Matroids Induced by Packing Subgraphs

Marek Janata
Department of Applied Mathematics, Charles University
Malostranské nám. 25, 11800 Praha 1, Czech Republic
e-mail: janata@kam.mff.cuni.cz

Abstract

The paper is concerned with the classification of families of graphs $\mathcal{T}$ with the property: For any graph G, the subsets of vertices of G that can be saturated by packing copies of graphs from $\mathcal{T}$ form a collection of independent sets of a matroid. From this point of view, we present a characterization of so called EHP-families of graphs (i.e families consisting of K_2, hypomatchable graphs and propellers). The main result is the following: For a matroid-inducing EHP-family $\mathcal{T}$, we characterize connected graphs H such that the family $\mathcal{T} \cup \{H\}$ is also matroid-inducing.

1 Introduction

Let $G = (V(G), E(G))$ be a graph. A matching of G can be viewed as a set of vertex disjoint subgraphs of G, each isomorphic to K_2. A natural generalization is a set of vertex disjoint subgraphs of G, each a member of a family $\mathcal{F}$ of subgraphs of G. This generalization is called an $[\mathcal{F}]$*-packing* of G. (We use brackets in this notation to avoid confusions with a more special generalization, which will be defined further).

analogously to matching which we are generalizing, let us introduce the following terminology: An $[\mathcal{F}]$-packing $\mathcal{Q}$ *covers* a vertex $v \in V(G)$ if one of the subgraphs included in $\mathcal{Q}$ contains v. Otherwise $\mathcal{Q}$ *skips* v. An $[\mathcal{F}]$-packing *saturates* a set of vertices $X \subseteq V(G)$ if it covers every member of X. For an $[\mathcal{F}]$-packing $\mathcal{Q}$, $V(\mathcal{Q})$ denotes the set of all vertices covered by $\mathcal{Q}$. An $[\mathcal{F}]$-packing $\mathcal{Q}$ of G is *maximal* if there is no $[\mathcal{F}]$-packing $\mathcal{Q}'$ of G with $V(\mathcal{Q}') \supsetneq V(\mathcal{Q})$ and is *perfect* if it covers all vertices of G. A graph G

is $[\mathcal{F}]$-*saturable* if it admits a perfect $[\mathcal{F}]$-packing and is *non*-$[\mathcal{F}]$-*saturable* if it has no perfect $[\mathcal{F}]$-packing. The $[\mathcal{F}]$-*packing problem* in G consists of finding an $[\mathcal{F}]$-packing of G saturating a set of maximum cardinality.

a special case of $[\mathcal{F}]$-packing appears when $\mathcal{F}$ consists of all subgraphs of G isomorphic to members of a fixed family $\mathcal{T}$ of graphs. In this special case, "$\mathcal{T}$" will be used instead of "$[\mathcal{F}]$" throughout the notation described above.

The $[\mathcal{F}]$-packing problem has been extensively studied from many points of view. The most important are the cases when the $[\mathcal{F}]$-packing problem can be solved in polynomial time (see Section 2 for examples). A common feature proved for many of the polynomially solvable cases is that the sets of vertices saturated by some $[\mathcal{F}]$-packing form a collection of independent sets of a matroid.

Let X be a set and let $\mathcal{M}$ be a nonempty hereditary system of subsets of X (i.e. if $A \in \mathcal{M}$ and $A' \subseteq A$ then $A' \in \mathcal{M}$). The maximal sets of $\mathcal{M}$ (under the set inclusion) are called *bases*. The pair $(X, \mathcal{M})$ is called a *matroid* if the set $\mathcal{B}$ of its bases satisfies the exchange axiom:

(EA) $\forall B, B' \in \mathcal{B}; \forall x \in B \setminus B'; \exists y \in B' \setminus B : (B' \setminus \{y\}) \cup \{x\} \in \mathcal{B}$.

If $M = (X, \mathcal{M})$ is a matroid then X is called the *ground set* of M and the subsets of X contained in $\mathcal{M}$ are called the *independent sets* of M. The exchange axiom implies the fact that all bases of a matroid have the same cardinality.

If G is a graph and $\mathcal{F}$ a family of its subgraphs then we denote by $M(G, [\mathcal{F}])$ the family of all subsets of $V(G)$ that can be saturated by some $[\mathcal{F}]$-packing. If $\mathcal{F}$ consists of all subgraphs of G isomorphic to members of a family $\mathcal{T}$ of graphs then $M(G, \mathcal{T})$ stands for $M(G, [\mathcal{F}])$. This paper is concerned with the classification of families $\mathcal{T}$ of connected graphs such that $M(G, \mathcal{T})$ is a collection of independent sets of a matroid in every graph G. A family $\mathcal{T}$ with this property is called *matroid-inducing*.

1.1 Notation and basic notions

For two graphs H, H', $H' \subseteq H$ denotes that H' is a subgraph of H. If $x \in V(H)$ then $H \setminus x$ is the graph obtained from H by deleting the vertex x. If $D \subseteq V(H)$ then $H \setminus D$ is the graph obtained from H by deleting every vertex $x \in D$. For a graph H, the number of vertices of H will be denoted by $|H|$.

Let H be a connected graph and let $u, v \in V(H)$. The *distance* $dist(u, v)$ is the minimum number of edges on a path from u to v. The distance between two subgraphs of H and between a vertex and a subgraph is defined analogously.

A graph H is *hypomatchable* if it has no perfect matching but for every $x \in V(H)$, $H \setminus x$ admits a perfect matching. A single vertex is considered hypomatchable. A common feature that we will use is that hypomatchable graphs do not contain vertices of degree 1.

A *star* S_k is a complete bipartite graph $K_{1,k}$, i.e. the graph with $k+1$ vertices $c, v_1, ..., v_k$ and k edges $cv_1, ..., cv_k$. The vertex c is called the *center* and k is the *index* of the star.

2 History and Results

The basic polynomially solvable cases of the $[\mathcal{F}]$-packing problem in which $M(G, [\mathcal{F}])$ is a matroid are the following:

(E) *Matching (Edge-packing) [3, 4].* $\mathcal{F}$ consists of all edges of G.

(EH) *Packing by edges and a set of hypomatchable graphs [1, 2, 5].* $\mathcal{F}$ consists of all edges of G and some hypomatchable subgraphs of G.

(S) *Packing by sequential sets of stars [7].* For some integer r, $\mathcal{F}$ consists of all subgraphs isomorphic to a star $S_i, 1 \leq i \leq r$.

In [8, 9], Loebl and Poljak studied the following case of the $[\mathcal{F}]$-packing problem:

(EHP) *Packing by edges, hypomatchable graphs and propellers.* $\mathcal{F}$ consists of all edges of G, some hypomatchable subgraphs of G and a family $\mathcal{R} \subsetneq \mathcal{F}$ of subgraphs of G named *propellers.* Let us introduce the notion of propeller:

A graph P is a *k-propeller* ($k \geq 1$ is the *index* of P), if it has a vertex c, called the *center*, such that $P \setminus c$ consists of $k+1$ components $D_0, ..., D_k$, where $|D_0| = 1$ and every D_i is hypomatchable. Note that every $(k+1)$-star is a k-propeller.

A propeller is called *rooted*, if it has one chosen vertex r of degree 1, called the *root.* We denote a rooted propeller P with root r by (P, r). If P has more vertices of degree one (like a star) then there are more possibilities of choosing the root and the corresponding rooted propellers are considered to be distinct. Since hypomatchable graphs have no vertices of degree one, the root r must be a neighbor of the center c. The edge rc is called the *stick* of (P, r). Let $D_0, ..., D_k$ be the components of $P \setminus c$. Without loss of generality we may suppose that $D_0 = \{r\}$. The remaining components

$D_1, ..., D_k$ are called the *blades* of (P, r). We denote the set of all blades of (P, r) by $D(P, r)$.

A family $\mathcal{R}$ of rooted propellers that are subgraphs of a graph G is called *G-closed* if it satisfies the following three axioms:

Heredity. If $(H, r) \in \mathcal{R}$ and (H', r) are rooted propellers with the same stick and $D(H', r) \subseteq D(H', r)$ then $(H', r) \in \mathcal{R}$.

Stick exchange. If $(H, r) \in \mathcal{R}$ is a rooted propeller with stick cr and r' is a vertex of $G \setminus H$ adjacent (in G) to c then $(H \setminus r) \cup cr'$ rooted in r' belongs to $\mathcal{R}$.

Blade exchange. Let $(H, r), (H', r) \in \mathcal{R}$ be rooted propellers with the same stick rc and $D(H', r) \nsubseteq D(H, r)$. Then for any blade D of (H, r) disjoint to all blades of (H', r) there is some blade $D' \in D(H', r) \setminus D(H, r)$ such that the rooted propeller (H'', r) with stick rc and blades $(D(H', r) \setminus D')) \cup D$ belongs to $\mathcal{R}$.

Note that the formulation of Blade exchange axiom pays no attention to the edges connecting the centers of the propellers with their blades. Blade exchange axiom itself does not say, *which* of the (possibly more) rooted propellers with stick rc and blades $(D(H', r') \setminus D')) \cup D$ has to belong to $\mathcal{R}$. Anyway, we do not have to be confused by this, since due to Heredity, if one such propeller belongs to $\mathcal{R}$, then all of such propellers do so. (This argumentation was consulted with the first author of [8].)

A family of graphs containing only edges, hypomatchable graphs and propellers is called an *EHP-family.* The result of Loebl and Poljak concerning the (EHP) problem is summarized in the following:

Theorem 2.1. *Let G be a graph and let $\mathcal{F} = E(G) \cup \mathcal{K} \cup \mathcal{R}$ be an EHP-family of its subgraphs, where $\mathcal{K}$ is a family of hypomatchable graphs and $\mathcal{R}$ is a G-closed family of rooted propellers. Then the $[\mathcal{F}]$-packing problem can be solved in polynomial time and $M(G, [\mathcal{F}])$ is a matroid.*

Let us observe that a sequential family of stars induces a closed family of propellers in every graph G. Hence the (EHP) case contains all of (E), (EH) and (S).

Note that (E) and (S) concern also the more special $\mathcal{T}$-packing problem: In (E), $\mathcal{T} = \{K_2\}$ and in (S), $\mathcal{T} = \{S_1, ..., S_r\}$ for some integer r. In both of these cases the $\mathcal{T}$-packing problem can be solved in polynomial time in any graph G and $\mathcal{T}$ is a matroid-inducing family.

In [8, 10], Loebl and Poljak studied the following case of the $\mathcal{T}$-packing problem:

(E+1) *Packing by edges and copies of a single graph.* $\mathcal{T}$ consists of K_2 and a fixed graph H. The complexity of this case was fully characterized in [10] and the matroid-inducing property of $\mathcal{T}$ in [8]. The two results show that in this case the $\mathcal{T}$-packing problem is polynomially solvable if and only if $\mathcal{T}$ is a matroid-inducing family (otherwise it is NP-complete). The characterization of the matroid-inducing property of $\mathcal{T}$ in this case is given by the following Theorem:

Theorem 2.2. *Let H be a connected graph. Then $\{K_2, H\}$ is a matroid-inducing family if and only if H is perfectly matchable, hypomatchable or a 1-propeller.*

The above result may be also viewed as follows: It is a characterization of graphs H that can be added to a matroid-inducing EHP-family $\{K_2\}$ such that $\{K_2\} \cup \{H\}$ is also a matroid-inducing family. The interesting property is that the resulting family is always an EHP-family. In the author's Master's Thesis [6], this result was extended to the following:

(S+1) *Packing by a sequential set of stars and copies of a single graph.* $\mathcal{T}$ consists of a sequential family $\mathcal{S} = \{S_1, S_2, ..., S_n\}$ of stars and of a fixed graph H. The characterization of the matroid-inducing property of $\mathcal{T}$ in this case is given by the following Theorem:

Theorem 2.3. *Let $\mathcal{S} = \{S_1, ..., S_k\}, k > 1$ be a family of stars and let H be a connected graph. Then $\mathcal{T} = \mathcal{S} \cup \{H\}$ is a matroid-inducing family if and only if H is $\mathcal{S}$-saturable, hypomatchable or the star S_{k+1}.*

This paper is concerned with matroid-inducing EHP-families $\mathcal{T}$. At first, we will reformulate the Loebl's and Poljak's necessary condition on EHP-families $\mathcal{F}$ of subgraphs of G, guaranteeing that $M(G, [\mathcal{F}])$ forms a matroid to a necessary condition guaranteeing that a general EHP-family $\mathcal{T}$ of graphs is matroid-inducing. Moreover, we will prove that this reformulated condition is sufficient, which is a new result: a full characterization of the (EHP)-case for $\mathcal{T}$-packing. This work will be done in Section 3.

The main purpose of this paper is to give a full characterization of individual graphs H that can be added to a general matroid-inducing EHP-family $\mathcal{T}$, such that $\mathcal{T} \cup \{H\}$ is also a matroid-inducing family. We will show that a graph H has this property if and only if H is $\mathcal{T}$-saturable or $\mathcal{T} \cup \{H\}$ is a matroid-inducing EHP-family, which shows that the property proved by Loebl and Poljak for the case $\mathcal{T} = \{K_2\}$ holds throughout the whole class of matroid-inducing EHP-families $\mathcal{T}$. The proof of this fact will be given in Section 4.

The results given in this paper are in fact generalizations of the resuts of Loebl and Poljak presented in [8]. In many places, we will be able to use proofs and arguments similar to those used in [8]. We will call the reader's attention to this correspondence on the appropriate places of the paper.

3 Matroid-Inducing EHP-Families

In this Section, a full characterization of EHP-families w.r.t. the matroid-inducing property will be presented. Let us start with a few more notions concerning propellers and families of propellers:

A propeller $P' \supseteq K_2$ is a *subpropeller* of a propeller P with center c, if P' arises from P by deleting one or more components of $P \setminus c$. Suppose P_1, P_2 are two disjoint graphs, such that each P_i is either a propeller with center c_i or an edge with one end-vertex c_i. We denote by $P_1 + P_2$ the graph that arises from P_1, P_2 by glueing vertices c_1, c_2 into one vertex c, arbitrary neighbors r_1, r_2 of c_1, c_2 with degree 1 into one vertex r and by deleting the multiple edge.

The following Claim will be used:

Claim 3.0.1. *The center c of a k-propeller P is the only vertex of P, such that $P \setminus c$ has more than one component with no perfect matching.*

Proof. $P \setminus c$ really has $k + 1 \geq 2$ such components $D_0, ..., D_k$ (every D_i is hypomatchable). If there exists a vertex $x \neq c$, such that $P \setminus x$ has at least two components with no perfect matching then $x \in D_j$ for some j. Let B be a component of $P \setminus x$ that has no perfect matching and does not contain c. Then B is also a component of $D_j \setminus x$, which is a contradiction, because D_j is hypomatchable. □

Let L be a graph and let $\mathcal{T}$ be a family of graphs. By "$[L] \in \mathcal{T}$" we will denote "there is some $L' \in \mathcal{T}$ isomorphic to L". The following definition mimics the definition of G-closed family of propellers that are subgraphs of G:

A family $\mathcal{R}$ of propellers is called *closed* if it satisfies the following two axioms:

Heredity. If $H \in \mathcal{R}$ and H' is a subpropeller of H then $[H'] \in \mathcal{R}$.

Blade exchange. If $H, H' \in \mathcal{R}$ are propellers and H' is isomorphic to $H_1 + H_2$, where $H_1 \in \mathcal{R}$ is a 1-propeller and H_2 is a subpropeller of H or an edge connecting the center of H to a vertex of degree one, then there exists a component B of $H \setminus H_2$ such that $[(H \setminus B) + H_1] \in \mathcal{R}$.

Let $\mathcal{T}'$ be a family of graphs. Note that in a $\mathcal{T}'$-packing, we may avoid the use of any member of $\mathcal{T}'$ that has a perfect packing by the remaining members. We say that $\mathcal{T} \subseteq \mathcal{T}'$ is a *sufficient subfamily* of $\mathcal{T}'$, if all graphs from $\mathcal{T}' \setminus \mathcal{T}$ have perfect $\mathcal{T}$-packing. The following Theorem characterizes matroid-inducing EHP-families:

Theorem 3.1. *An EHP-family $\mathcal{T}$ of graphs is matroid-inducing if and only if $\mathcal{T}$ is a sufficient subfamily of some EHP-family $\mathcal{T}' = \{K_2\} \cup \mathcal{H} \cup \mathcal{R}$, where $\mathcal{H}$ is a family of hypomatchable graphs and $\mathcal{R}$ is a closed family of propellers.*

Proof. The "if" part of the proof is a simple application of Theorem 2.1. Let $\mathcal{T}$ be a sufficient subfamily of a family $\mathcal{T}'$ consisting of K_2, hypomatchable graphs and a closed family of propellers. Let G be an arbitrary graph. We will construct a family $\mathcal{F}$ of subgraphs of G in two steps: In the first step we will insert in $\mathcal{F}$ all subgraphs of G isomorphic to members of $\mathcal{T}'$ with all possible selections of roots for every propeller. In the second step we will enlarge $\mathcal{F}$ to include every rooted propeller that is a subgraph of G and has a common center, root and blades with any of the rooted propellers added in the first step (the only difference may be in the edges connecting the blades and the center of the propeller). Obviously $M(G, [\mathcal{F}]) = M(G, \mathcal{T})$. It can be simply verified that $\mathcal{F}$ satisfies the supposition of Theorem 2.1. Thus $M(G, [\mathcal{F}]) = M(G, \mathcal{T})$ is a matroid and since G was arbitrary, $\mathcal{T}$ is a matroid-inducing family.

We will prove the "only if" part by constructing counter-examples. We want to prove that if a family $\mathcal{T}$ cannot be enlarged by adding $\mathcal{T}$-saturable graphs to a family $\mathcal{T}'$ consisting of K_2, hypomatchable graphs and a closed family of propellers, then there exists a counter-example G such that $M(G, \mathcal{T})$ is not a matroid.

Let $\mathcal{T}$ be a family that cannot be enlarged this way. There are two cases:

(i) $\mathcal{T}$ *violates Heredity.* There exists a propeller $P \in \mathcal{T}$ such that one of its subpropellers P' is not $\mathcal{T}$-saturable (thus $[P'] \notin \mathcal{T}$ and we cannot add P' into $\mathcal{T}$)

(ii) $\mathcal{T}$ *violates Blade exchange.* There are two propellers $H, H' \in \mathcal{T}$ such that for some 1-propeller $H_1 \in \mathcal{T}$, H' is isomorphic to $H_1 + H_2$, where H_2 is a subpropeller of H or an edge connecting the center of H to a vertex of degree one, and for no component B of $H \setminus H_2$, the propeller $(H \setminus B) + H_1$ is $\mathcal{T}$-saturable.

In the following two Lemmas 3.2, 3.3 we will show counter-examples to both (i) and (ii). For each case we will construct a counter-example G, such that $M(G,\mathcal{T})$ is not a matroid (we will find two bases of different cardinality). That will be the proof of the "only if" part of Theorem 3.1. $\square$

Lemma 3.2. Counter-example for an EHP-family violating Heredity (i). *Let $\mathcal{T}$ be an EHP-family. If there exists a $\mathcal{T}$-saturable propeller P and a non-$\mathcal{T}$-saturable subpropeller P' of P, then $\mathcal{T}$ is not matroid-inducing.*

Proof. Let P be a $\mathcal{T}$-saturable propeller with center c and P' its non-$\mathcal{T}$-saturable subpropeller. Without loss of generality we may suppose that P is a propeller with minimum $|P|$ and that P' is a k-propeller with maximum k.

With this supposition, for one component D of $P \setminus c$, $P' = P \setminus D$. Let $r \in V(P')$ be an arbitrary neighbor of c with degree one (such r exists since P' is a propeller). Let us denote by b the number of non-$\mathcal{T}$-saturable components of $P' \setminus cr$. We know that $b \geq 1$, otherwise the whole P' would be $\mathcal{T}$-saturable. Let us observe the following:

Claim 3.2.1. *There is no $\mathcal{T} \setminus \{P\}$-packing of P saturating the whole $P' = P \setminus D$.*

Proof of Claim 3.2.1. Let $\mathcal{Q}$ be a $\mathcal{T} \setminus \{P\}$-packing of P saturating P' and let B be a non-$\mathcal{T}$-saturable component of $P' \setminus cr$. Let $W \in \mathcal{Q}$ be the graph containing the center c of P'. It follows that $r \in W$ and that $W \cap B$ is not empty and non-$\mathcal{T}$-saturable.

$W \setminus c$ is a graph with at least two non-$\mathcal{T}$-saturable components. Using Claim 3.0.1, we conclude that W is a propeller with center c. Let $\mathcal{A}$ be the family of all components of $W \setminus cr$ intersecting D and let W' be a propeller that arises from W by deleting all members of $\mathcal{A}$. W' is a subpropeller of W and due to minimality of P, W' has a perfect $\mathcal{T}$-packing $\mathcal{Q}'$ (if $[W'] \in \mathcal{T}$ then trivially $\mathcal{Q}' = \{W'\}$). Let $\mathcal{Q}_1$ be the part of $\mathcal{Q}$ intersecting $D \setminus W$. We may construct a perfect $\mathcal{T}$-packing $((\mathcal{Q} \setminus \{W\}) \cup \mathcal{Q}') \setminus \mathcal{Q}_1$ of P', which is a contradiction.
End of Proof of Claim 3.2.1.

Let us construct a counter-example G and prove that $M(G,\mathcal{T})$ is not a matroid by introducing two bases of different cardinality. G arises from two copies P_1, P_2 of P (vertices of each P_i will be denoted by the index $_i$) by glueing the center c_2 of P_2 to an arbitrary vertex x_1 from the component D_1 of $P_1 \setminus c_1$ (see Figure 2a).

Consider a base B_2 of $M(G, \mathcal{T})$ containing $V(P_2)$. We may construct a $\mathcal{T}$-packing $\mathcal{Q}$ of G saturating $V(P_2)$ as follows: $\mathcal{Q}$ uses a copy P_2 of P, a perfect matching of $D_1 \setminus x_1$, a copy $c_1 r_1$ of K_2 and maximum $\mathcal{T}$-packings of all components of $P_1 \setminus (D_1 \cup \{r_1, c_1\})$. Hence $|B_2| \geq |G| - b$, where b is the number of non-$\mathcal{T}$-saturable components of $P' \setminus cr$ (we know that $b \geq 1$).

Let B_1 be a base containing $V(P_1)$. If $\mathcal{N}$ is a $\mathcal{T}$-packing corresponding to B_1, then $\mathcal{N}$ uses a graph W covering the center c_1 of P_1. Obviously $W \setminus c_1$ has at least $b+1$ non-$\mathcal{T}$-saturable components and so, according to Claim 3.0.1, W is a propeller and c_1 its center.

If $c_2 \notin W$ then let L be the graph covering c_2 (there exists such graph since $V(P_1) \subseteq B_1$). We know that c_1 is contained in the graph W and thus $L \cap P_1$ contains only vertices from D_1. Let us delete L from $\mathcal{N}$ and consider the induced $\mathcal{T}$-packing $\mathcal{N}'$ of P_1. $\mathcal{N}'$ saturates the whole $P_1 \setminus D_1$ and by a cardinality argument $\mathcal{N}'$ does not use any copy of P, which is a contradiction by Claim 3.2.1.

If $c_2 \in W$ then every component of $W \setminus c_2$ intersecting P_2 is $\mathcal{T}$-saturable (W is a propeller with center c_1). Neither of the $b+1$ non-$\mathcal{T}$-saturable components of $P_2 \setminus c_2$ can be saturated by $\mathcal{N}$ and thus $|B_1| \leq |G| - (b+1) < |B_2|$. We have found two bases of different cardinality and so G is a counter-example showing that $\mathcal{T}$ is not a matroid-inducing family. $\square$

We have shown a counter-example for an EHP-family violating Heredity. In the next Lemma we will complete the proof of Theorem 3.1 by introducing a counter-example for an EHP-family violating Blade exchange.

Lemma 3.3. Counter-example for an EHP-family violating Blade exchange (ii).
Let $\mathcal{T}$ be an EHP-family. Let $\mathcal{T}$ fulfill Heredity, i.e. if a propeller $R \in \mathcal{T}$ then let all subpropellers of R be $\mathcal{T}$-saturable.

If there are two propellers $P, P' \in \mathcal{T}$ such that for some 1-propeller $P_1 \in \mathcal{T}$, P' is isomorphic to $P_1 + P_2$, where P_2 is a subpropeller of P or an edge connecting the center of P to a vertex of degree one, and for no component B of $P \setminus P_2$, the propeller $(P \setminus B) + P_1$ is $\mathcal{T}$-saturable, then $\mathcal{T}$ is not matroid-inducing.

Proof. Let P, P', P_1, P_2 be the propellers described above. Let us consider the graph $G = P + P_1$. Note that G is a propeller: let us denote by c its center and by r the glued-together neighbor of c of degree one (see Figure 2b). Without loss of generality assume that $r \in V(P_2)$.

Let B_1 be a base of $M(G, \mathcal{T})$ containing $V(P)$. Then $|B_1| \geq |G| - 1$, since we may construct a $\mathcal{T}$-packing of G saturating $V(P)$ using a copy of P and a maximum matching of $P_1 \setminus cr$.

Let B_2 be a base containing $V(P_1) \cup V(P_2)$. If $|B_2| = |G| - 1$ then the $\mathcal{T}$-packing $\mathcal{Q}$ corresponding to B_2 skips exactly one vertex in one component W of $P \setminus P_2$. If $|B_2| = |G|$ then let W be an arbitrary component of $P \setminus P_2$. We will prove that $G \setminus W$ is $\mathcal{T}$-saturable:

Let us pay attention to the graph $L \in \mathcal{Q}$ covering the center c of G. L covers the whole edge cr and so L is a copy of K_2 or a propeller with center c.

If L is a copy cr of K_2 then every component of $G \setminus (W \cup \{c, r\})$ is $\mathcal{T}$-saturable. Thus the whole $G \setminus W$ is $\mathcal{T}$-saturable.

If L is a propeller then let $\mathcal{A}$ be the set of all components of $L \setminus c$ intersecting W. Let L' be the graph that arises from L by deleting all components from $\mathcal{A}$. L is a copy of K_2 or a propeller and due to Heredity, L' has a perfect $\mathcal{T}$-packing $\mathcal{Q}'$. Denote by $\mathcal{Q}_1$ the part of $\mathcal{Q}$ intersecting $W \setminus L$. We may construct a perfect $\mathcal{T}$-packing $((\mathcal{Q} \setminus \{L\}) \cup \mathcal{Q}') \setminus \mathcal{Q}_1$ of $G \setminus W$.

We have proved that $G \setminus W$ is $\mathcal{T}$-saturable. It gives us a contradiction, because W is a component of $P \setminus P_2$ and $(G \setminus W) = (P \setminus W) + P_1$, which has no perfect $\mathcal{T}$-packing by the assumption. Thus $|B_2| < |G| - 1 \leq |B_1|$, which proves that G is a counter-example showing that $\mathcal{T}$ is not a matroid-inducing family. $\square$

4 (EHP+1)-Packing

In this Section we will introduce a characterization of the graphs H that can be added to a matroid-inducing EHP-family $\mathcal{T}$ such that $\mathcal{T} \cup \{H\}$ is a matroid-inducing family. We will prove the following extension of Theorem 2.2:

Theorem 4.1. *Let $\mathcal{T}$ be a matroid-inducing EHP-family and let H be a graph. Then the family $\mathcal{T} \cup \{H\}$ is matroid-inducing if and only if H is $\mathcal{T}$-saturable or $\mathcal{T} \cup \{H\}$ is a matroid-inducing EHP-family.*

The "if" part of the Theorem is trivial. Moreover, the characterization of matroid-inducing EHP-families (Theorem 3.1), gives us a partial negative result for adding a propeller H such that $\mathcal{T} \cup \{H\}$ is not matroid-inducing.

For proving the "only if" part of the Thoerem 4.1 it remains to show that if a matroid-inducing EHP-family $\mathcal{T}$ is enlarged by a graph H that is not hypomatchable, $\mathcal{T}$-saturable or a propeller, then $\mathcal{T} \cup \{H\}$ is not matroid-inducing. We will show it at the end of this Section by discussing the structure of H and by showing counter-examples.

Before proving Theorem 4.1, some auxiliary notions and lemmas concerning the structure of packings will be introduced. This technical work will be done in Subsections 4.1 and 4.2. The proof of Theorem 4.1 will be given in Subsection 4.3.

4.1 Economical packings

Economical packing is a notion first defined by Loebl and Poljak in [8] for packing by edges and hypomatchable subgraphs. Economical packings try to cover as many vertices by copies of K_2 as possible. We will define a similar notion for $\mathcal{T}$-packing by matroid-inducing EHP-families $\mathcal{T}$ and prove a short Lemma needed in the proof of Theorem 4.1.

Let $\mathcal{T}$ be a matroid-inducing EHP-family. Consider a graph G and a $\mathcal{T}$-packing $\mathcal{Q}$ of G. We denote by $V(\mathcal{Q})$ and $E(\mathcal{Q})$ the set of all vertices and edges of G belonging to some graph of $\mathcal{Q}$. Thus, $G_{\mathcal{Q}} = (V(\mathcal{Q}), E(\mathcal{Q}))$ is a subgraph of G whose components are copies of graphs from $\mathcal{T}$, and conversely, this graph uniquely determines $\mathcal{Q}$.

A vertex $v \in V(G)$ is called *fixed* w.r.t. $\mathcal{Q}$, if it is covered by a copy of K_2 or by a center of a propeller in $\mathcal{Q}$. For $\mathcal{Q}$, we define a packing $\mathcal{Q}_h$ by the following: $G_{\mathcal{Q}_h}$ is the subgraph of $G_{\mathcal{Q}}$ induced by non-fixed vertices. Obviously $\mathcal{Q}_h$ consists only of hypomatchable graphs (all hypomatchable graphs in $\mathcal{Q}$ and for each propeller $P \in \mathcal{Q}$ with center c, all components of $P \setminus c$).

Definition 4.2. We say that a $\mathcal{T}$-packing $\mathcal{Q}$ of G is *economical*, if there is no $\mathcal{T}$-packing $\mathcal{Q}'$ of G such that $\forall v \in V(G)$: if $deg_{\mathcal{Q}}(v) = 1$ then $deg_{\mathcal{Q}'}(v) = 1$, and

(i) $V(\mathcal{Q}') \supsetneq V(\mathcal{Q})$ and $\mathcal{Q}'_h \subseteq \mathcal{Q}_h$ or
(ii) $V(\mathcal{Q}') = V(\mathcal{Q})$ and $\mathcal{Q}'_h \subsetneq \mathcal{Q}_h$

Note that if a set $X \subseteq V(G)$ is saturable by some $\mathcal{T}$-packing then there exists an economical $\mathcal{T}$-packing saturating X. Let $\mathcal{Q}$ be a $\mathcal{T}$-packing of G, such that each propeller used in $\mathcal{Q}$ is assumed rooted in an arbitrarily selected root ($\mathcal{Q}$ is an *arbitrarily rooted $\mathcal{T}$-packing*). For $\mathcal{Q}$, we define two packings $\mathcal{Q}_e, \mathcal{Q}_b$ by the following: $G_{\mathcal{Q}_e}$ is the subgraph of $G_{\mathcal{Q}}$ induced by fixed vertices and roots of propellers and $G_{\mathcal{Q}_b}$ is the subgraph of $G_{\mathcal{Q}}$ induced by vertices covered by hypomatchable graphs and blades of rooted propellers. Obviously $\mathcal{Q}_e$ contains only edges (copies of K_2 involved in $\mathcal{Q}$ and sticks of propellers from $\mathcal{Q}$) and $\mathcal{Q}_b \subseteq \mathcal{Q}_h$.

Given two rooted packings $\mathcal{Q}, \mathcal{Q}'$, we say that C is a *component* of $\mathcal{Q} \cup \mathcal{Q}'$, if it is a component of the graph $G_{\mathcal{Q}_e} \cup G_{\mathcal{Q}_b} \cup G_{\mathcal{Q}'_e} \cup G_{\mathcal{Q}'_b}$. Note that the components of $\mathcal{Q} \cup \mathcal{Q}'$ are not necesarilly components of the graph $G_{\mathcal{Q}} \cup G_{\mathcal{Q}'}$, since they do not contain the edges connecting the centers and blades of propellers from $\mathcal{Q}, \mathcal{Q}'$ (in [8], where only edges and hypomatchable graphs are used in the packings, these two types of components are the same). The following Lemma (in fact an extension of Theorem 3 from [8] proved by a similar technique) describes the components of $\mathcal{Q} \cup \mathcal{Q}'$ for two rooted economical $\mathcal{T}$-packings $\mathcal{Q}, \mathcal{Q}'$:

Lemma 4.3. *Let $\mathcal{T}$ be a matroid-inducing EHP-family and let G be a graph. Let $\mathcal{Q}$ and $\mathcal{Q}'$ be two rooted economical $\mathcal{T}$-packings of G and let C be a component of $\mathcal{Q} \cup \mathcal{Q}'$. Then*

(i) C contains at most 1 vertex that is uncovered by $\mathcal{Q}$

(ii) C contains at most 1 graph from $\mathcal{Q}_b$

(iii) if C contains exactly one graph from $\mathcal{Q}_b$ then $\mathcal{Q}$ saturates C

Proof. Let us call a path P *alternating* w.r.t. $\mathcal{Q}$ if it contains alternately edges of $\mathcal{Q}_e$ and $E(G) \setminus E(Q)$. An alternating path P with end vertices u and v is called *augmenting*, if $\{u, v\} \cap V(\mathcal{Q}_e) = \varnothing$ and $\{u, v\} \subseteq H$ for no $H \in \mathcal{Q}_b$ (i.e., the end vertices of P do not belong to the same hypomatchable graph of $\mathcal{Q}_b$). It is easy to see that an economical packing does not admit an augmenting path.

For a contradiction, let C contain two graphs H_1, H_2, such that each H_i is either a single vertex uncovered by $\mathcal{Q}$, or a member of $\mathcal{Q}_b$. Assume that the distance between H_1, H_2 is a minimum. Then C contains a path P, such that P has end vertices in H_1 and H_2, and P does not contain any edge of a graph from $\mathcal{Q}_b$.

Let us denote by $j(P)$ the maximum number of consecutive edges of a graph from $\mathcal{Q}'_b$, that do not belong to any graph from $\mathcal{Q}$, involved in P. Let us assume that P is a path with minimum $j(P)$.

If $j(P) \leq 1$ then P is an augmenting path w.r.t. $\mathcal{Q}$, which is a contradiction. If P contains edges of at least two distinct graphs from $\mathcal{Q}'_b$ then P contains an augmenting path w.r.t. $\mathcal{Q}'$, which is a contradiction. Thus P contains edges of exactly one graph $H' \in \mathcal{Q}'_b$.

Let x be the first vertex (in the direction from H_1 to H_2) adjacent to two consecutive edges of H' in P that do not belong to $\mathcal{Q}$. Note that $x \in \mathcal{Q}_e$, otherwise the beginning of P would be an augmenting path. Let $\mathcal{Q}''$ be the packing that arises from $\mathcal{Q}'$ by substituting H' with a perfect matching of $H' \setminus x$. Let P' be the (unique) path starting in x by the (unique) edge from $\mathcal{Q}_e$ and alternating with respect to both $\mathcal{Q}$ and $\mathcal{Q}''$. P' starts in x; denote by z the other end vertex. If $z \in H_1$ then $j(P)$ is not a minimum, otherwise we may find an augmenting path w.r.t. $\mathcal{Q}$ or $\mathcal{Q}'$, which is a contradiction. □

4.2 Structural lemma

Let H be a graph. For a vertex $x \in V(H)$, we will denote by H_x the graph that arises from H by adding a new vertex x_a and a new edge xx_a.

For a matroid-inducing EHP family $\mathcal{T}$, $k(\mathcal{T})$ denotes the maximum index of a star included in $\mathcal{T}$. Note that if $k(\mathcal{T}) \geq 2$ ($\mathcal{T}$ contains the star S_2) then all hypomatchable graphs are $\mathcal{T}$-saturable. Hence all hypomatchable graphs and propellers different from stars may be avoided in every $\mathcal{T}$-packing.

If $\mathcal{T}$ is a family of graphs and H is a graph then we will denote by $\mu_{\mathcal{T}}(H)$ the maximum number of vertices of H saturated by some $\mathcal{T}$-packing of H. If $\mathcal{T}$ is a matroid-inducing family then it follows that every maximal $\mathcal{T}$-packing of H saturates exactly $\mu_{\mathcal{T}}(H)$ vertices.

Lemma 4.4. *Let $\mathcal{T}$ be a matroid-inducing EHP-family. Let H be a graph that is neither $\mathcal{T}$-saturable, nor hypomatchable. Then there exists a vertex $x \in V(H)$, such that $\mu_{\mathcal{T}}(H_x) \leq |H_x| - 2$.*

Proof. We will prove this Lemma in two steps: In the first step we will find a vertex $x \in V(H)$ and an economical $\mathcal{T}$-packing $\mathcal{Q}$ of H_x with $|V(\mathcal{Q})| \leq |H_x| - 2$. In the second step we will prove that $\mathcal{Q}$ has to be maximal w.r.t. H_x. Because $\mathcal{T}$ is matroid-inducing, we will have $\mu_{\mathcal{T}}(H_x) = |V(\mathcal{Q})| \leq |H_x| - 2$.

(i) Let $\mathcal{N}$ be an economical maximal $\mathcal{T}$-packing of H. Because H is not $\mathcal{T}$-saturable, there exists a vertex $w \in V(H) \setminus V(\mathcal{N})$. We will color w red. Let v be a neighbor of w. Obviously $\mathcal{N}$ covers v by a copy of K_2 or by a center of a propeller.

Consider the graph H_v. If $\mathcal{N}$ is economical w.r.t. H_v then we set $x = v$ and $\mathcal{Q} = \mathcal{N}$ and we are finished with the first part of the proof.

If $\mathcal{N}$ is not economical w.r.t. H_v then there exists a $\mathcal{T}$-packing $\mathcal{N}'$ of

H_v with

(o) $\forall v \in V(H)$: if $deg_{\mathcal{N}}(v) = 1$ then $deg_{\mathcal{N}'}(v) = 1$, and
(i) $V(\mathcal{N}') \supsetneq V(\mathcal{N})$ and $\mathcal{N}'_h \subseteq \mathcal{N}_h$ or
(ii) $V(\mathcal{N}') = V(\mathcal{N})$ and $\mathcal{N}'_h \subsetneq \mathcal{N}_h$

If $v_a \notin V(\mathcal{N}')$ then $\mathcal{N}'$ is a $\mathcal{T}$-packing of H proving that $\mathcal{N}$ is not economical w.r.t. H, which is a contradiction. Therefore $v_a \in V(\mathcal{N}')$.

If the star $S_2 \notin \mathcal{T}$ then every propeller from $\mathcal{T}$ has a unique vertex of degree one. Hence $\mathcal{N}, \mathcal{N}'$ may be unambiguously viewed as rooted $\mathcal{T}$-packings. Note that due to (o), $V(\mathcal{N}_e) \subseteq V(\mathcal{N}'_e)$. Let J be the (unique) maximal alternating path starting in v_a by the edge $v_a v$ and containing alternately edges of $\mathcal{N}'_e$ and $\mathcal{N}_e$. Since $V(\mathcal{N}_e) \subseteq V(\mathcal{N}'_e)$, J has an odd number of edges. Let z be the last vertex of J. We know that z is uncovered by $\mathcal{N}$ or covered by $\mathcal{N}_b$. If $z \neq w$ then by substituting the first edge $v_a v$ in J by the edge wv we get an augmenting path w.r.t. $\mathcal{N}$ in H, which is a contradiction. Therefore the last vertex of J is w.

On the other hand, if $S_2 \in \mathcal{T}$ then we may assume that $\mathcal{N}, \mathcal{N}'$ consist of stars only. In this case, consider a sequence of edges J of maximal possible length starting in v_a by the edge $v_a v$ and containing alternately (uniquely determined) edges of $\mathcal{N}' \setminus \mathcal{N}$ and edges of $\mathcal{N} \setminus \mathcal{N}'$ leading to vertices of degree one. Let z be the last vertex of J. Due to (o), J cannot have an even number of edges. Thus J has an odd number of edges and z is either uncovered by $\mathcal{N}$ or a member of $\mathcal{N}_b$ or a center of a star $S_j, 1 \leq j < k(\mathcal{T})$ in $\mathcal{N}$ with all edges covered by $\mathcal{N}'$ or already in J. If the last possibility appers then we may construct a new $\mathcal{T}$-packing of H contradicting the maximality of $\mathcal{N}$ by swapping edges and non-edges of $\mathcal{N}$ along $(J \setminus \{v_a v\}) \cup \{wv\}$. Otherwise, similarly as above, we conclude that $z = w$. Without loss of generality assume that J is a path (cycles in J may be skipped).

In both cases we have found an odd cycle C in H. C consists of the edge wv and the path $J \setminus v_a$. We can conclude that none of the edges of C belongs to a propeller from $\mathcal{N}$ ($\mathcal{N}$ would not be economical w.r.t. H). Thus C contains only copies of K_2 from $\mathcal{N}$ and edges uncovered by $\mathcal{N}$. Let us color all vertices of C red. Note that for every red vertex y there exists an economical maximal $\mathcal{T}$-packing $\mathcal{N}^y$ of H skipping y and covering all other red vertices by copies of K_2.

Let us take a red vertex w', that has a non-red neighbor v'. Let $\mathcal{N}^{w'}$ be an economical maximal $\mathcal{T}$-packing of H skipping w' and covering all red vertices by copies of K_2. As above, we will either find out that $\mathcal{N}^{w'}$ is economical also w.r.t. $H_{v'}$, or we will find an odd cycle C' containing v' and

w' and consisting of copies of K_2 from $\mathcal{N}^{w'}$ and edges uncovered by $\mathcal{N}^{w'}$. In the latter case let us again color all vertices of C' red and observe that (still) for every red vertex y there is an economical maximum $\mathcal{T}$-packing $\mathcal{N}^y$ of H skipping y and covering all other red vertices by copies of K_2.

Continuing analogously, we cannot finish with all vertices colored red. Then H would be hypomatchable (for every [red] vertex there would be a $\mathcal{T}$-packing skipping it and covering all other [red] vertices by copies of K_2). Hence in i-th step we will find a red vertex $w^{(i)}$, its non-red neighbor $v^{(i)}$ and a $\mathcal{T}$-packing $\mathcal{N}^{w^{(i)}}$ of H, which is economical w.r.t $H_{v^{(i)}}$ and skips $w^{(i)}$ and the newly added vertex $v_a^{(i)}$. Let us set $x = v^{(i)}$ and $\mathcal{Q} = \mathcal{N}^{w^{(i)}}$. The first step of the proof is finished.

(ii) We have found a vertex $x \in V(H)$ and an economical maximal $\mathcal{T}$-packing $\mathcal{Q}$ of H, which is economical also w.r.t. H_x and skips at least the newly added vertex x_a and one more neighbor u of x. If $\mathcal{Q}$ is not maximal w.r.t. H_x then there is a $\mathcal{T}$-packing $\mathcal{R}'$ of H_x with $V(\mathcal{R}') \supsetneq V(\mathcal{Q})$. Let $\mathcal{R}$ be an economical $\mathcal{T}$-packing of H_x covering the same set of vertices as $\mathcal{R}'$. If $x_a \notin V(\mathcal{R})$ then $\mathcal{R}$ proves that $\mathcal{Q}$ is not maximal w.r.t. H, which is a contradiction. Thus $\mathcal{R}$ covers x_a. Assume $\mathcal{R}$ and $\mathcal{Q}$ are arbitrarily rooted. Let B be the component of $\mathcal{R} \cup \mathcal{Q}$ containing x_a. According to Lemma 4.3 B does not contain any other vertex, that is uncovered by $\mathcal{Q}$. In particular, $u \notin V(B)$. Let D be the component containing u. We know that $D \setminus u$ is saturated by $\mathcal{Q}$ and that $G_{\mathcal{Q}} \cap D$ does not contain any graph from $\mathcal{Q}_b$. Let $\mathcal{Q}'$ be a $\mathcal{T}$-packing that arises from $\mathcal{R}$ by replacing the edge xx_a by xu, by replacing all graphs of $\mathcal{R}$ intersecting D with graphs (edges) of $\mathcal{Q}_e$ intersecting D and by substituting the newly constructed graphs with their perfect $\mathcal{T}$-packings where necessary. It may be simply observed that $\mathcal{Q}'$ is a correctly defined $\mathcal{T}$-packing of H with $V(\mathcal{Q}') \supseteq V(\mathcal{Q})$ and so $\mathcal{Q}$ is not maximal w.r.t. H, which is a contradiction.

Thus $\mathcal{Q}$ has to be maximal w.r.t. H_x. Because $\mathcal{Q}$ skips at least two vertices of H_x, the Lemma 4.4 is proved. $\square$

4.3 Proof of the negative part of Theorem 4.1

Let $\mathcal{T}$ be a matroid-inducing EHP-family and let H be a graph that is neither $\mathcal{T}$-saturable nor hypomatchable nor a propeller. For proving Theorem 4.1 we need to show that $\mathcal{T} \cup \{H\}$ is not a matroid-inducing family. We will proceed by discussing the structure of H and introducing counter-examples. For each type of a bad graph H we will find a counter-example G, such that

$M(G, \mathcal{T} \cup \{H\})$ is not a matroid. Moreover, we will always find two bases of different cardinality. Let us introduce an auxiliary Claim about bases, analogous to the Claim introduced on page 344 in [8]:

CLAIM 4.4.1. *Let H be a graph, let $x \in V(H)$ be a vertex with less than $k(\mathcal{T})$ neighbors of degree one. Let $G = H_x$ and let B_2^x be a base of $M(G, \mathcal{T} \cup \{H\})$ such that $x, x_a \in B_2^x$ and for each $w \in V(H_x)$ with $j_w > 0$ neighbors of degree one, there are $n = min(j_w, k(\mathcal{T}))$ vertices $y_1, .., y_n \in V(H_x)$ of degree one, such that $wy_i \in E(H)$ and $y_i \in B_2^x$ for each i.*

If $\mathcal{Q}$ is a $\mathcal{T} \cup \{H\}$-packing of G corresponding to B_2^x then $\mathcal{Q}$ uses no copy of H.

Proof of Claim 4.4.1. Assume $\mathcal{Q}$ uses a copy H' of H. As $|H'| \leq |B_2^x| \leq |H| + 1$, we have $B_2^x = V(H')$. If the vertex x has h neighbors of degree one in H $(0 < h < k(\mathcal{T}))$, then we get a contradiction, because H' would have more vertices with $h + 1$ neighbors of degree one than H.

If x has no neighbors of degree one then let us denote by $c(H)$ the set of all vertices of H that have neighbors of degree one. Let $dst(H) = \sum_{u,v \in c(H)} (1 + dist(u, v))$, where $dist(u, v)$ is the distance between u and v. We get $dst(H') > dst(H)$, which is again a contradiction.
End of Proof of Claim 4.4.1.

Let us start discussing the structure of H: According to Lemma 4.4 we know that if H is neither $\mathcal{T}$-saturable nor hypomatchable then there exists a vertex $x \in V(H)$, such that $\mu_{\mathcal{T}}(H_x) \leq |H_x| - 2$. Let us denote by $A(H)$ the set of all such vertices of H. There are several cases:

Case 1 (analogous to Case 1 of [8]): There exists a vertex $a \in A(H)$ with less than $k(\mathcal{T})$ neighbors of degree one.
Let $G = H_a$ (see Figure 2c). Let B_1 be a base of $M(G, \mathcal{T} \cup \{H\})$ containing $V(H)$. We know that $|B_1| \geq |G| - 1$ using a copy of H. Let B_2^a be the base defined in Claim 4.4.1. By this Claim B_2^a uses no copy of H and thus every $(\mathcal{T} \cup \{H\})$-packing corresponding to B_2^a is in fact a $\mathcal{T}$-packing. Because $a \in A(H)$, we have $|B_2^a| \leq |H_a| - 2 < |B_1|$ and so G is a counter-example with two bases of different cardinality.

If Case 1 does not occur then every vertex from $A(H)$ has at least $k(\mathcal{T})$ neighbors of degree one. Let us denote the set of all such neighbors by $B(H)$. On the other hand, also the following Claim holds:

CLAIM 4.4.2. *If $a \in V(H)$ has $k \geq k(\mathcal{T})$ neighbors of degree one, then $a \in A(H)$.*

Proof of Claim 4.4.2. Let B be the set of $k \geq k(\mathcal{T})$ neighbors of a of degree one in H. If $a \notin A(H)$ then $\mu_{\mathcal{T}}(H_a) \geq |H_a| - 1$. Let $\mathcal{Q}$ be a maximal $\mathcal{T}$-packing of H_a covering all vertices from B. We know that $\mathcal{Q}$ skips a_a, otherwise $S_{k+1} \in \mathcal{T}$. Hence $\mathcal{Q}$ saturates H, which is a contradiction.
End of Proof of Claim 4.4.2.

Let us follow the discussion and introduce other Cases:

Case 2 (analogous to Case 2 of [8]): $\mu_{\mathcal{T}}(H \setminus b) \leq |H| - 3$ for some $b \in B(H)$.
Denote by a the unique neighbor of b in H. Consider the graph H_b (vertex b_a and edge bb_a were added to H). Let $\mathcal{Q}$ be a maximal $\mathcal{T}$-packing of H_b covering b_a.

If $\mathcal{Q}$ covers the edge bb_a by a copy of K_2 then $|V(\mathcal{Q})| = |H_b| - 2$ and so $b \in A(H)$. Since b has no neighbors of degree one, we may use Case 1.

If $S_2 \in \mathcal{T}$ and $\mathcal{Q}$ covers b_a by a copy abb_a of S_2 then $\mathcal{Q}$ skips all of the $k(\mathcal{T}) - 1 \geq 1$ neighbors of a with degree one different from b. Moreover, we may observe that the number of vertices skipped by $\mathcal{Q}$ is strictly greater than 1, otherwise H is $\mathcal{T}$-saturable. Hence $|V(\mathcal{Q})| \leq |H_b| - 2$ and so $b \in A(H)$, which leads to Case 1.

It remains to inspect maximal $\mathcal{T}$-packings of H_b that cover bb_a by a 1-propeller different from S_2. Without loss of generality we may suppose that $S_2 \notin \mathcal{T}$ (otherwise the use of propellers different from stars could be eliminated). Let $\mathcal{Q}$ be a maximal $\mathcal{T}$-packing of H_b covering bb_a by a 1-propeller $P \neq S_2$. We know that $|V(\mathcal{Q})| \geq |H_b| - 1$, otherwise we can use Case 1. Let $\mathcal{N}$ be a $\mathcal{T}$-packing of H that arises from $\mathcal{Q}$ by substituting P with a perfect matching of $P \setminus b_a$. Obviously $|V(\mathcal{N})| \geq |H| - 1$ and because H is not $\mathcal{T}$-saturable, $\mathcal{N}$ is maximal w.r.t. H and $|V(\mathcal{N})| = |H| - 1$.

Let $d \neq b$ be an arbitrary vertex adjacent in P to a. We may observe that d has no neighbor of degree one in H. Such a vertex d' could not be in P, because hypomatchable graphs do not contain vertices of degree 1, but we could easily enlarge $V(\mathcal{Q})$ by d', which is a contradiction with the maximality of $\mathcal{Q}$. Consider the graph H_d. $\mathcal{N}$ is a $\mathcal{T}$-packing of H_d skipping 2 vertices. We will show that $\mathcal{N}$ is maximal w.r.t. H_d and so $d \in A(H)$:

If $\mathcal{N}$ is not maximal w.r.t. H_d then there exists a $\mathcal{T}$-packing $\mathcal{L}$ of H_d with $V(\mathcal{L}) \supsetneq V(\mathcal{N})$. If $d_a \notin V(\mathcal{L})$ then $\mathcal{L}$ proves that $\mathcal{N}$ is not maximal w.r.t. H, which is a contradiction. So $d, d_a \in V(\mathcal{L})$ and because $|V(\mathcal{N})| = |H| - 1$, we get $|V(\mathcal{L})| \geq |H_d| - 1$. Let us pay attention to the edge ab: $\mathcal{L}$ must cover the edge ab by a copy of K_2 or by a propeller R with $d \notin V(R)$. We will construct a new $\mathcal{T}$-packing $\mathcal{L}'$: $\mathcal{L}'$ arises from $\mathcal{L}$ by replacing the edges

$d_a d$ and ab with the edge ad and by substituting the new graph covering a with its perfect $\mathcal{T}$-packing if necessary. It can be simply verified that $\mathcal{L}'$ is a correctly defined $\mathcal{T}$-packing skipping b with $|V(\mathcal{L}')| \geq |H| - 2$, which contradicts the assumption of Case 2.

We have shown that $\mathcal{N}$ is maximal w.r.t. H_d. Since $|V(\mathcal{N})| = |H_d| - 2$ and $\mathcal{T}$ is a matroid-inducing family, we get $d \in A(H)$. Because d has no neighbor of degree one in H, we may use Case 1. This completes Case 2.

Case 3 (analogous to Case 3 of [8]): $\mu_{\mathcal{T}}(H \setminus b) = |H| - 2$ for every $b \in B(H)$.
Let t be the maximum integer such that H can be decomposed as in Fig. 1, where $z \in B(H)$ and $\bigcup_{i=s}^{t}(N_i)$ is not $\mathcal{T}$-saturable for any $s = 1, ..., t$. Let us construct a counter-example G such that $M(G, \mathcal{T} \cup \{H\})$ is not a matroid: G arises from two copies H_1, H_2 of H (vertices of each H_i are indexed by $_i$) by glueing the vertex z_2 of H_2 to the unique neighbor c_1 of z_1 in H_1. (The construction from Case 3 of [8] was used - see Figure 2d).

Let B_1 be a base of $M(G, \mathcal{T} \cup \{H\})$ containing $V(H_1)$. Then $|B_1| \geq |G| - 1$ using a copy H_1 of H and a maximum $\mathcal{T}$-packing of $H_2 \setminus z_2$. Let B_2 be a base containing $V(H_2)$. Assume that $|B_2| = |B_1|$. Because neither H_2 nor $H_2 \setminus z_2$ is $\mathcal{T}$-saturable, we have to use a copy H' of H with $H' \cap H_2 \neq \varnothing$. Denote $T_2 = H' \cap H_2$. Let $(z_2, N_1, ..., N_t)$ be the decomposition of H_2 as in Figure 1. Denote $N_i' = N_i \cap T_2$. Because $H_2 \setminus T_2$ must be $\mathcal{T}$-saturable, T_2 has to intersect every N_i, and so $N_i' \neq \varnothing$, $i = 1, ..., t$. Moreover, $\bigcup_{i=s}^{t}(N_i')$ is not $\mathcal{T}$-saturable for any $s = 1, ..., t$ (in particular, $T_2 = \bigcup_{i=1}^{t}(N_i')$ is not $\mathcal{T}$-saturable).

By a cardinality argument H' contains the vertex $x = z_2 = c_1$. Denote $T_1 = H' \cap H_1$. If x has $k(\mathcal{T})$ neighbors $r_1, ..., r_k$ of degree one in T_1, then $x \in A(H')$ and $(r_1, T_1 \setminus r_1, N_1', ..., N_t')$ is a decomposition of H' into $t + 1$ parts.

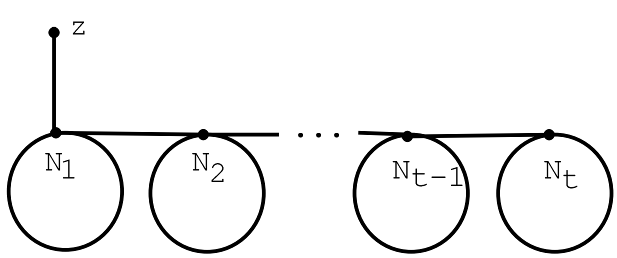

Figure 1: Case 3 - Decomposition of H

Because $r_1 \in B(H')$, by the assumption of Case 3 $H' \setminus r_1 = (T_1 \setminus r_1) \cup \bigcup_1^t (N_i')$ is not $\mathcal{T}$-saturable and we get a contradiction with the maximality of t.

If x has less than $k(\mathcal{T})$ neighbors of degree one in T_1 then without loss of generality assume that H' does not contain z_1. Thus $(H_1 \setminus z_1) \setminus H'$ has to be $\mathcal{T}$-saturable and so $H' \cap (H_1 \setminus \{z_1, x\})$ is non-$\mathcal{T}$-saturable. Hence $\mu_{\mathcal{T}}(H' \setminus x) = |H'| - 3$. The graph $T_1 = H' \cap H_1$ is in this case also non-$\mathcal{T}$-saturable since $\mu(H_1 \setminus z_1) = |H_1| - 2$. We will prove that $x \in A(H')$ and then, because x has less then $k(\mathcal{T})$ neighbors of degree 1 in H', we will use Case 1:

For a contradiction, let $x \notin A(H')$. Then there exists a maximal $\mathcal{T}$-packing $\mathcal{Q}$ of H_x' with $|V(\mathcal{Q})| \geq |H_x'| - 1$. The added vertex $x_a \in V(\mathcal{Q})$, otherwise H' is $\mathcal{T}$-saturable. If $\mathcal{Q}$ covers the new edge xx_a by a copy of K_2 then either T_1 or T_2 is $\mathcal{T}$-saturable, which is a contradiction. If $\mathcal{Q}$ covers the new edge xx_a by a propeller P then by Heredity of $\mathcal{T}$, one of the graphs T_1, T_2, $T_1 \setminus x$, $T_2 \setminus x$ is $\mathcal{T}$-saturable, which is a contradiction. This completes Case 3.

Case 4. $\mu_{\mathcal{T}}(H) = |H| - 1$ *and there exists a vertex* $a \in A(H)$ *and a component* D *of* $H \setminus a$ *that is saturated by every maximal* $\mathcal{T}$*-packing of* H*.* Note that in this case $H \setminus D$ cannot be saturated by any $\mathcal{T}$-packing of H, otherwise a base containing $V(H \setminus D)$ would lead to a contradiction. Because all vertices from $A(H)$ have neighbors of degree one, there must exist a vertex $d \in D \setminus A(H)$. Thus $\mu_{\mathcal{T}}(H_d) \geq |H_d| - 1$. Assume that the distance between a, d is maximum possible. Let us construct a graph G such that $M(G, \mathcal{T} \cup \{H\})$ is not a matroid: G arises from two copies H_1, H_2 of H (vertices of each H_i are indexed by $_i$) and one new vertex r_0 by adding the edge $r_0 d_1$ and edges $a_2 x_1$ for every $x_1 \in V(H_1)$ such that $x_1 d_1 \in E(H_1)$ (see Figure 2e). Let us find two bases of $M(G, \mathcal{T} \cup \{H\})$ with different cardinality:

Let B_2 be a base containing $V(H_2)$. Then $|B_2| \geq |G| - 1$ using two copies H_1, H_2 of H (vertex r_0 will remain uncovered). Let B_1 be a base containing $V(H_1) \cup \{r_0\}$. Since $H_1 \setminus D_1$ is not saturated by any $\mathcal{T}$-packing of H_1, B_1 has to use a copy H' of H intersecting $H_1 \setminus D_1$ in a nonempty and non-$\mathcal{T}$-saturable subgraph. Obviously $a_1 \in V(H')$, and so all $k(\mathcal{T})$ neighbors of a_1 with degree one from H_1 are in $V(H')$. According to Claim 4.4.2 $a_1 \in A(H')$. Denote $T_1 = H' \cap (H_1 \setminus D_1)$. We know that every maximal $\mathcal{T}$-packing of H' skips exactly one vertex. The skipped vertex is always in T_1, otherwise there exists a $\mathcal{T}$-packing of H_1 saturating $H_1 \setminus D_1$, which is a contradiction. So $H' \setminus T_1$ is saturated by every maximum $\mathcal{T}$-packing of H'.

If H' does not intersect H_2 then by a cardinality argument $d_1 \in V(H')$. Thus $r_0 \in V(H')$ and there has to be a vertex $y \in V(H_1) \setminus V(H')$. Note that the degree of y in H_1 must be 1, otherwise similarly as in Claim 4.4.1, $dst(H') > dst(H)$. Vertex y has to be covered by a graph containing the edge ya_2 and so y has to be a neighbor of d_1. If $\mu_{\mathcal{T}}(H_{1,y}) \geq |H_{1,y}| - 1$ then we have a contradiction with the supposed maximality of the distance between a, d. Thus $\mu_{\mathcal{T}}(H_{1,y}) \leq |H_{1,y}| - 2$ and so $y \in A(H_1)$. Because y has no neighbors of degree one in H_1, we may use Case 1.

If H' intersects H_2 then $a_2 \in V(H')$. If $|B_1| = |B_2| \geq |G| - 1$ then $H' \setminus a_2$ must have a non-$\mathcal{T}$-saturable component $B \subseteq H_2 \setminus a_2$. Let $\mathcal{Q}$ be a maximal $\mathcal{T}$-packing of H'. We know that $\mathcal{Q}$ saturates all the vertices of $H' \setminus T_1$. Hence $\mathcal{Q}$ saturates B and so there is a graph $L \in \mathcal{Q}$ covering a_2 and intersecting B in a nonempty subgraph. If $|L \cap B| > 1$ then $L \cap B$ is a part of a hypomatchable subgraph or a propeller and there exists a vertex $w \in L \cap B$ such that $\mu_{\mathcal{T}}(H' \setminus w) \geq |H' \setminus w| - 1$. Therefore $\mu_{\mathcal{T}}(H'_w) \geq |H'_w| - 1$, which contradicts the supposed maximality of the distance between a and d. Thus $L \cap B = \{v\}$ (a single vertex). The vertex v has no neighbors of degree one in H' (these could not be covered by $\mathcal{Q}$). If $\mu_{\mathcal{T}}(H'_v) \geq |H'_v| - 1$ then v contradicts the supposed maximality of the distance between a and d, otherwise $v \in A(H')$ and because v has no neighbors of degree one, we may use Case 1. We have proved that $|B_1| \neq |B_2|$, which proves that $\mathcal{T}$ is not a matroid-inducing family. This is the end of Case 4.

Case 5. $\mu_{\mathcal{T}}(H) = |H| - 1$, and there are vertices $b \in B(H)$ and $x \in V(H)$, such that $\mu_{\mathcal{T}}(H \setminus b) = |H| - 1$ and $\mu_{\mathcal{T}}(H \setminus x) = |H| - 2$.

Let $a \in A(H)$ be the (unique) neighbor of b. We know that $x \neq a$, otherwise H is $\mathcal{T}$-saturable. The counter-example graph G is constructed as follows: G arises from two copies H_1, H_2 of H (vertices of each H_i are denoted by the index $_i$) by glueing the vertex a_2 to x_1 (see Figure 2f). Let B_2 be a base containing $V(H_2)$. $|B_2| \geq |G| - 1$ using a copy of H and a maximal $\mathcal{T}$-packing of $H_1 \setminus x_1$. Let B_1 be a base containing $V(H_1)$ and let $\mathcal{N}$ be a $\mathcal{T} \cup \{H\}$-packing associated with B_1. Because neither H_1 nor $H_1 \setminus a_2$ is $\mathcal{T}$-saturable, $\mathcal{N}$ has to use a copy H' of H intersecting $H_1 \setminus a_1$ in a non-empty and non-$\mathcal{T}$-saturable subgraph. By a cardinality argument, $a_2 \in V(H')$. If H' does not contain all the neighbors of a_2 from $B(H_2)$ then without loss of generality $\mathcal{N}$ skips b_2. If $|B_1| \geq |G| - 1$ then $V(\mathcal{N}) = V(G \setminus b_2)$. Let $\mathcal{Q}$ be a maximal $\mathcal{T}$-packing of H'. We know that $\mathcal{Q}$ skips exactly one vertex of H' and so either $H' \cap H_1 \setminus a_2$ or $H' \cap H_2 \setminus a_2$ is saturated by $\mathcal{Q}$. By Heredity, one of the graphs $H_1, H_2, H_1 \setminus a_2, H_2 \setminus a_2 b_2$ is $\mathcal{T}$-saturable, which is a contradiction.

If H' contains all the neighbors of a_2 from $B(H_2)$ then $a_2 \in A(H')$. Note that every $\mathcal{T}$-packing of H' skips exactly one vertex in $H' \cap (H_1 \setminus x_1)$, otherwise H_1 or $H_1 \setminus x_1$ is $\mathcal{T}$-saturable by Heredity. Thus $H' \cap (H_2 \setminus a_2) \neq \varnothing$ is saturated by every maximal $\mathcal{T}$-packing of H' and we may use Case 4. This completes Case 5.

Case 6 (analogous to Case 4 of [8]): There are two vertices $a, a' \in A(H)$, such that if $b \in B(H)$ is a neighbor of a or a' then $H \setminus b$ is $\mathcal{T}$-saturable.
If L is a graph and $\mathcal{Q}$ is its $\mathcal{T}$-packing then we call the size of $|V(L) \setminus V(\mathcal{Q})|$ the *defect of* $\mathcal{Q}$. The *defect of* L is the defect of a maximum $\mathcal{T}$-packing of L. Let $b, b' \in B(H)$ be neighbors of a, a', respectively. Let us denote by m the defect of $H \setminus \{a, b, a', b'\}$, and by n, n' the defects of $H \setminus \{a, b\}$ and $H \setminus \{a', b'\}$, respectively.

At first assume that $m < n + n'$ and let us construct a counter-example graph G: G arises from three copies H_0, H_1, H_2 of H (vertices of each H_i are indexed by $_i$) by glueing vertices a_0 to a_1, b_0 to b_1, a'_0 to a'_2 and b'_0 to b'_2 (and deleting the multiple edges). For each $i \in \{1, 2\}$ we will denote $I_i = H_i \setminus H_0$. (The construction from Case 4 of [8] was used - see Figure 2g).

Let B_1 be a base containing $V(H_1) \cup V(H_2)$. Then $|B_1| \geq |G| - m$. Let B_2 be a base containing $V(H_0)$. Because H_0 is not $\mathcal{T}$-saturable, B_2 has to use a copy H' of H intersecting H_0. If $H' = H_0$ then $|B_2| = |G| - (n + n') < |G| - m \leq |B_1|$, which proves that $\mathcal{T}$ is not matroid-inducing.

If $H' \neq H_0$ then $|H' \cap I_1| \geq 1$ or $|H' \cap I_2| \geq 1$. Without loss of generality let $|H' \cap I_1| \geq 1$. Note that H' has to cover all neighbors of a_0 of degree one in H_0 and so according to Claim 4.4.2, $a_0 \in A(H')$. If there does not exist a $\mathcal{T}$-packing of H' saturating $H' \cap H_0$ then $H' \cap I_1$ is saturated by every maximum $\mathcal{T}$-packing of H'. Because $H' \cap I_1$ contains one or more components of $H' \setminus a_0$ and $a_0 \in A(H')$, we can use Case 4. If there exists a $\mathcal{T}$-packing of H' saturating $H' \cap H_0$ then by Heredity of $\mathcal{T}$, $H' \cap H_0$ is $\mathcal{T}$-saturable. Hence $H_0 \setminus H'$ is not $\mathcal{T}$-saturable and B_2 has to use another copy H'' of H with $|H'' \cap H_2| > 0$, $|H'' \cap H_0| > 0$ and $(H'' \cap H_0)$ not $\mathcal{T}$-saturable. Similarly as above, $a'_0 \in A(H'')$. It follows that $H'' \cap H_2$ is saturated by every maximum $\mathcal{T}$-packing of H'' and so we may use Case 4.

It remains to prove that $m \geq n + n'$ does not occur: If Case 5 does not hold then for every $x \in V(H)$, the defect of $H \setminus x$ is either 0 or at least 2. Note that if the defect is at least 2 then x is covered by an edge or by a center of a propeller in each maximal $\mathcal{T}$-packing of H. Let $\mathcal{Q}$ be an economical maximal $\mathcal{T}$-packing covering both ab and $a'b'$. Assuming that K_2 is a "0-propeller", the vertices a, a' have to be covered by centers of propellers in $\mathcal{Q}$. Let us denote the propellers covering a, a' by P, P', respectively and assume that P, P' are rooted in b, b', respectively and that $\mathcal{Q}$ is a $\mathcal{T}$-packing with minimum sum of indexes of P, P'. Let $\mathcal{Q}'$ be a $\mathcal{T}$-packing of H constructed from $\mathcal{Q}$ by substituting all blades of the two mentioned rooted propellers by their maximum matchings (one vertex in every blade will remain uncovered). It is simple to prove that $\mathcal{Q}'$ is an economical $\mathcal{T}$-packing of H (generally, every $\mathcal{T}$-packing that arises from an economical $\mathcal{T}$-packing by substituting some hypomatchable graphs and blades of propellers by their maximum matchings is economical). Note that there is one more vertex $y \notin V(P) \cup V(P')$ skipped by $\mathcal{Q}'$: y is the vertex skipped by the original $\mathcal{T}$-packing $\mathcal{Q}$. If the sum of the indexes of P, P' was a minimum then the defect of $\mathcal{Q}'$ is exactly m, since in this case $\mathcal{Q}'$ induces a maximal $\mathcal{T}$-packing in $H \setminus \{a, b, a', b'\}$.

Similarly, we will construct two other $\mathcal{T}$-packings $\mathcal{N}_1, \mathcal{N}_1'$ of $H \setminus \{ab\}$ and $H \setminus \{a'b'\}$, respectively. $\mathcal{N}_1$ and $\mathcal{N}_1'$ arise from $\mathcal{Q}$ by replacing blades of only one rooted propeller P or P', respectively, by their maximum matchings and by skipping the edge $ab, a'b'$, respectively. If both $\mathcal{N}_1, \mathcal{N}_1'$ are maximal, then obviously $m < n + n'$.

Let us assume that without loss of generality $\mathcal{N}_1$ is not maximal. Let $\mathcal{N}_2 \supseteq \mathcal{N}_1$ be an economical maximal $\mathcal{T}$-packing of $H \setminus ab$. The $\mathcal{T}$-packing $\mathcal{N} = \mathcal{N}_2 \cup \{ab\}$ is an economical $\mathcal{T}$-packing of the whole H. Assume that $\mathcal{Q}$ and $\mathcal{N}$ are arbitrarily rooted.

Subcase 1. If $y \in \mathcal{N}$ then every vertex skipped by $\mathcal{N}$ lies in a component of $\mathcal{Q} \cup \mathcal{N}$ intersected by P in $\mathcal{Q}$. Let $\mathcal{C} = \{C_1, ..., C_t\}$ be a collection of all such components and let $V(\mathcal{C})$ denote the set of all vertices in all C_i. We may construct a perfect $\mathcal{T}$-packing of H by exchanging the graphs of $\mathcal{N}$ intersecting $V(\mathcal{C}) \cup \{a, b\}$ with graphs of $\mathcal{Q}$ intersecting $V(\mathcal{C}) \cup \{a, b\}$ (we will use a subpropeller of P) and by replacing the newly constructed graphs by their perfect $\mathcal{T}$-packings where necessary, which is a contradiction.

Subcase 2. If $y \notin \mathcal{N}$ then $\mathcal{N}$ covers a vertex z covered by a blade of P in $\mathcal{Q}$. Since Case 5 does not occur, $H \setminus z$ is $\mathcal{T}$-saturable. Let $\mathcal{D}$ be an economical maximal rooted $\mathcal{T}$-packing of H saturating $V(H) \setminus z$. We will subsequently modify $\mathcal{N}$ and $\mathcal{Q}$ to obtain a contradiction. Throughout the sequence of modifications, we will maintain the following invariant:

(I) At most one vertex skipped by $\mathcal{N}$ (denoted by y) is not covered by a propeller with center a in $\mathcal{Q}$. If y exists then z is covered by a propeller with center a in $\mathcal{Q}$.

At the beginning, (I) is satisfied. There are several situations:

Situation A. If y, z are in the same component C of $\mathcal{D} \cup \mathcal{N}$ then by replacing all graphs (edges) of $\mathcal{N}$ in C with the graphs (edges) of $\mathcal{D}$ in C, we get Subcase 1.

Situation B. If the component C of $\mathcal{D}\cup\mathcal{N}$ containing y is intersected by a hypomatchable graph in $\mathcal{D}$ then we get a contradiction with the maximality of $\mathcal{N}_2$.

Situation C. If the component C of $\mathcal{D}\cup\mathcal{N}$ containing y is intersected by a blade B of a rooted propeller (W, r) with center $c \neq a$ in $\mathcal{D}$ then let us focus on the vertex c. If c is uncovered by $\mathcal{N}$ or covered by a hypomatchable graph, by a root or a blade of a rooted propeller or by a copy of K_2 in $\mathcal{N}$ then we get a contradiction with the maximality of $\mathcal{N}_2$. Thus c is a center of a rooted propeller (W', r') in $\mathcal{N}$. If $D(W', r') \subseteq D(W, r)$ then we may enlarge $\mathcal{N}_2$ to saturate $V(C)$, which is a contradiction. Otherwise, by Blade exchange there exists a blade $B' \in D(W', r') \setminus D(W, r)$ such that the graph induced by $V(W' \setminus B') \cup V(B)$ is $\mathcal{T}$-saturable. Let us modify $\mathcal{N}$ by replacing W' with a perfect $\mathcal{T}$-packing of $V(W' \setminus B') \cup V(B)$, by replacing the graphs of $\mathcal{N}$ in C with the graphs of $\mathcal{D}$ in C and by replacing B' with a maximum matching of $B' \setminus x_1$, where $x_1 \in V(B')$ is arbitrary. After this modification, let us newly consider $y = x_1$, observe that (I) is satisfied and continue our sequence of modifications according to the current Situation.

Situation D. The last situation occurs when the component C of $\mathcal{D} \cup \mathcal{N}$ containing y is intersected by a blade B of a rooted propeller (W, r) with center a in $\mathcal{D}$. Without loss of generality we may suppose that $y \in V(B)$ and $C = B$, since $\mathcal{N}$ may be simply modified to satisfy this condition.

If there is a blade Y of P with $V(Y) \cap V(B) \neq \varnothing$ then let $x_2 \in V(Y) \cap V(B)$ and let us modify $\mathcal{N}$ by replacing the graphs of $\mathcal{N}$ in B with a perfect matching of $B \setminus x_2$. After this modification we are in Subcase 1.

If B does not intersect any blade of P then according to Blade exchange, there is a blade D of P such that the graph induced by $V(P \setminus D) \cup V(B)$ is $\mathcal{T}$-saturable. Let us modify $\mathcal{Q}$ by replacing the graphs of $\mathcal{Q}$ with the graphs of $\mathcal{D}$ in the component of $\mathcal{Q} \cup \mathcal{D}$ containing B and by replacing P with a perfect $\mathcal{T}$-packing of the graph induced by $V(P \setminus D) \cup V(B)$ and with a perfect matching of $D \setminus x_3$ where $x_3 \in V(D)$ is arbitrarily selected if $\mathcal{N}$ saturates D and is the unique vertex of D skipped by $\mathcal{N}$ otherwise.

If $\mathcal{N}$ saturates D (this occurs e.g. when $z \in V(D)$) then this modification leads to Subcase 1. If $\mathcal{N}$ does not saturate $V(D)$ then let us newly consider

$y = x_3$ and observe that (I) is satisfied. Hence we may continue our sequence of modifications according to the current Situation.

For a rooted $\mathcal{T}$-packing $\mathcal{L}$, we denote by $D(\mathcal{L})$ the set of all graphs induced by a blade and a center of any propeller included in $\mathcal{L}$. The above sequence of modifications is finite, since in each step the size of one of $D(\mathcal{D}) \setminus D(\mathcal{Q})$, $D(\mathcal{D}) \setminus D(\mathcal{N})$ decreases. We end our sequence in Subcase 1 or by a contradiction with the maximality of $\mathcal{N}_2$. This concludes the proof of Case 6.

It remains to prove that the list of cases is complete. If H is a connected graph which is neither $\mathcal{T}$-saturable nor hypomatchable then according to Lemma 4.4, $A(H) \neq \varnothing$. If neither of Cases 1,2,3 holds then there is a vertex $b \in B(H)$ such that $H \setminus b$ is $\mathcal{T}$-saturable. Because H is not $\mathcal{T}$-saturable, we have $\mu_{\mathcal{T}}(H) = |H| - 1$. Let $a \in A(H)$ be the (unique) neighbor of b in H. If any of the components of $H \setminus a$ is saturated by every maximum $\mathcal{T}$-packing of H, then we can use Case 4. So for every component of $H \setminus a$ there exists a maximum $\mathcal{T}$-packing of H skipping one of its vertices. Let us iterate through all the components of $H \setminus a$. For every component D we will take a maximum $\mathcal{T}$-packing of H skipping one of the vertices $x \in V(D)$. We will color x red and follow the coloring algorithm described informally in the proof of Lemma 4.4 until all vertices of D are red or until we find a vertex $a' \in V(D) \cap A(H)$. If we finish with all vertices in all components colored red then every component of $H \setminus a$ is hypomatchable and H is a propeller, which was completely addressed in Section 3. Note that this way we can find not only a vertex $a' \in A(H), a' \neq a$, but also a maximal $\mathcal{T}$-packing $\mathcal{Q}$ of H skipping exactly one of the neighbors of a'. Let us observe that for every neighbor $b' \in B(H)$ of a', $\mathcal{Q}$ can be easily changed to a $\mathcal{T}$-packing skipping exactly b'. We have found two vertices $a, a' \in A(H)$, such that if $b \in B(H)$ is a neighbor of a or a' then $H \setminus b$ is $\mathcal{T}$-saturable, which can be solved by Case 6. This concludes the proof of Theorem 4.1.

5 Conclusion

We have introduced a full characterization of EHP-families of graphs $\mathcal{T}$ such that $M(G, \mathcal{T})$ is a matroid for every graph G. Moreover, we have fully characterized the enlargements of matroid-inducing EHP-families by one graph H by proving that $\mathcal{T} \cup \{H\}$ is a matroid-inducing family if and only if H is $\mathcal{T}$-saturable or $\mathcal{T} \cup \{H\}$ is a matroid-inducing EHP-family.

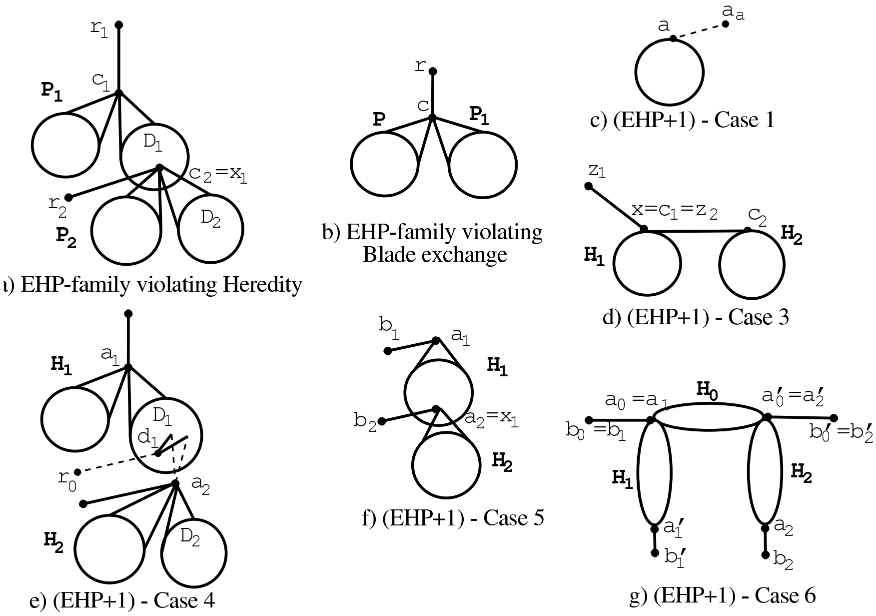

Figure 2: Summary of counter-examples

The paper studies the matroidal aspects of the $\mathcal{T}$-packing problem. Many other results have been recently extended from matching to packing by EHP-families. The most important results are those concerning complexity. These were introduced by Loebl and Poljak in [9] and [10].

Acknowledgement
The author thanks the referees for many helpful comments and Martin Loebl for helpful consultations.

References

[1] G. Cornuejols and D. Hartvigsen. An extension of matching theory. *J. Comb. Theory*, B(40):285–296, 1986.

[2] G. Cornuejols, D. Hartvigsen, and W. R. Pulleybank. Packing subgraphs in a graph. *Oper. Res. Lett.*, (1):449–467, 1982.

[3] J. Edmonds. Paths, trees and flowers. *Canad. J. Math*, (17):449–467, 1965.

[4] J. Edmonds and D. R. Fulkerson. Transversals and matroid partition. *J. Res. Nat. Bur Standards*, B(69):147–153, 1965.

[5] P. Hell and D. Kirkpatrick. Packing by cliques and by finite families of graphs. *Discrete Math.*, (49):118–133, 1984.

[6] M. Janata. Packing matroids. Master's thesis, Charles University, Prague, April 2000.

[7] M. Las Vergnas. An extension of Tutte's 1-factor problem. *Discrete Math*, (23):241–255, 1978.

[8] M. Loebl and S. Poljak. On matroids induced by packing subgraphs. *J. Comb. Theory*, B(44):338–355, 1988.

[9] M. Loebl and S. Poljak. Good family packing. In J. Nešetřil and M. Fiedler, editors, *Fourth Czechoslovak Symposium on Combinatorics, Graphs and Complexity*, pages 181–186. Elsevier Science Publishers, B.V., 1992.

[10] M. Loebl and S. Poljak. Efficient subgraph packing. *J. Comb. Theory*, 59(1):106–121, 1993.