Antisymmetric Flows in Matroids

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Abstract

We present a seemingly new definition of flows and flow numbers for oriented matroids and prove that the *flow number* $\Phi_{\mathcal{L}}$ and the *antisymmetric flow number* $\Phi_{\mathcal{L}as}$ of an oriented matroid are bounded with its rank. In particular we show that if $\mathcal{O}$ is an oriented matroid of rank r then $\Phi_{\mathcal{L}}(\mathcal{O}) \leq r+2$ and $\Phi_{\mathcal{L}as}(\mathcal{O}) \leq 3^{\lfloor \frac{9}{2} r \rfloor + 1}$.

Furthermore, we introduce the notion of a semiflow and show that each oriented matroid has an antisymmetric 3-NZ-semiflow.

1 Introduction

When considering flows in matroids the main focus has been on the existence of packings of paths under capacity restrictions [8, 9]. Far less attention has been payed to a possible generalization of the theory of nowhere-zero flows.

Goddyn, Tarsi and Zhang [3] introduced a generalization of the circular flow number to regular and to oriented matroids. Goddyn, Hliněný and Hochstättler [2] renamed this parameter into *oriented flow number* Φ_o and showed that it is bounded for oriented matroids of bounded rank.

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Here, we present a different generalization of the flow number to oriented matroids, denoted by $\Phi_{\mathcal{L}}$, using the integer lattice generated by the characteristic vectors of oriented circuits. In general, this parameter differs from the oriented flow number of [2]. In particular, we compute $\Phi_{\mathcal{L}}$ for uniform oriented matroids and show that it is a matroid invariant in this case. Similarly, we generalize the notion of an *antisymmetric flow*, introduced by Nešetřil and Raspaud [5], from digraphs to oriented matroids and show that the corresponding flow number is well defined and bounded by a function of the rank of the underlying matroid.

Finally, we consider the integer lattice generated by oriented circuits and cocircuits, yielding sums of flows and coflows, which we call *semiflows*. We show that any oriented matroid for any $l \in \mathbb{Z}$ has a semiflow using only the values l and $l+1$.

Our notation is fairly standard. We assume familiarity with basics of oriented matroid theory and of matroid theory, standard references are [1, 6]. We say that a matroid is *cosimple* if every cocircuit has at least three elements. By $r : E \to \mathbb{N}$ we denote the rank function of the matroid in discussion.

2 A Remark on Kirchhoff's Law

Hartmann and Schneider [4] generalized max-balanced flows to oriented matroids by requiring that a flow $\vec{v}$ satisfies the "max-version of Kirchhoff's law" for all oriented cocircuits $\vec{X} = (X^+, X^-)$, i.e. that

$$\max_{e \in X^+} \vec{v}(e) = \max_{e \in X^-} \vec{v}(e). \tag{1}$$

In the setting of nowhere-zero flows an attempt to proceed in a similar fashion by requiring

$$\sum_{e \in X^+} \vec{v}(e) = \sum_{e \in X^-} \vec{v}(e). \tag{2}$$

has the following drawback.

Example 1 *Let $\mathcal{O}$ be the rank two oriented matroid associated to n points on the real line. Then up to symmetry $\mathcal{O}$ has n cocircuits $X_1, \dots, X_n$ where $X_i^+ = \{1, \dots, i-1\}$ and $X_i^- = \{i+1, \dots, n\}$. The conditions from (2) can be written as $Af = 0$ where A is the square matrix that has -1s under, 0s on and 1s above the diagonal. Note, that A is unimodular, it has determinant 1, if n is even and singular if n is odd.*

Since a flow $\vec{v}$ that satisfies (2) for all cocircuits induces such a flow on each contraction minor, the former implies that such an $\vec{v}$ must be zero on each set of points P that can be contracted to an even line. To be more precise: For some set $D \subseteq E$: $E \setminus D$ has to consist of the even line P and possibly some loops in $\mathcal{O}/D$.

3 Nowhere-Zero Flows in Orientable Matroids

Defining flows as integer sums of circuits instead, seems to be more appropriate in our setting.

Definition 1 *Let $\mathcal{O}$ denote an oriented matroid on a finite set E with circuits $\vec{\mathcal{C}}$. We denote by $\chi_{\vec{C}} : E \to \{0, 1, -1\}$ the characteristic vector of $\vec{C} \in \vec{\mathcal{C}}$ and by $\vec{\mathcal{F}}$ the integer lattice (free integer module) generated by the characteristic vectors of circuits.*

$$\vec{\mathcal{F}} := \{\sum_{\vec{C} \in \mathcal{C}} \lambda_{\vec{C}} \chi_{\vec{C}} \mid \lambda_{\vec{C}} \in \mathbb{Z}\}. \tag{3}$$

A flow *in $\mathcal{O}$ is any $\vec{v} \in \vec{\mathcal{F}}$. The flow is said to be a k-*flow, *if $|\vec{v}(e)| \leq k - 1$ for each $e \in E$, it is* nowhere zero *or an* NZ-flow *if $\vec{v}(e) \neq 0$ for all $e \in E$. The* flow number *$\Phi_{\mathcal{L}}(\mathcal{O})$ of an oriented matroid is the smallest $k \in \mathbb{N}$ such that there exists a NZ-k-flow.*

Remark 1 *The existence of a NZ-k-flow is invariant under reorientation of the oriented matroid. It might still vary for different reorientation classes of an orientable matroid, though.*

Even for orientable matroids with a unique reorientation class, e.g. regular matroids, it is crucial to define the parameter via some orientation and characteristic functions of oriented circuits. Using the characteristic functions of circuits of the underlying matroid instead, does not suffice, e.g. the Petersen graph does not have a NZ-4-flow, but admits a cycle double cover.

As a continuation of Example 1 we compute $\Phi_{\mathcal{L}}(\mathcal{O})$ for uniform oriented matroids.

Theorem 1 *Let $\mathcal{O}$ be a uniform oriented matroid on $E = \{1, \ldots, n\}$ of rank $d \leq n + 1$. Then*

$$\Phi_{\mathcal{L}}(\mathcal{O}) = \begin{cases} 2 & \text{if} \quad nd \text{ is even,} \\ 3 & \text{if} \quad nd \text{ is odd.} \end{cases}$$

In particular $\Phi_{\mathcal{L}}(\mathcal{O})$ is matroid invariant for uniform oriented matroids.

Proof. The claim is obvious, if $n-d=1$. First, we show by induction on $n-d \geq 2$ that

Claim 1 *If d is even, then the flow lattice is trivial, i.e. $\vec{\mathcal{F}} = \mathbb{Z}^n$.*

If $n = d+2$ then the reorientation class of $\mathcal{O}$ is unique, namely the dual of the n point line. Choosing a proper reorientation we may assume that the circuits, up to symmetry, are C_l for $l = 1, \ldots, n$

$$C_i = \begin{cases} + & \text{if} \quad i < l \\ 0 & \text{if} \quad i = l, \\ - & \text{if} \quad i > l. \end{cases}$$

By Example 1 their characteristic vectors form a unimodular matrix, which generates $\mathbb{Z}^n$ (see e.g. [7]). Now if $n-d > 3$, using inductive assumption for the deletion minors $\mathcal{O} \setminus \{1\}$ resp. $\mathcal{O} \setminus \{n\}$, we conclude that all unit vectors e_i are flow vectors implying the assertion. □

Now, assume d is odd.

Claim 2 *For all $e \neq f \in \{1, \ldots, n\}$ there is a flow vector $\vec{v} \in \vec{\mathcal{F}}$ such that $|\vec{v}(e)| = |\vec{v}(f)| = 1$ and $\vec{v}(g) = 0$ for all $g \in \{1, \ldots, n\} \setminus \{e, f\}$.*

Proof. Again we proceed by induction on $n-d \geq 2$. For $n = d+2$, as above we consider the matrix A which now has rank $n-1$. Its kernel is spanned by $k = (1, -1, 1, -1, \ldots, -1, 1)$. From this it is immediate, that $\vec{\mathcal{F}} = (k)^{\perp} \cap \mathbb{Z}^n$, implying the claim. Now, if $n-d > 2$, and $e, f \in E$, choose $g \in E \setminus \{e, f\}$. By inductive assumption there is a flow vector as desired in $\mathcal{O} \setminus \{g\}$ and the claim follows. □

The two Claims above imply that

$$\Phi_{\mathcal{L}}(\mathcal{O}) \leq \begin{cases} 2 & \text{if} \quad d \text{ is even,} \\ 2 & \text{if} \quad d \text{ is odd and} \quad n \text{ is even,} \\ 3 & \text{if} \quad d \text{ is odd and} \quad n \text{ is odd.} \end{cases}$$

Finally, we remark that if d is odd then any circuit has even length and thus $1^{\top} v$ must be an even number for each characteristic function of a signed circuit and thus also for flow vectors. This implies that $\Phi_{\mathcal{L}}(\mathcal{O}) \geq 3$ if d is odd and n is odd. □

We are not aware of an example of an orientable matroid with more than one reorientation class and where the flow numbers differ. The above theorem suggests that $\Phi_{\mathcal{L}}$ – contrary to Φ_o of [2] – might be matroid invariant.

Problem 1 *Does there exist an orientable matroid with orientations $\mathcal{O}_1, \mathcal{O}_2$ and $\Phi_{\mathcal{L}}(\mathcal{O}_1) \neq \Phi_{\mathcal{L}}(\mathcal{O}_2)$?*

We conclude this section with a straightforward upper bound on the flow number of oriented matroids of bounded rank.

Theorem 2 *Let $\mathcal{O}$ be an oriented matroid of rank r without a coloop. Then $\mathcal{O}$ has a NZ-$(r+2)$-flow.*

Proof. Let $E' \subseteq E$ be a maximal set that can be covered by pairwise disjoint oriented circuits $\vec{C}_1, \dots, \vec{C}_l$ and $\vec{D} = \vec{C}_1 \circ \dots \circ \vec{C}_l$. Then $|E \setminus E'| \leq r$. Furthermore, for each $e \in E \setminus E'$ there exists an oriented circuit $e \in \vec{C}_e$ conformal to D (see eg. [1] 3.7.6), i.e. $\mathrm{sep}(\vec{D}, \vec{C}_e) = \emptyset$. Then $\vec{f} = \sum_{i=1}^{l} \chi_{\vec{C}_i} + \sum_{e \in E \setminus E'} \chi_{\vec{C}_e}$ is a NZ-(r+2)-flow. $\square$

4 Antisymmetric Flows in Oriented Matroids

In this section we generalize the notion of antisymmetric flows, introduced by Nešetřil and Raspaud [5] for digraphs, to oriented matroids.

Definition 2 *A flow $\vec{v} \in \vec{\mathcal{F}}$ in an oriented matroid $\mathcal{O}$ is* antisymmetric *or an ASF if $\vec{v}(e) \neq -\vec{v}(g)$ holds for every pair $e \neq g \in E$. The* asymmetric flow number $\Phi_{\mathcal{L}as}(\mathcal{O})$ *of an oriented matroid is the smallest $k \in \mathbb{N}$ such that there exists an antisymmetric NZ-k-flow.*

Example 2 *We consider the uniform oriented matroids on $n \geq d+2$ elements but for even rank d only. As an immediate consequence of Claim 1 in the proof of Theorem 1 we have $\Phi_{\mathcal{L}as}(\mathcal{O}) = 2$. For odd d the situation becomes more difficult and $\Phi_{\mathcal{L}as}$ depends on the specific reorientation. As another consequence of the considerations in the proof of Theorem 1 we conclude that $(1,2,1,1,1)$ is a flow in the dual of the five-point-line. But if we reorient the second and fourth element then the lattice becomes $\mathcal{F} = (1,1,1,1,1)^{\perp} \cap \mathbb{Z}^5$. Since 5 is not divisible by 3 $\mathcal{F}$ does not contain a 3-ASF, but a 4-ASF $(3,3,-2,-2,-2) \in \mathcal{F}$.*

Next we show that the antisymmetric flow number is well defined and give a first upper bound.

Theorem 3 *Let $\mathcal{O}$ be an oriented matroid on a finite set E without a directed cocircuit of size 1 or 2. Then $\Phi_{\mathcal{L}as}(\mathcal{O}) \leq 3^{|E|-1}$.*

We proceed similar to the proof of Theorem 4 in [5]. We will use the following two lemmas that require a definition first:

Definition 3 *Let M be a matroid. A family $C_1, \ldots, C_t$ of circuits of M is* distinguishing, *if for any pair $\{e, e'\}$ of elements of E there exists $1 \leq i \leq t$ such that $|C_i \cap \{e, e'\}| = 1$.*

Lemma 1 *Let M be an orientable matroid and $\mathcal{O}$ one of its orientations. If M has a family $C_1, \ldots, C_t$ of distinguishing circuits, then $\mathcal{O}$ has a 3^t-ASF.*

Proof. Let $\vec{v} = \sum_{i=1}^{t} 3^{i-1} \chi_{\vec{C}_i}$. Let $e, e' \in E$ and C_j be distinguishing for this pair. Then

$$\begin{aligned} \vec{v}(e) + \vec{v}(e') &= \underbrace{\sum_{i=1}^{j-1} 3^{i-1}(\chi_{\vec{C}_i}(e) + \chi_{\vec{C}_i}(e'))}_{-3^{j-1} < \ldots < 3^{j-1}} + 3^{j-1}\chi_{\vec{C}_j}(e') \\ &+ \sum_{i=j+1}^{t} 3^{i-1}(\chi_{\vec{C}_i}(e) + \chi_{\vec{C}_i}(e')) \neq 0. \end{aligned}$$

□

In the next step we show that, given a basis B of M, the set of fundamental circuits can be augmented to a family of distinguishing circuits.

Lemma 2 *Let M be a cosimple matroid and B a basis. Then the set of fundamental circuits $(C_e)_{e \in E \setminus B}$ can be augmented to an distinguishing family $C_1, \ldots, C_t$ where $t \leq |E| - 1$.*

Proof. We consider the equivalence relation

$$e \sim e' \quad \Leftrightarrow \quad \forall d \in E \setminus B : |\{e, e'\} \cap C_d| \neq 1.$$

Clearly, no two elements in $E \setminus B$ are equivalent. Assume there exist $e \in B, e' \in E \setminus B$ such that $e \sim e'$. This means, the only fundamental circuit containing e is $C_{e'}$ implying that $r(E \setminus \{e, e'\}) < r(E)$ and thus, M is not cosimple, contradicting our assumptions on M. Therefore, any non-trivial class consists solely of elements of B.

Let $e \sim e'$. As M is cosimple, there exists a circuit $C_{\{e,e'\}}$ such that $|\{e, e'\} \cap C_{\{e,e'\}}| = 1$. Thus by adding at most $r(E) - 1$ circuits we can augment the set of fundamental circuits to a distinguishing family. The claim follows. □

Proof of Theorem 3. If $\mathcal{O}$ has a cocircuit $\{e, e'\}$ of size 2, by assumption it is not directed. Thus any flow satisfies $\vec{f}(e) = \vec{f}(e')$. Therefore, we may contract e' and assume without loss of generality that the matroid M underlying $\mathcal{O}$ is cosimple. Now, the assertion is an immediate consequence of Lemmas 1 and 2. $\square$

5 Oriented Matroids of Bounded Rank

We improve on the result of the last section and show that any oriented matroid without positive cocircuit of size 1 or 2 has a $3^{\lfloor\frac{9}{2}r(E)\rfloor+1}$-ASF.

We shall need the following:

Lemma 3 *Let M be a cosimple matroid on a finite set E and $I \subseteq E$. There exists $I \subseteq E' \subseteq E$ such that the restriction $M_{|E'}$ of M is of full rank, cosimple and $|E'| \leq 3r(E) + |I|$.*

Proof. Choose a basis B. Since M has no coloop, for each $b \in B$ there exists a $b' \in E \setminus B$ such that $B \setminus \{b\} \cup \{b'\}$ is a basis. If we augment B by these elements to get B', then B' has no coloop. Every cocircuit of size 2 forms a pair of parallel elements in the dual of $M_{|B'}$. Thus, by adding at most $r(E)$ further elements from E, since M is cosimple, we can eliminate all cocircuits of size 2 to get B''. Finally, since B'' contains a basis of M, $E' := B'' \cup I$ is still cosimple. $\square$

Remark 2 *Note, that the bound in Lemma 3 as stated is sharp. To see this consider a tree plus two parallels for each element and I a set of loops. The corresponding matroid is cosimple but no proper subset of the edges has the required property. But using the structure of the set E' of the following Theorem 4, it may well be possible to improve on the general bound for an ASF, presented here, using a similar technique.*

Theorem 4 *Let $\mathcal{O}$ be an oriented matroid without a directed cut of size 1 or 2. Then $\mathcal{O}$ has a $3^{\lfloor\frac{9}{2}r(E)\rfloor+1}$-ASF.*

Proof. Again we may assume that the matroid M underlying $\mathcal{O}$ is cosimple. We proceed in two steps.

Consider a maximal set $\vec{C}_1, \ldots, \vec{C}_k$ of disjoint circuits in $\mathcal{O}$ and set

$$\vec{v}_1 := \sum_{i=1}^{k} \chi_{\vec{C}_i}.$$

Let $F_1 := \vec{v}_1^{-1}(\{0,-1\})$. By choosing a proper orientation of the circuits we may assume that $|F_1| \le \frac{1}{2}(|E|-r)+r$. Let $\vec{D}_1,\ldots,\vec{D}_l$ denote a maximal set of pairwise disjoint circuits in the restriction $\mathcal{O}_{|F_1}$ to F_1 and

$$\vec{v}_2 := \vec{v}_1 + 3\sum_{i=1}^{l} \chi_{\vec{D}_i}.$$

Let $F_2 := \vec{v}_2^{-1}(\{0,-1,-3\})$. Note that $\vec{v}_2(E) \subseteq \{0,\pm 1, 2, \pm 3, -4\}$. By construction we have $\vec{v}_2^{-1}(\{0,-3,3\}) \le r$ and $\vec{v}_2^{-1}(\{0,-1\}) \le r$. Thus, choosing a proper orientation again, we may assume that $|F_2| \le \lfloor \frac{3}{2} r(E) \rfloor$. By Lemma 3 there exists a set $F_2 \subseteq E' \subseteq E$ such that $M_{|E'}$ is cosimple and $|E'| \le \lfloor \frac{9}{2} r(E) \rfloor$.

By Theorem 3 there exists a $3^{\lfloor \frac{9}{2} r(E) \rfloor -1}$-ASF $\vec{w}'$ for the oriented matroid $\mathcal{O}_{|E'}$. We extend this to a flow $\vec{w}$ for $\mathcal{O}$ by putting $\vec{w}(e) = 0$ on all $e \in E \setminus E'$. Finally, we set

$$\vec{v} := \vec{v}_2 + 3^2 * \vec{w}. \tag{4}$$

Then, clearly $\vec{v}$ is an $3^{\lfloor \frac{9}{2} r(E) \rfloor +1}$-flow and nowhere zero. We claim that $\vec{v}$ is antisymmetric. Let $e \neq e' \in E$ and consider the expression $\vec{v}(e)+\vec{v}(e') = \vec{v}_2(e)+\vec{v}_2(e')+9(\vec{w}(e)+\vec{w}(e'))$. Since $\vec{g}$ is antisymmetric $\vec{w}(e)+\vec{w}(e') \neq 0$ or $\{\vec{v}_2(e), \vec{v}_2(e')\} \subset \{1,2,3,-4\}$ and thus $|\vec{v}_2(e)+\vec{v}_2(e')| \le 8$. We conclude that $\vec{v}(e)+\vec{v}(e') \neq 0$.

□

6 Semiflows

Since the flow number is unbounded for cographic matroids, it is clear that even the minimum of the flow number of an oriented matroid and its dual is not bounded, even for regular matroids. Things get completely different when we allow sums of flows and *coflows (tensions)*, i.e. flows in the dual.

Definition 4 *Let $\mathcal{O}$ be an oriented matroid $\mathcal{O}$ with circuits $\mathcal{C}$ and cocircuits $\mathcal{C}^*$. A* semiflow *is any vector from*

$$\vec{\mathcal{F}} + \vec{\mathcal{F}}^* := \{\sum_{\vec{C}\in\mathcal{C}} \lambda_{\vec{C}}\chi_{\vec{C}} + \sum_{\vec{D}\in\mathcal{C}^*} \mu_{\vec{D}}\chi_{\vec{D}} \mid \lambda_{\vec{C}}, \mu_{\vec{D}} \in \mathbb{Z}\}. \tag{5}$$

*A semiflow $\vec{f}$ is said to be a k-*semiflow, *if $|\vec{f}(e)| \le k-1$ for each $e \in E$, it is* nowhere zero *or an* NZ-semiflow *if $\vec{f}(e) \neq 0$ for all $e \in E$.*

Example 3 *We consider uniform oriented matroids of odd dimension d and an odd number $n \geq d+2$ of elements. As the dimension of the dual matroid now is even, its flow lattice is trivial. Thus, all uniform oriented matroids admit a NZ-2-semiflow.*

It is well-known that each binary matroid can be partitioned into circuits and cocircuits. As a consequence, each regular matroid has a NZ-2-semiflow, too. Actually, we are not aware of any oriented matroid without a NZ-2-semiflow.

Problem 2 *Does there exist an oriented matroid which does not admit a NZ-2-semiflow?*

What we can show instead, is that given $l \in \mathbb{Z}$ each oriented matroid has a semiflow using only the numbers $\{l, l+1\}$. This is an easy consequence of the following lemma:

Lemma 4 *Let $\mathcal{O}$ be an oriented matroid on a finite set E, $e \in E$ and $k, l \in \mathbb{Z}$. There exists $\vec{f} \in \vec{\mathcal{F}} + \vec{\mathcal{F}}^*$ such that $\vec{f}(e) = k$ and $\vec{f}(g) \in \{l, l+1\}$ for $g \in E \setminus \{e\}$.*

Proof. We proceed by induction on $n = |E|$. If $n = 1$ then e is either a loop or a coloop and the assertion clearly is true. Thus assume $n \geq 2$. By inductive assumption $\mathcal{O} \setminus \{e\}$ has a semiflow using only the values l and $l+1$. This yields an $\vec{f}' \in \vec{\mathcal{F}} + \vec{\mathcal{F}}^*$ such that $\vec{f}'(g) \in \{l, l+1\}$ for $g \in E \setminus \{e\}$. We choose such an $\vec{f}'$ with $|k - \vec{f}'(e)|$ minimal. Assume $\vec{f}'(e) < k$ and let $I = \{g \in E \setminus e \mid f'(g) = l+1\}$. Applying Farkas' Lemma (see [1] 3.4.6) to ${}_{-I}\mathcal{O}$, i.e. the oriented matroid that arises from $\mathcal{O}$ by reorientation on I, yields a circuit $\vec{C}$ or a cocircuit $\vec{C}^*$ that is positive on e, non-negative for all g in $E \setminus \{e\}$ satisfying $\vec{f}'(g) = l$ and non-positive on h if $\vec{f}'(h) = l+1$. Then $\vec{f} := \vec{f}' + \chi_{\vec{C}}$ resp. $\vec{f} := \vec{f}' + \chi_{\vec{D}}$ is a semiflow satisfying $\vec{f}(g) \in \{l, l+1\}$ for $g \in E \setminus \{e\}$ and $\vec{f}'(e) < \vec{f}(e) \leq k$ contradicting our choice of $\vec{f}'$. For the case $\vec{f}'(e) > k$ consider $J := \{g \in E \setminus e \mid \vec{f}'(g) = l\}$, apply Farkas' Lemma to ${}_{-J}\mathcal{O}$ and subtract the resulting characteristic vector. □

This not only yields a NZ-3-semiflow but also an antisymmetric one. Call a semiflow f *antisymmetric* if $\vec{f}(E) \cap -\vec{f}(E) = \emptyset$.

Theorem 5 *Let $\mathcal{O}$ be an oriented matroid. Then $\mathcal{O}$ has an antisymmetric NZ-3-semiflow.*

□

Note, that this result is best possible even for graphs, as is easily seen considering an acyclic orientation of a triangle.

7 Remarks and Further Open Problems

We are not aware of previous studies on the integer lattice of circuits of an oriented matroid. Thus, it seems to be an open field to figure out, which concepts of flow theory generalize to this setting.

There are also several open problems that are related directly to the work presented here. Explicitely, we mention two further problems.

- Is Φ_o of [2] related to our flow number, e.g. do there exist constants $c_1, c_2 \in \mathbb{Q}$ such that $c_1\Phi_o \leq \Phi_{\mathcal{L}} \leq c_2\Phi_o$? Note, that $\Phi_o \leq \Phi_{\mathcal{L}}$, as valid in the regular case, is no longer true. As an example consider the 6 point line which has a NZ-2-flow in our setting but its oriented flow number is 2.5.
- Give lower bounds on the flow number and the asymmetric flow number for the class of oriented matroids of bounded rank. Can it be much worse than for graphic and cographic matroids, i.e. can the flow number become significantly larger than $\sqrt{r(M)}$?

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