldots, 8$ contains an n-cycle; subsequently, in $\widetilde{G}$, there is no n-cycle C^* such that $E(C^*) \cap E(C_n) = \emptyset$. But then (since the distance of x_i, x_{i+1}, $i \in \{1, \ldots, n\}$ in the graph $\widetilde{G} - E(C_n)$ is at least 3, and the distance of x_i, x_{i+2} in $\widetilde{G} - E(C_n)$ is at least 9) each n-cycle of $\widetilde{G}$ uses all four edges of the original C_n. Hence, the only n-cycle of $\widetilde{G}$ consists only of big vertices.

For the 8-cycle, the construction proceeds in the following way: for each $j = 1, \ldots, m$, take a new copy C^j of the configuration C of the Figure 2. Next, for $k = 1, \ldots, 5$ and $j = 1, \ldots, m-1$, connect the half-edges f^j_k with e^{j+1}_k (the counterpart half-edges in C^j and C^{j+1} of f_k and e_k) and identify all vertices x^l, $l = 1, \ldots, m$ (the counterparts in S_l of x); let x be the vertex resulted from this identification. The configuration S obtained in this way consists of 5- and 6-gons and it contains half-edges $\widehat{e}_k = e^1_k$, $\widehat{f}_k = f^m_k$, $k = 1, \ldots, 5$ lying in the "outerface".

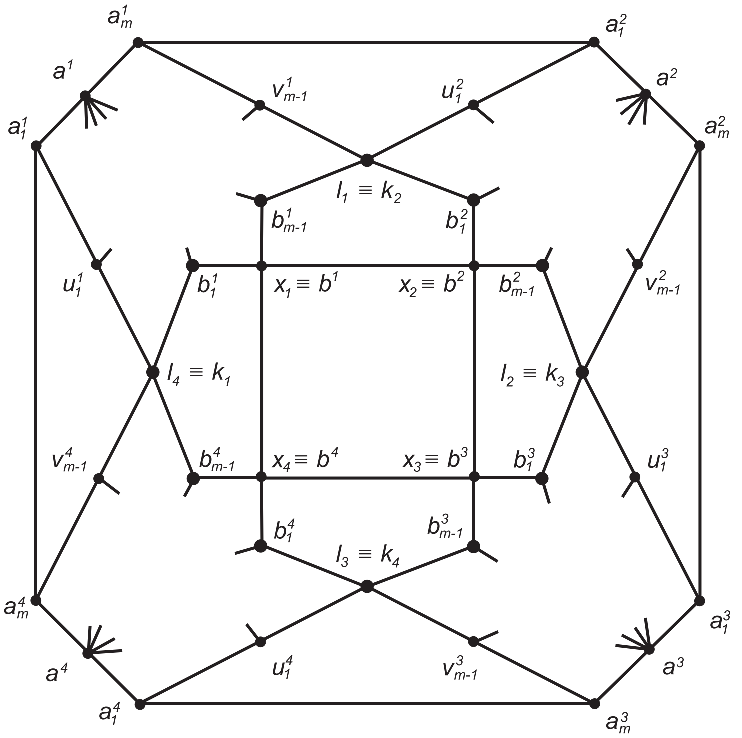

Figure 1: Graph $\widetilde{G}$ for $n = 4$

Now, take an 8-cycle $C_8 = x_1 x_2 \ldots x_8$ and for each x_i take a new copy S^i of S. Next, identify x_i with the vertex x^i (the counterpart vertex in S^i of x). In addition, connect the half-edges $\widehat{f}_k^i$ with the half-edges $\widehat{e}_k^{i+1}$ for each $k = 1, \ldots, 5$ (index i is taken modulo 8), as depicted on Figure 3. The resulting plane 3-connected graph $\widetilde{G}$ consists of 5-, 6- and 7-faces, exactly one 40-face and exactly one 8-face (which is the only 8-cycle in this graph), all of which vertices are big; the argument to show this is the similar like the one for the case $n = 4$ or 6.

For the 10-cycle, the construction of the configuration S is the same as for the 8-cycle (with the configuration C' of the Figure 4 instead of C). Also, the construction of the graph $\widetilde{G}$ is similar; just, when connecting the half-edges of S^i and S^{i+1}, additional vertices have to be introduced, see

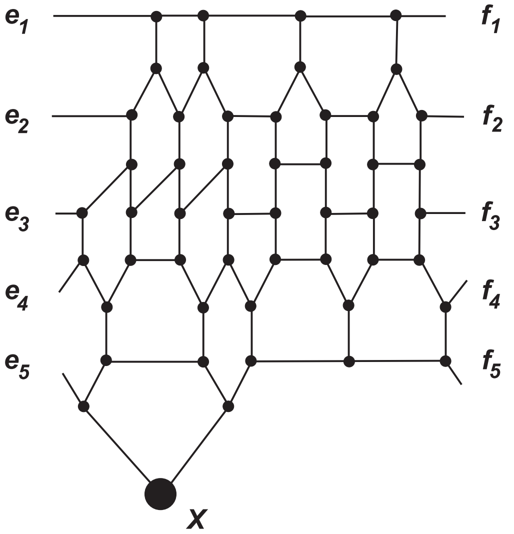

Figure 2: Configuration C

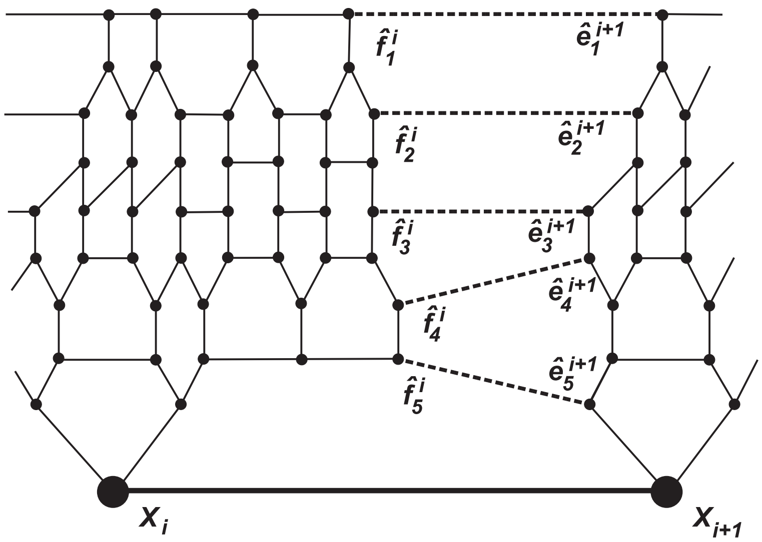

Figure 3: Interconnection of two C-configurations

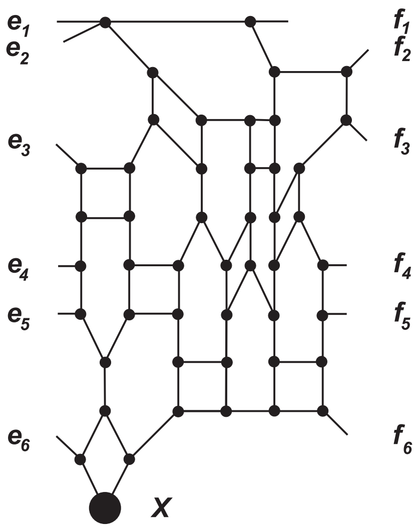

Figure 4: Configuration C'

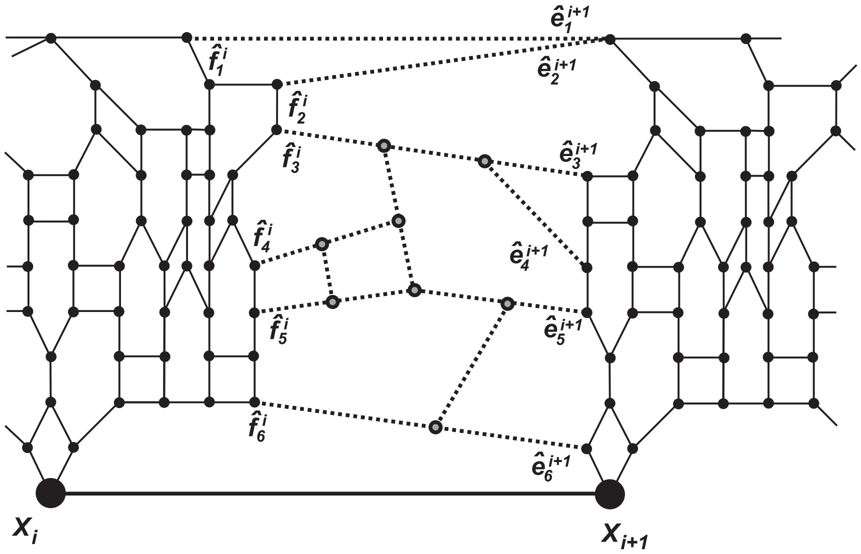

Figure 5: Interconnection of two C'-configurations

Figure 5. The graph $\widetilde{G}$ consists of 4-, 7- and 9-faces, exactly one 40-face and exactly one 10-face (which is the only 10-cycle in this graph), all of which vertices are big.

$\square$

5 Some problems

One possible further work is to study the gravity of the paths in $\mathcal{P}_3, \mathcal{P}_4$ or $\mathcal{P}_5$ as well as polishing the few left cases from the Table 2.3. However, we conclude the paper by posing few problems about the gravity of the graphs in $\wp$.

Problem 5.1 *For given integer n, are there infinitely many n-heavy graphs in the family $\wp$?*

Regading above problem, consider all k-stars with a quadrangle attached to one leaf. In $\wp$, each such graph has gravity ≥ 2 (see the dual of m-sided antiprism). Regarding the next problem, notice that by Theorem 4.1(e), the open cases are the even cycles of length $g \geq 12$.

Problem 5.2 *Is every cycle absolutely heavy in the family $\wp$?*

An important example of a family of plane graphs with infinite set of light graphs is $\wp$ (see [3]), where $\mathcal{L}(\wp) = \{P_k, k \geq 1\}$. Hence, $\overline{\mathcal{L}(\wp)} = \{P_k, k \geq 1\} \cup \{C_k, k \geq 1\} \cup \{S_3\}$ and by Theorem 4.1 (d), $\mathcal{AL}(\wp) \subseteq \{S_3\} \cup \{C_k, k \geq 12 \text{ even}\}$.

Problem 5.3 *Find all almost-light graphs and find all 1-heavy graphs in the family $\wp$.*

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