Coloring graphs from lists with bounded size of their union

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Abstract

A graph G is k-choosable if its vertices can be colored from any lists $L(v)$ of colors with $|L(v)| \geq k$ for all $v \in V(G)$. A graph G is said to be (k,u)-choosable if its vertices can be colored from any lists $L(v)$ with $|L(v)| \geq k$, for all $v \in V(G)$, and with $|\bigcup_{v \in V(G)} L(v)| \leq u$. For each $3 \leq k \leq u$, we construct a graph G which is (k,u)-choosable but not $(k,u+1)$-choosable. On the other hand, it is proven that each $(k,2k-1)$-choosable graph G is $O(k \cdot \ln k \cdot 2^{4k})$-choosable.

Key words: Graph coloring, list coloring.

1 Introduction

An undirected simple graph $G=(V,E)$ is said to be *k-choosable* if for every list assignment $L(v)$ of lists of at least k colors to each vertex $v \in V(G)$, there is a proper vertex coloring c with $c(v) \in L(v)$. The smallest number k for which a graph G is k-choosable is also called the *list chromatic number* or the *choice number* of G. A list assignment L with $|L(v)| \geq k$ is called a k-list assignment and a list assignment L from which a graph G cannot be colored is called *bad*. Hence a graph is not k-choosable if and only if there is a bad k-list assignment for it. The concepts of choosability and

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list coloring were introduced independently by Erdős, Rubin and Taylor [3] and Vizing [6]. Since then, they have received a considerable amount of attention as witnessed by several recent surveys on the subject [2, 4, 5, 7].

By the definition, the list chromatic number of a graph G is at least its chromatic number and there are graphs for which the inequality is strict. There are even bipartite (2-colorable) graphs whose list chromatic number is arbitrarily large. In fact, Erdős, Rubin and Taylor [3] showed that the list chromatic number of the complete bipartite graph $K_{n,n}$ is $(1+o(1))\cdot\log_2 n$. Actually, they proved the following:

Proposition 1 *For the complete bipartite graph $K_{n,n}$ with $n \geq \binom{2k-1}{k}$, there is a bad k-list assignment L such that the size of the union of all the lists $L(v)$ is $2k-1$.*

On the other hand, it is easy to see that each bipartite graph can be colored from lists of sizes k whose union is at most $2k-2$ for $k \geq 2$. Indeed, consider a bipartite graph G with vertex parts V_1 and V_2 and a k-list assignment L such that the union L_0 of all the lists $L(v)$, $v \in V(G)$, has size at most $2k-2$. Partition the set L_0 to two non-empty disjoint sets A and B such that $|A|, |B| \leq k-1$. Observe that each list $L(v)$ has a non-empty intersection with both sets A and B as otherwise $|L(v)| \leq k-1$. Color every vertex v of V_1 by any color from $L(v) \cap A$ and every vertex v of V_2 by any color from $L(v) \cap B$. Since the vertices of V_1 are colored only with the colors from the set A and the vertices of V_2 only with the colors from the set B, the obtained coloring is a proper coloring of G.

This motivates the following problem which we address in this paper:

Problem 2 *Given a number k, does there exist a number u_k such that if a graph G can be colored from any lists whose sizes are k and whose union has size at most u_k, then the graph G is k-choosable?*

In order to make our presentation clearer, we introduce the following two definitions: A list assignment $L(v)$ of colors to the vertices of a graph G is said to be an (k,u)*-list assignment* if $|L(v)| \geq k$ for each $v \in V(G)$ and $|\bigcup_{v \in V(G)} L(v)| \leq u$. A graph G is (k,u)*-choosable* if there is a proper vertex coloring c with $c(v) \in L(v)$ for every (k,u)-list assignment $L(v)$. Clearly, a graph G is k-colorable iff it is (k,k)-choosable and G is k-choosable iff it is (k,u)-choosable for all $u \geq k$. Proposition 1 can now be reformulated to our new notation as follows:
The complete bipartite graph $K_{n,n}$ with $n \geq \binom{2k-1}{k}$ is not $(k, 2k-1)$-choosable.

Similarly, Problem 2 can be reformulated as follows:
Given a number k, does there exist a number u_k such that a graph G is k-choosable iff it is (k, u_k)-choosable?
As noted above, each bipartite graph is $(k, 2k-2)$-choosable for all $k \geq 2$ and hence if such a number u_k exists, it must be at least $2k-1$.

In this paper, we completely resolve Problem 2 and show that the answer is mostly negative. Namely, we show that a graph G is 2-choosable iff it is $(2, 4)$-choosable, i.e., the number u_2 exists and it is equal to 4 (Theorem 4). We further construct, for each $3 \leq k \leq u$, a (small) graph G which is (k, u)-choosable but which is not $(k, u+1)$-choosable (Theorem 6). This shows that a number u_k does not exist for any $k \geq 3$. For the sake of completeness, we remark that a graph G is 1-choosable iff it has no edges. Hence, the number u_1 also exists and it is equal to 1.

In addition, we consider the following question suggested to us by Tomasz Łuczak. First, let us reformulate the statement Theorem 6 as follows: For each $k \geq 3$ and $u \geq k$, the fact that G is (k, u)-choosable does not imply that G is k-choosable. This leads to the problem whether at least the following holds:

Problem 3 *Is it true that for each $k \geq 3$ there exist some numbers u and K such that each (k, u)-choosable graph G is K-choosable?*

Clearly, if such numbers K and u exist, the number K must be at least $k+1$ by Theorem 6 and u must be at least $2k-1$. The latter follows from Proposition 1 and the fact that each bipartite graph is $(k, 2k-2)$-choosable. We focus on the case $u = 2k-1$ and show the following: Each $(k, 2k-1)$-choosable graph G is $O(k \cdot \ln k \cdot 2^{4k})$-choosable (Theorem 7). This can be improved by a more careful argument to $O(k \cdot 2^{4k})$ (and even slightly more), but we do not know whether a subexponential bound can be proven.

2 2-choosable graphs

In this section, we prove that the number u_2 as defined in the introduction is equal to 4. Since the complete bipartite graph $K_{2,4}$ is $(2, 3)$-choosable but not 2-choosable, the number u_2 must be at least 4. The upper bound follows from the next theorem:

Theorem 4 *Each $(2, 4)$-choosable graph is 2-choosable.*

Proof: It was shown in [3] that if a graph G is not 2-choosable, then it contains one of the following subgraphs:

- an odd cycle,
- two vertex-disjoint (even) cycles joined by a path,
- two (even) cycles sharing a single vertex,
- two vertices joined by three vertex-disjoint paths (one of them might be an edge) such that at least two paths have two or more internal vertices, or
- two vertices joined by four vertex-disjoint paths.

It is easy to see that each of these graphs is not $(2,4)$-choosable. Actually, all of them except for the graph $K_{2,4}$ (covered by the last case) are not even $(2,3)$-choosable.

Hence, if a graph G is $(2,4)$-choosable, it cannot contain any of the above graphs as a subgraph. Then G must be 2-choosable.

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3 The negative result

In this section, we construct graphs which are (k,u)-choosable but not $(k,u+1)$-choosable for all $3 \leq k \leq u$. As the first and most important step of our construction, we prove the following version with precolored vertices:

Lemma 5 *For any $k \geq 3$, $t \geq 2$, there exists a graph $H_k^t = (V,E)$ with t distinguished vertices $T = \{v_1, \ldots, v_t\} \subseteq V$ which has the properties (1) and (2) below. Let W denote the set $V \setminus T$.*

1. *For any precoloring c_0 of the vertices T by t distinct colors and any set $\{\alpha_1, \alpha_2, \ldots,\}$ of $\max\{2, k-t\}$ of additional colors (not used by c_0), there exists a $(k, \max\{t+2,k\})$-list assignment L for vertices of W such that c_0 cannot be extended to a proper coloring c of the whole graph with $c(w) \in L(w)$ for $w \in W$. Moreover, the lists $L(w)$ contain only the t colors used by c_0 and the colors from A.*

2. *Any precoloring c_0 of the vertices T by at most $t-1$ distinct colors can be extended to a proper coloring c of the graph H_k^t with $c(w) \in L(w)$, $w \in W$, for any k-list assignment L for vertices of W.*

Moreover, the order of the graph H_k^t is at most $O(t^2/k+k)$.

Proof: In the proof, we distinguish three cases.

The first case: $t \leq k-1$. We construct the graph H_k^t as follows: The set W consists of $k+1-t$ vertices $w_1, \ldots, w_{k+1-t}$ and edges of the graph H_k^t are all edges $v_i w_j$ for $1 \leq i \leq t$ and $1 \leq j \leq k+1-t$ and all edges $w_i w_j$ for $1 \leq i < j \leq k+1-t$.

We now verify that the graph H_k^t has the desired properties:

1. Consider a precoloring c_0 of the vertices of T by t distinct colors and, for all vertices $w_j \in W$, set the lists $L(w_j)$ to the set $\{c_0(v_1), \ldots, c_0(v_t), \alpha_1, \ldots, \alpha_{k-t}\}$ of k colors. The vertices of W can be colored only by colors which are different from the colors $c_0(v_1), \ldots, c_0(v_t)$, i.e., only by the $k-t$ colors $\alpha_1, \ldots, \alpha_{k-t}$. Since W induces a clique on $k-t+1$ vertices, this is impossible and the precoloring c_0 cannot be extended to a proper coloring c with $c(w_j) \in L(w_j)$ for all $w_j \in W$.

2. If c_0 is a precoloring of the vertices of T by at most $t-1$ distinct colors and L a list assignment with $|L(w_j)| = k$ for all $w_j \in W$, then c_0 can be extended to a coloring c of the whole graph H_k^t. Simply remove the (at most $t-1$) colors used by a precoloring c_0 from the lists L. The vertices of W form a complete graph (of order $k+1-t$) and each of them has a list of size at least $k-(t-1) = k+1-t$. Hence, this complete graph can be properly colored from the shortened lists and such a coloring is an extension of the precoloring c_0.

The second case: $k = 3$ and $t \geq 3$. We construct the graph H_3^t as follows: Let X be an odd-cardinality set of vertices and connect each of them to exactly two nodes $v_i, v_{i'}$, chosen so that each pair of vertices v_i and $v_{i'}$, $1 \leq i < i' \leq t$, is connected to at least one vertex $x \in X$. Note that the set X can be chosen to have size $\binom{t}{2}$ or $\binom{t}{2}+1$ (a single pair can be used twice to get $|X|$ odd). Add an odd cycle C of length $|X|$ on a new set of vertices and form a perfect matching between the vertices of C and X. Take now $W = X \cup C$ and let E be the edges described above.

Again, we verify that the graph $H_{3,u}^t$ has the two claimed properties:

1. For each vertex $x \in X$ whose neighbors are vertices v_i and $v_{i'}$, set its list $L(x) = \{c_0(v_i), c_0(v_{i'}), \alpha_1\}$. For each vertex $w \in C$, set its list $L(w) = \{\alpha_1, c_0(v_1), c_0(v_2)\}$. A proper extension of the precoloring c_0 must color all the vertices of X by the color α_1. Then the odd cycle C must be colored using only the colors $c_0(v_1)$ and $c_0(v_2)$ which is impossible.

2. Consider now a precoloring c_0 of the vertices of T using at most $t-1$ colors and a list assignment L with $|L(w)| = k$ for all $w \in W$. Since the precoloring c_0 uses less than t colors, there exists a vertex $x_0 \in X$ such that its two neighbors v_i and $v_{i'}$ have the same color assigned by c_0. Color the vertices of $X \setminus \{x_0\}$ from their lists: This is possible since each of them has two precolored neighbors and its list is of size three. Let $w_0 \in C$ be the neighbor of the vertex x_0 in the cycle C. For each vertex $w \in C \setminus \{w_0\}$, remove from its list the color of its single neighbor in X (if the color is contained in the list $L(w)$). This leaves lists of at least two colors available for each $w \in C \setminus \{w_0\}$ and a list of three colors for the vertex w_0. Any such lists allow a proper coloring of the cycle C. Finally, color the vertex x_0. Note that the two neighbors of x_0 in T have the same color and hence it is possible to color x_0 by a color from its list.

The third case: $k \geq 4$ and $t \geq k$. The structure of the graph H_k^t is the following: W is an odd set of vertices, each of them connected to exactly $k-2$ nodes from $\{v_1, \ldots, v_t\}$. The edges are chosen so that each pair of vertices v_i and $v_{i'}$, $1 \leq i < i' \leq t$, has a common neighbor in W. In addition to these edges, the graph H_k^t contains an odd cycle formed on the vertices W.

A straightforward way to construct such a graph of order $O(t^2/k)$ is to partition the vertices $v_1, \ldots, v_t$ into $\lceil t/(k-3) \rceil$ sets T_j with $|T_j| \leq k-3$ and for each $1 \leq i \leq t$ and $1 \leq j \leq \lceil t/(k-3) \rceil$, create a vertex $w \in W$ whose neighbors are the vertex v_i and all the vertices of T_j (and, if necessary, some other vertices of T so that w has exactly $k-2$ neighbors in T). Possibly add one vertex to get $|W|$ odd.

We verify that the constructed graph H_k^t has the two desired properties:

1. For a vertex $w \in W$ with neighbors $v_{i_1}, \ldots, v_{i_{k-2}}$, we set the list $L(w)$ to be $\{c_0(v_{i_1}), \ldots, c_0(v_{i_{k-2}}), \alpha_1, \alpha_2\}$. In any proper coloring from lists L which extends the precoloring c_0, all the vertices in W must be

colored by the colors α_1 and α_2 which is impossible as the vertices W induce an odd cycle in H_k^t.

2. Consider now a precoloring c_0 of the vertices of T using at most $t-1$ colors and a list assignment L with $|L(w)| = k$ for all $w \in W$. Since the precoloring c_0 uses at most $t-1$ colors and each pair of vertices of T is contained in at least one set T_j, there exists a vertex $w_0 \in W$ such that its $k-2$ neighbors in T have at most $k-3$ distinct colors. Thus the vertex w_0 has at least three colors not conflicting with the given precoloring c_0 in its list $L(w_0)$ and each vertex $w \in W \setminus \{w_0\}$ has at least two non-conflicting colors in its list $L(w)$. Since W is a cycle, it can be properly colored by non-conflicting colors. Such a coloring is an extension of the precoloring c_0.

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We can now prove the main theorem of this section:

Theorem 6 *For each $u \geq k \geq 3$, there exists a (k,u)-choosable graph G which is not $(k,u+1)$-choosable. Moreover, the order of G is at most $O(u^2)$ (the constant is independent of k).*

Proof: Let $t = u+2-k \geq 2$. The graph G consists of a vertex r, k cliques S_i $(i = 1,\dots,k)$ on $k-1$ vertices each, and k copies of the graph H_k^t from Lemma 5. Let T_i denote the t distinguished vertices of the ith copy of the graph H_k^t. In addition to the edges of the graphs S_i and H_k^t, the graph G contains, for each $i = 1,\dots,k$, all edges between r and S_i and all edges between S_i and T_i. Since G consists of k cliques of order $k-1$ and k copies of the graph H_k^t whose order is at most $O(t^2/k + k)$, the order of G is at most $O(k^2 + t^2) = O(u^2)$.

First, we show that G is not $(k,u+1)$-choosable. Consider the following list assignment L: The vertex r and the vertices of cliques S_i have the same list $\{1,\dots,k\}$. For an integer $i = 1,\dots,k$, associate to each vertex $v \in T_i$ one of the t colors $i, k+1, k+2, \dots, u+1$ (each color to a single vertex of T_i). The list of the vertex v then consists of the colors $\{1,\dots,k\} \setminus \{i\}$ and the color associated to it. The remaining vertices contained in copies of H_k^t have colors like in Lemma 5(1) for $\alpha_1, \alpha_2, \dots, \alpha_{\max\{2,k-t\}} \in \{1,2,3,\dots,k\} \setminus \{i\}$ and a precoloring c_0 where each vertex of T_i is precolored by its associated color.

Suppose that G can be properly colored from the lists described above. Let i be the color of r. Then the $k-1$ vertices of the clique S_i are colored by all the colors $\{1, \ldots, k\} \setminus \{i\}$. Thus each vertex $v \in T_i$ is colored by its associated color. By Lemma 5 and the choice of lists in copies of the graph H_k^t, the considered coloring of G cannot be proper.

Second, we show that the graph G is (k,u)-choosable. Fix a (k,u)-list assignment L. First color the vertex r, then the vertices of cliques S_i and then the vertices contained in the sets T_i. Each vertex has at most $k-1$ of its neighbors colored before itself, thus each vertex can be colored from its list. Since the number of vertices contained in $S_i \cup T_i$ is $(k-1)+t = u+1$, two of the vertices of $S_i \cup T_i$ have the same color for each i (recall that L is a (k,u)-list assignment). However, the vertices of S_i form a clique and each of them is adjacent to all the vertices of T_i. Thus the two vertices colored by the same color must be in T_i. By Lemma 5(2), the precoloring of the vertices can be extended to the whole copy of the graph H_k^t. Hence, the graph G is indeed (k,u)-choosable.

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4 The positive result

In contrast with the negative result of the previous section that (k,u)-choosability does not imply k-choosability for any u, we show in this section that if a graph G is $(k, 2k-1)$-choosable graph, then it is also $O(k \cdot \ln k \cdot 2^{4k})$-choosable. We follow ideas of a proof from [1] that a graph with minimum degree at least $\Theta(k^4 \cdot 2^{2k})$ is (k, k^2)-choosable.

Theorem 7 *For all $k \geq 2$, each $(k, 2k-1)$-choosable graph is $O(k \cdot \ln k \cdot 2^{4k})$-choosable.*

Proof: Let $D = (k+1) \cdot \ln 2 \cdot \ln k \cdot 2^{4k+1}$. We show that if a graph G is $(k, 2k-1)$-choosable, then G is also D-choosable for each $k \geq 2$. Assume for the sake of contradiction that G is a graph with the smallest order n which is $(k, 2k-1)$-choosable but not D-choosable. Since G has the smallest possible order, its minimum degree is at least D.

Our proof is probabilistic. We first choose a small random subset of vertices denoted by A and assign each vertex of A a random list of k colors. Then, with a positive probability, at least $n/2$ vertices v have the following property: the neighbors of v that are in A have been assigned all possible lists. We fix such a set A with such a list assignment and show that the remaining vertices cannot be colored from random lists with positive probability. We note that in the rest of the argument we do not use the assumption that G is not D-choosable; we only use the assumption that all vertices have degree at least D.

Let $p = 1/(\ln k \cdot 2^{2k+1})$. Note that $p < 1/8$ for each $k \geq 2$. Let A be a subset of vertices of the graph G where each vertex of G is chosen to be in A randomly and independently with the probability p. Since the expected size of A is pn, the probability that the size of A is larger than $2pn$ is at most $1/2$ by Markov's inequality. Let Σ be the set consisting of all k-element subsets of the set $\{1, \ldots, 2k-1\}$. Note that $|\Sigma| = \binom{2k-1}{k} < 2^{2k-1}$. Assign each vertex of A a list of colors equal to a set from Σ randomly and independently. Let L_A be the resulting $(k, 2k-1)$-list assignment for vertices of A. We say that a vertex v of G is *good* if $v \notin A$ and for each set $S \in \Sigma$, there is a neighbor v' of v in G such that $v' \in A$ and $L_A(v') = S$.

The probability that a vertex $v \in V(G) \setminus A$ is not good is the probability that no neighbor of v in A has its list equal to a set S for some $S \in \Sigma$. The probability that none of at least D neighbors of v both is in A and has a list equal to a fixed set $S \in \Sigma$ is at most

$$\left(1 - \frac{p}{|\Sigma|}\right)^D < \left(1 - \frac{p}{2^{2k-1}}\right)^D < e^{-\frac{pD}{2^{2k-1}}} = e^{-2(k+1)\cdot\ln 2} = 2^{-2k-2}.$$

Hence the probability that there is a set $S \in \Sigma$ such that no neighbor of v in A has a list equal to S is at most $|\Sigma| \cdot 2^{-2k-2} < 2^{2k-1} \cdot 2^{-2k-2} = 1/8$.

The probability that a given vertex v is not good is at most $1/8$ from the previous calculation plus the probability that v is in A, which is p. Hence the probability that a vertex v is not good is at most $1/8 + p < 1/4$. It follows by Markov's inequality that the number of vertices which are not good is at most $n/2$ with probability strictly larger than $1/2$. Thus there exists a set $A \subseteq V(G)$ and a $(k, 2k-1)$-list assignment L_A such that $|A| \leq 2pn$ and the number of good vertices is at least $n/2$. Fix such a set A and a list assignment L_A for the rest of the proof. Let B be the set of good vertices.

Fix a coloring c of vertices of A from their lists and assign each vertex of B a list from the set Σ randomly and independently. Let L_B be the resulting $(k, 2k-1)$-list assignment for vertices of B. Consider a vertex $v \in B$. Since for each set $S \in \Sigma$, there is a neighbor $v' \in A$ of the vertex v with $L_A(v') = S$, the coloring c assigns neighbors of the vertex v at least k different colors. Hence, the probability that v cannot be colored from its list $L_B(v)$ is at least $1/|\Sigma|$. Since the choice of sets $L_B(v)$ for vertices $v \in B$ was independent, the probability that all vertices of B can be colored from their lists for a fixed coloring c is at most:

$$\left(1-\frac{1}{|\Sigma|}\right)^{\frac{n}{2}} < \left(1-\frac{1}{2^{2k-1}}\right)^{\frac{n}{2}} < e^{-\frac{n}{2^{2k}}}$$

Hence, the probability that for a random list assignment L_B, there is a coloring c of vertices of A from their lists which can be extended to a coloring of vertices of B from L_B is at most the product of the number of such colorings c and the probability that a single one can be extended. This probability is strictly smaller than

$$k^{|A|} \cdot e^{-\frac{n}{2^{2k}}} \leq k^{2pn} \cdot e^{-\frac{n}{2^{2k}}} = e^{\frac{n}{2^{2k}}} \cdot e^{-\frac{n}{2^{2k}}} = 1$$

Thus, with a positive probability, there exists a list assignment L_B such that no coloring c of vertices of A from their lists can be extended to a coloring of all vertices of B from the list assignment L_B. Fix such a list assignment L_B. Complete now the $(k, 2k-1)$-list assignments L_A and L_B to a $(k, 2k-1)$-list assignment L for all vertices of the graph G by choosing lists for vertices of $V(G) \setminus (A \cup B)$ arbitrarily from the set Σ. Such a list assignment L is a bad $(k, 2k-1)$-list assignment for the graph G (the vertices of A and B cannot be colored from their lists) which contradicts the assumption that the graph G is $(k, 2k-1)$-choosable.

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5 Conclusion

We have proved that in order to show that a graph G is not k-choosable it is not enough to restrict one's attention to lists whose union is bounded by some function of k. We also settled Problem 3 by showing that each $(k, 2k-1)$-choosable graph G is $O(k \cdot \ln k \cdot 2^{4k})$-choosable. Recall that for each $k \geq 2$ and $K \geq 2$, there exist a graph G which is $(k, 2k-2)$-choosable but which is not K-choosable (consider a sufficiently large complete bipartite graph). Hence, the constraint on the union of lists cannot be weakened in Theorem 7. However, it would be interesting to investigate the other extremal case in Problem 3, namely the following:

Problem 8 *Is it true that for each $k \geq 3$ there exists a number u such that each (k, u)-choosable graph G is $(k+1)$-choosable?*

It would be also interesting to find (at least an asymptotic) solution to the following problem:

Problem 9 *For which pairs $K > k \geq 3$ does there exist a number u such that each (k,u)-choosable graph G is K-choosable? What is the smallest number u for a given pair k and K?*

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References

[1] N. Alon, Degrees and choice numbers, *Random Structures Algorithms* **16** (2000) 364-368

[2] N. Alon, Restricted colorings of graphs, in: *Surveys in Combinatorics 1993* (K. Walker, ed.), London Math. Soc. Lecture Notes Series 187 (Cambridge Univ. Press, 1993) 1–33.

[3] P. Erdős, A. L. Rubin, H. Taylor, Choosability in graphs, *Congress. Numer.* **26** (1980) 122–157.

[4] J. Kratochvíl, Z. Tuza, M. Voigt, New trends in the theory of graph colorings: Choosability and list coloring, in: *Contemporary Trends in Discrete Mathematics (from DIMACS and DIMATIA to the future)* (R. L. Graham et. al., eds.), DIMACS Series in Discrete Mathematics and Theoretical Computer Science, vol. 49 (AMS, Providence, 1999) 183–197.

[5] Z. Tuza, Graph colorings with local constraints — a survey, *Discuss. Math., Graph Theory* **17 (2)** (1997) 161–228.

[6] V. G. Vizing, Colouring the vertices of a graph in prescribed colours (in Russian), *Diskret. Anal.* **29** (1976) 3–10.

[7] D. R. Woodall: List colourings of graphs, in: *Surveys in Combinatorics 2001* (J. W. P. Hirschfeld, ed.), London Math. Soc. Lecture Notes Series 288 (Cambridge Univ. Press, 2001) 269-301.