About a new class of matroid-inducing packing families

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Abstract

Let $\mathcal{T}$ be a family of graphs. A $\mathcal{T}$-packing of a graph G is a set of disjoint subgraphs of G, each isomorphic to one of the members of $\mathcal{T}$. We are concerned with families $\mathcal{T}$, such that in every graph G, the subsets of vertices that can be saturated by some $\mathcal{T}$-packing form a collection of independent sets of a matroid. The main purpose of this paper is to introduce a new class of families with this property.

1 Introduction

A matching of a graph G can be viewed as a set of disjoint subgraphs of G, each isomorphic to K_2. A natural generalization, called a *$\mathcal{T}$-packing* of G is a set of disjoint subgraphs of G, each isomorphic to a member of a fixed family $\mathcal{T}$ of graphs. We consider only connected graphs G and connected members of $\mathcal{T}$.

If G is a graph then we denote by $V(G)$ the set of its vertices and by $E(G)$ the set of its edges. Analogously to matching which we are generalizing, let us introduce the following terminology: A $\mathcal{T}$-packing *covers* a vertex $v \in V(G)$ if one of the subgraphs included in the packing contains v. A $\mathcal{T}$-packing *saturates* a set of vertices $X \subseteq V(G)$ if it covers every member of X. A $\mathcal{T}$-packing Q is *maximal* if there is no packing Q' with $V(Q') \supsetneq V(Q)$ and is *perfect* if it covers all vertices of G. A $\mathcal{T}$-packing problem consists of finding a $\mathcal{T}$-packing saturating a set of a maximum cardinality.

We are concerned with families $\mathcal{T}$ of graphs such that in every graph G, the subsets of $V(G)$ that can be saturated by some $\mathcal{T}$-packing form a matroid.

Definition 1.1. Let X be a set and let $\mathcal{M}$ be a nonempty hereditary system of subsets of X (if $B \subseteq A \subseteq X$ and $A \in \mathcal{M}$ then $B \in \mathcal{M}$). The maximal sets of $\mathcal{M}$ (under the set inclusion) are called *bases*. The system $\mathcal{M}$ is called a *matroid* if the set $\mathcal{B}$ of its bases satisfies the exchange axiom:
(EA) $\forall B, B' \in \mathcal{B}; \forall x \in B \setminus B'; \exists y \in B' \setminus B : (B' \setminus \{y\}) \cup \{x\} \in \mathcal{B}$.

The exchange axiom implies the fact that all bases of a matroid have the same cardinality. We will often shortly write only $B \setminus y \cup x$ instead of $(B \setminus \{y\}) \cup \{x\}$.

We call a family $\mathcal{T}$ of graphs *matroid-inducing*, if for every graph G, the system of all subsets of $V(G)$ that can be saturated by some $\mathcal{T}$-packing is a matroid.

1.1 Notation and basic notions

If $x \in V(H)$ then we denote by $H \setminus x$ the graph obtained from H by deleting the vertex x.

If Q is a $\mathcal{T}$-packing of a graph G then we denote by $V(Q), E(Q)$ the sets of all vertices and edges of G belonging to some graph of Q. Thus, $G_Q = (V(Q), E(Q))$ is a subgraph of G whose components are graphs isomorphic to members of $\mathcal{T}$, and conversely, this graph uniquely determines Q. The degree of a vertex $x \in V(G)$ in the graph G_Q will be denoted by $deg_Q(x)$.

If G is a graph, Q its $\mathcal{T}$-packing and P a path in G then $Q \triangle P$ will denote the packing (not necessarily a $\mathcal{T}$-packing) constructed by swapping edges and non-edges of Q on P ($\triangle$ is the familliar symmetric difference).

A graph H is called *hypomatchable* if it has no perfect matching but for every vertex x, $H \setminus x$ has a perfect matching. A single vertex is considered hypomatchable.

A *star* S_k is a complete bipartite graph $K_{1,k}$, i.e. the graph with $k+1$ vertices $c, v_1, ..., v_k$ and k edges $cv_1, ..., cv_k$. The number k is called the *power* of the star, the vertex c is called the *center* of the star and the vertices $v_1, ..., v_k$ are called *tips* of the star.

A *two-star* $S_{k,l}$ is a graph constructed from two stars S_k, S_l by connecting their centers by a new edge called a *spine*. The vertices of the two stars used are still called centers and tips. A center of a two-star with m neighboring tips will be sometimes called an *m-center* or a *center of power*

m. A tip of a two-star adjacent to a center c of power m will be sometimes called a *c-tip* or a *tip of power m*.

A *k-propeller* ($k \geq 1$) is a graph obtained from a star S_{k+1} by replacing k of its tips by hypomatchable graphs (arbitrarily conected to the center).

1.2 History and results

We are concerned with families $\mathcal{T}$ containing K_2. Considering such families, the matroid-inducing property was proved for all of the basic polynomially solvable cases of the $\mathcal{T}$-packing problem:

(i) Matching ($\{K_2\}$-packing) - Edmonds and Fulkerson, [2]

(ii) Packing by K_2 and hypomatchable graphs - Cornuejols and Hartvigsen, [1]

(iii) Packing by a sequential set of stars $\{S_1, ..., S_r\}$ - Las Vergnas, [5]

A case generalizing all of the previous cases is Packing by an EHP-family of graphs, studied by Loebl and Poljak [6, 7, 8] and by the author [3, 4]. An *EHP-family* is a family consisting of K_2, hypomatchable graphs and propellers. Not all EHP-families are matroid-inducing. For a full characterization of EHP-families with respect to the matroid-inducing property see [3, 4]. The matroid-inducing EHP-families turn out to have a structure generalizing the sequential set of stars. A polynomial algorithm for solving the $\mathcal{T}$-packing problem for matroid-inducing EHP-families was introduced in [7, 8]. The following result about matroid-inducing EHP-families was introduced in [4]:

Theorem 1.2. *[4] Let $\mathcal{T}$ be a matroid-inducing EHP-family and let H be a graph. If $\mathcal{T}_1 = \mathcal{T} \cup \{H\}$ is matroid-inducing then $\mathcal{T}_1$ is a matroid-inducing EHP-family.*

In other words, if we want to enlarge a matroid-inducing EHP-family $\mathcal{T}$ by one graph H to get another matroid-inducing family, we have to use a graph of the same type as the graphs already included in $\mathcal{T}$ (hypomatchable or a propeller such that $\mathcal{T} \cup \{H\}$ is a matroid-inducing EHP family).

It might be expected that a similar situation occurs when we try to enlarge a matroid-inducing EHP-family by a family of more graphs. We will show that this expectation is false by introducing a rather simple class of non-EHP matroid-inducing families.

2 The Matroid-Inducing Families

Theorem 2.1. *The family* $\mathcal{F}_n = \{S_1, ..., S_n, S_{n+2}, S_{n+1,1}, ..., S_{n+1,n+1}\}$ *is matroid-inducing for all* $n \geq 1$.

In other words: In a sequential set of stars $S_1, ..., S_{n+2}$ which is known to be a matroid-inducing family [5], the replacement of the second largest star S_{n+1} by a sequence $S_{n+1,1}, ..., S_{n+1,n+1}$ of two-stars results in a matroid-inducing family. Figure 1 shows the family $\mathcal{F}_1$.

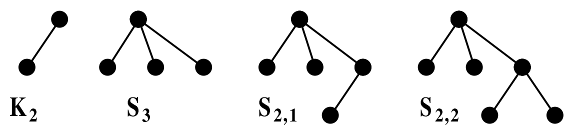

Figure 1: Family $\mathcal{F}_1$

If L is an $\mathcal{F}_n$-packing of a graph G, then every vertex of $V(G)$ covered by a copy of a star $S_k, k < n$, by a tip of S_n (for $n > 1$), by a tip of a two-star or by an m-center of a two-star with $1 \leq m < n+1$ in L, will be called *empty with respect to* L. Let us make an easy observation and then start the proof of Theorem 2.1:

OBSERVATION 2.2. *Let* G *be a graph and* L *an* $\mathcal{F}_n$*-packing of* G. *If a vertex uncovered by* L *is a neighbor (in* G*) of an empty or uncovered vertex with respect to* L *then* L *is not maximal.*

Proof. (Theorem 2.1) For a contradiction, assume that for some n, $\mathcal{F}_n$ is not matroid-inducing. Hence there exists a graph G and two bases $B, B' \subseteq V(G)$, for which the exchange axiom (EA) does not hold. Thus there is a vertex $x \in B \setminus B'$, such that $\forall y \in B' \setminus B : B' \setminus y \cup x$ is not a base.

Let us denote by Q, Q' the maximal $\mathcal{F}_n$-packings corresponding to B, B', respectively. Without loss of generality we may suppose that G is a graph with minimum $|V(G)|$ and that B, B' and Q, Q' are selected such that the distance $dist(Q, Q') = |E(Q) \setminus E(Q')| + |E(Q') \setminus E(Q)|$ is a minimum. Let us observe that G has no perfect $\mathcal{F}_n$-packing.

Let N, N' be two $\mathcal{F}_n$-packings of G. We say that N' is an *exchange-product* of N, if $V(N') = V(N) \setminus a \cup b$ for some $a \in V(N)$ and $b \notin V(N)$. In the minimum graph G, the following Claim holds:

CLAIM 2.2.1. *Let* N *be a maximal* $\mathcal{F}_n$*-packing of* G. *If* N' *is an exchange-product of* N *then* N' *is maximal.*

Proof of Claim 2.2.1. Let $V(N') = V(N) \setminus a \cup b$. For a contradiction, let M be an $\mathcal{F}_n$-packing of G with $V(M) \supsetneq V(N')$. Hence $|V(M)| > |V(N')| = |V(N)|$. If $a \in V(M)$ then $V(M) \supsetneq V(N)$, which contradicts the maximality of N. Thus $a \notin V(M)$.

If $|V(M)| < |V(G)| - 1$ then the subgraph of G given by $M \cup N$ or one of its components is also an example of graph with two bases contradicting (EA), which contradicts the minimality of $|V(G)|$. Thus $|V(M)| \geq |V(G)| - 1$ and because G has no perfect $\mathcal{F}_n$-packing, we have $|V(M)| = |V(G)| - 1$.

In the following discussion, we will construct a sequence $(M = M_0, M_1, ...)$ of $\mathcal{F}_n$-packings, such that for every i the following holds:

(i) $|V(M_i)| = |V(G)| - 1 > |V(N)|$

(ii) $\exists v_i \in V(N) \setminus V(M_i)$

(iii) if $i > 0$ then $dist(N, M_i) < dist(N, M_{i-1})$

Note that due to (i), each M_i must be maximal (because G admits no perfect $\mathcal{F}_n$-packing). At the beginning, $M = M_0$ surely satisfies (i) and (ii) (put $v_0 = a$). The last of the sequence of M_is will give us a contradiction.

The sequence is constructed by induction as follows:

Let M_i be the yet last constructed member of the sequence. Let P be an alternating path starting at v_i by an edge of $E(N) \setminus E(M_i)$ and containing alternately edges of $E(N) \setminus E(M_i)$ leading to a vertex of smallest possible degree in N and edges of $E(M_i) \setminus E(N)$ leading to vertex of degree 1 in M_i. Assume P is a path with maximum possible number of edges.

If P has an even number of edges and the last vertex p of P is uncovered by N then let $M' = M_i \triangle P$. Obviously M' is an $\mathcal{F}_n$-packing with $V(M') \supsetneq V(N)$, which contradicts the maximality of N. If p is covered by N (e.g. by an edge leading from a former vertex of P) then let $M_{i+1} = M_i \triangle P$ and set $v_{i+1} = p$. We may see that $dist(M_{i+1}, N) < dist(M_i, N)$ and $v_{i+1} \in V(N) \setminus V(M_{i+1})$.

Let P have an odd number of edges. Let p be the last vertex of P, rp the last edge of P and denote by P' the path P without its last edge. Note that due to (i) and (ii), M_i covers p.

If p is empty with respect to M_i then let $M' = M_i \triangle P'$. M' is an $\mathcal{F}_n$-packing containing an uncovered vertex r, that is a neighbor of an empty or uncovered vertex p. By Observation 2.2, M' is not maximal. Since $|V(M')| = |V(M_i)| = |V(G)| - 1$, G must have a perfect $\mathcal{F}_n$-packing which is a contradiction.

If p is covered by a center of S_{n+2} in M_i then by maximality of P it follows that $deg_N(p) \geq n + 3$, which is a contradiction.

If p is covered by a tip of S_{n+2} with center c in M_i then $M_i \triangle P$ contradicts the maximality of M_i.

If p is an $(n+1)$-center of a two-star $S_{n+1,m}, 1 \leq m < n+1$ with spine pq in M_i then $M_i \triangle (P \cup \{pq\})$ contradicts the maximality of M_i.

If p is a center of a copy T of $S_{n+1,n+1}$ with spine pq in M_i then let y be an q-tip of T. Obviously, all edges of M_i leading from p to a p-tip of T are in $E(N)$, otherwise we could continue with P. Thus the edge $pq \notin E(N)$. Let us construct a $\mathcal{F}_n$-packing $L = M_i \triangle (P \cup \{pq, qy\})$. If y is uncovered with respect to N then L contradicts the maximality of N. If $y \in V(N)$ then let us put $M_{i+1} = L$ and $v_{i+1} = y$ and observe that (i), (ii), (iii) are satisfied.

The last case occurs when p is a center of an n-star H in M_i. Obviously, $deg_N(p) \geq n+1$, otherwise we could continue with P to one of the tips of H. If $deg_N(p) = n+1$ then p is an n-center of a copy T of a two-star $S_{n,n+1}$ in N. If the last-but-one vertex r of P is the other center of T then there was a mistake in construction of P (we want the edges of $E(N) \setminus E(M)$ to lead to a vertex with smallest possible degree). Thus r is a tip of T. Let pq be the spine of T and let y be a q-tip of T. Obviously, the edge $qy \notin E(M_i)$. Because $|V(M_i)| = |V(G)| - 1$ and $v_i \notin V(M_i)$, y is covered by M_i. If y is empty with respect to M_i then $M' = M_i \triangle P \cup \{pq\}$ is a $\mathcal{F}_n$-packing with an uncovered vertex q neighboring with an empty vertex y and so there exists a $\mathcal{F}_n$-packing M'' with $V(M'') \subsetneq V(M')$ (contradiction). If y is not empty with respect to M_i then there exists an edge $yy_1 \in E(M_i)$ such that $deg_{M_i}(y_1) = 1$. Let us put $L = M_i \triangle (P \cup \{pq, qy, yy_1\})$. Observe that $yy_1 \notin E(N)$ and so $dist(N, L) < dist(N, M_i)$. If y_1 is uncovered by N then L contradicts the maximality of N, otherwise L satisfies (i), (ii), (iii) and we may put $M_{i+1} = L$.

On the other hand, if $deg_N(p) = n+2$ then there is a vertex $a \neq r$, such that the edge $pa \in E(N) \setminus E(M_i)$. Let us focus on the vertex a:

If a is uncovered by M_i or covered by a center of one of $S_1, ..., S_n$ in M_i then $M_i \triangle (P \cup \{pa\})$ contradicts the maximality of M_i.

If a is a tip of a star $S_k, 1 < k <= n$ with center c in M_i then $M_i \triangle (P \cup \{pa, ac\})$ contradicts the maximality of M_i.

If a is a c-tip of power $k <= n+1$ of $S_{n+1,k}$ in M_i then $M_i \triangle (P \cup \{pa, ac\})$ contradicts the maximality of M_i.

If a is a c-tip of power $n+1$ of $S_{n+1,k}, k < n+1$ with spine cd in M_i, then $M_i \triangle (P \cup \{pa, ac, cd\})$ contradicts the maximality of M_i.

If a is an $(n+1)$-center of a two-star $S_{n+1,l}, 1 \leq l < n+1$ with spine ac in M_i, then $M_i \triangle (P \cup \{pa, ac\})$ contradicts the maximality of M_i.

If a has a neighbor c of degree $n+2$ in M_i (thus a is a tip of S_{n+2} and c is its center or a is an l-center of a two-star $S_{n+1,l}, l \leq n+1$ and c is its $(n+1)$-center) then let y be a neighbor of c different from a. Obviously $deg_{M_i}(y) = 1$. Consider the packing $L = M_i \triangle (P \cup \{pa, ac, cy\})$. Obviously at least one of the edges ac, cy is not in $E(N)$ and so $dist(N, L) < dist(N, M_i)$. If $y \notin V(N)$ then L contradicts the maximality of N. If $y \in V(N)$ then L satisfies (i), (ii), (iii) and we may put $M_{i+1} = L$ and $v_{i+1} = y$.

The last possibility is that a is a center of a copy of S_{n+2} in M_i. Then there exists a tip y such that the edge $ay \in E(M_i) \setminus E(N)$. Consider the packing $L = M_i \triangle (P \cup \{pa, ay\})$. If $y \notin V(N)$ then L contradicts the maximality of N. If $y \in V(N)$ then L satisfies (i), (ii), (iii) and we may put $M_{i+1} = L$ and $v_{i+1} = y$.

We have inspected all of the cases and subcases and found out that either there is an $\mathcal{F}_n$-packing M_{i+1} with smaller distance from N than M_i or we get a contradiction. Thus, if we are at the end of our sequence of $\mathcal{F}_n$-packings $(M_0, M_1, ...)$ then there is a contradiction by an $\mathcal{F}_n$-packing L' with either $V(L') = V(G)$ or $V(L') \supsetneq V(N)$.
End of Proof of Claim 2.2.1.

Let us continue the proof of Theorem 2.1 and focus on the two bases B, B' contradicting the (EA) and on the two corresponding $\mathcal{F}_n$-packings Q, Q'. Let $x \in B \setminus B'$ be a vertex contradicting (EA). Let $p \in B$ be a neighbor of x (in Q) with minimum $deg_Q(p)$. Note that if the $deg_Q(p) > 1$ then $deg_Q(x) = 1$. There are several cases:

Case 0: There exists an alternating path P with an even number of edges starting at x with an edge of $E(Q) \setminus E(Q')$, ending with an edge $yq \in E(Q') \setminus E(Q)$, and containing alternately edges of $E(Q)$ and edges of $E(Q')$ that lead to vertex of degree 1 in Q', such that $Q'' = Q' \triangle P$ is an $\mathcal{F}_n$-packing with distance $dist(Q, Q'') < dist(Q, Q')$.

Let $B'' = V(Q'')$. The packing Q'' is an exchange-product of Q' and therefore it is maximal. Hence B'' is a base. If $q \notin Q$ then $B'' = B' \setminus q \cup x$ contradicts the choice of x. Thus $q \in Q$. Because $dist(Q, Q'') < dist(Q, Q')$, (EA) holds for B, B''. Thus, for $q \in B \setminus B''$, there exists $z \in B'' \setminus B$ (thus $z \in B' \setminus B$) such that $B'' \setminus z \cup q = B' \setminus z \cup x$ is a base, which is a contradiction.

Case 1: $1 \leq deg_Q(p) \leq n$. Because Q' is maximal, due to Observation 2.2 we know that p is not empty with respect to Q'. Thus Q' covers p by a center of S_n, by a center of S_{n+2} or by an $(n+1)$-center of a two-star. In all of the cases we may find a neighbor q of p with $deg_{Q'}(q) = 1$, such that the edge $pq \notin E(Q)$. Considering the path xpq, we get Case 0.

Case 2: $deg_Q(p) = n+1$. Thus p is an n-center of a copy H of a two-star $S_{n+1,n}$ in Q and x is a p-tip of H. Let pr be the spine of H and let a be an arbitrary r-tip of H.

We know that either $deg_{Q'}(p) = n+2$ or p is covered by a center of S_n in Q'. If $deg_{Q'}(p) = n+2$ then there is a neighbor q of p, such that $deg_{Q'}(q) = 1$ and $pq \notin E(Q)$. Hence, considering path xpq, we have Case 0.

If p is covered by a center of a copy T of S_n in Q' and Case 0 does not occur then T covers all p-tips of H different from x and the edge pr. Let us observe that the edge $ra \notin E(Q')$. If a is empty with respect to Q' then we may construct an $\mathcal{F}_n$-packing Q'' with $Q'' \supseteq Q' \cup \{x\}$ contradicting the maximality of Q'. Thus a has a neigbour b in Q', such that $deg_{Q'}(b) = 1$. Obviously the edge $ab \notin E(Q)$ and so, considering the alternating path $xprab$, we are in Case 0.

Case 3: $deg_Q(p) = n+2$. Again, we know that either p is a center of S_n in Q' or $deg_{Q'}(p) = n+2$. If $deg_{Q'}(p) = n+2$ and Case 0 does not occur then p is an $(n+1)$-center of a copy H of a two-star in Q' and the only edge of $E(Q') \setminus E(Q)$ neighboring with p is the spine pr of H. If H is not a copy of $S_{n+1,n+1}$ then $Q' \triangle \{xp, pr\}$ contradicts the maximality of Q'. If H is a copy of $S_{n+1,n+1}$ then let b be an arbitrary r-tip of H. Let us construct a new packing $Q'' = Q' \triangle \{xp, pr, rb\}$. Define $B'' = V(Q'')$. Q'' is an exchange-product of Q' and so Q'' is a maximal $\mathcal{F}_n$-packing. If $b \notin Q$ then $B'' = B' \setminus b \cup x$ contradicts the choice of x. If $b \in Q$ then note that $pr \notin E(Q)$ and so $dist(Q, Q'') < dist(Q, Q')$. Because Q'' is a maximal $\mathcal{F}_n$-packing of G with $dist(Q, Q'') < dist(Q, Q')$, we get a contradiction as in Case 0.

If p is covered by a center of S_n in Q' and Case 0 does not occur then there is an edge $pq \in E(Q) \setminus E(Q'), x \neq q$. Let us focus on the vertex q and distinguish the following situations:

If q is uncovered by Q', covered by a star $S_j, 1 \leq j \leq n$ in Q' or covered by a tip of a two-star in Q' then it is easy to construct a packing saturating $V(Q') \cup \{x\}$ which contradicts the maximality of Q'.

If q is covered by a center of a copy T of S_{n+2} in Q' then certainly there exists a tip a of T, such that the edge $qa \notin E(Q)$. Let us define a new $\mathcal{F}_n$-packing Q'' by $E(Q'') = E(Q') \triangle \{xp, pq, qa\}$. Q'' is an $\mathcal{F}_n$-packing of G with $dist(Q, Q'') < dist(Q, Q')$. Obviously Q'' is an exchange-product of Q' and so it is maximal. We get a contradiction as in Case 0.

If q is a $(n+1)$-center of a two-star $S_{n+1,l}, 1 \leq l < n+1$ with spine qw in Q' then the packing $Q' \triangle \{xp, pq, qw\}$ contradicts the maximality of Q'.

The last possibility is that q has a neighbor w of degree $n+2$ in Q' (thus q is a tip of S_{n+2} with center w or q is a m-center of a copy T of a two-star $S_{n+1,m}, 1 \leq m \leq n+1$ in Q' and qw is the spine of T). Let $z \neq q$ be a neighbor of w and let us construct a new packing $Q'' = Q' \triangle \{xp, pq, qw, wz\}$. If $z \notin Q$ then $B'' = B' \setminus z \cup x$ is a base which is a contradiction. Thus $z \in Q'$. Note that Q'' is an exchange-product of Q' and that at least one of the edges qw, wz is not in $E(Q)$. Thus Q'' is maximal and $dist(Q, Q'') < dist(Q, Q')$ and again we get a contradiction as in Case 0.

We have inspected all the possible cases and subcases and always got a contradiction. This proves that each $\mathcal{F}_n$ is a matroid-inducing family. □

3 Conclusion

We have introduced a new class of families $\mathcal{F}_n$ of graphs, such that in every graph G, the sets of vertices saturated by some $\mathcal{F}_n$-packing form a matroid.

In the future work, it would be interesting to seek for a polynomial algorithm for finding a maximum $\mathcal{F}_n$-packing. In many of the known algorithms for various cases of the $\mathcal{T}$-packing problem, the polynomial recognition of critical graphs is one of crucial steps. Concerning our class of families $\mathcal{F}_n$, we already know that the only $\mathcal{F}_n$-critical graphs are odd cycles (in case of $\mathcal{F}_1$) and a single vertex.

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