

Rainbow Arithmetic Progressions

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Abstract

Consider a coloring of $\{1, 2, \dots, n\}$ in 3 colors, where $n \equiv 0 \pmod{3}$. If all the color classes have the same cardinality, then there is a 3-term arithmetic progression whose elements are colored in distinct colors. This rainbow variant of van der Waerden's theorem proves the conjecture of the second author.

1 Introduction

Given a coloring of a set of natural numbers, we say that a subset is *monochromatic* if all of its elements have the same color and we say that it is *rainbow* if all of its elements have distinct colors. A classical result in Ramsey theory is van der Waerden's theorem [vW27], which states that for every k and t and sufficiently large n , every k -coloring of $[n] := \{1, 2, \dots, n\}$ contains a monochromatic arithmetic progression of length t . Jungić et. al. [JLMNR] considered, for the first time in the literature, a rainbow counterpart of van der Waerden's theorem. They proved that every 3-coloring of the set of natural numbers \mathbb{N} with the upper density of each color greater than $1/6$ contains a rainbow AP(3). They also asked whether the “finite” version of their theorem also holds, and, backed by the computer evidence ($n \leq 56$), posed the following conjecture.

Conjecture 1 *For every $n \geq 3$, every partition of $[n]$ into three color classes R , G , and B with $\min(|R|, |G|, |B|) > r(n)$, where*

$$r(n) := \begin{cases} \lfloor (n+2)/6 \rfloor & \text{if } n \not\equiv 2 \pmod{6} \\ (n+4)/6 & \text{if } n \equiv 2 \pmod{6} \end{cases} \quad (1)$$

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contains a rainbow $AP(3)$.

Moreover, they constructed a 3-coloring of $[n]$ with $\min(|R|, |G|, |B|) = r(n)$, where r is the function defined in (1), that contains no rainbow $AP(3)$. This shows that Conjecture 1, if true, is the best possible.

A weaker form of this conjecture, due to the second author, was posed at the open problem session of the 2000 MIT Combinatorics Seminar [JLMNR].

Conjecture 2 *Let $n \equiv 0 \pmod{3}$. For every equinumerous 3-coloring of $[n]$, that is, a coloring in which all color classes have the same cardinality, there exists a rainbow $AP(3)$.*

In this paper, we prove Conjecture 2.

2 Proof of Conjecture 2

Given a 3-coloring of $[n]$ with colors Red (R), Green (G) and Blue (B), a B -block is a string of consecutive elements of $[n]$ that are colored Blue. R -block and G -block are defined similarly. We say that the coloring is *rainbow-free* if it contains no rainbow $AP(3)$.

First, we show that every rainbow-free 3-coloring contains a **dominant color**, that is, a color $z \in \{B, G, R\}$ such that for every two consecutive numbers that are colored with different colors, one of them is colored with z . We will need the following two lemmata.

Lemma 1 *Let $c : [n] \rightarrow \{B, G, R\}$ be a 3-coloring of $[n]$ such that every two different colors appear next to each other. Then there exist $p, r \in [n]$, $p < r$, such that*

1. $c(p) = c(r)$,
2. $c(p+1) \neq c(p)$,
3. $c(r-1) \notin \{c(p), c(p+1)\}$, and
4. no element in the interval $[p+1, r-1]$ is colored by the color $c(p)$.

Proof: Let G be the first color to appear and let the first G -block be followed by an R -block. Since G appears next to B , there exists a G -block that is next to a B -block. If this G -block is preceded by a B -block,

$$\begin{array}{ccccccc}
G \dots G & R \dots R & \dots & B \dots B & G \dots G & \dots & \\
\uparrow & & & & \uparrow & & \\
p & & & & r & &
\end{array}$$

then the lemma follows. So, suppose that \mathcal{G} , the first G -block that is next to a B -block, is preceded by an R -block and followed by a B -block \mathcal{B} .

$$\begin{array}{ccccccc}
G \dots G & R \dots R & \dots & R \dots R & G \dots G & B \dots B & \dots \\
& & & & \uparrow & \uparrow & \\
& & & & \mathcal{G} & \mathcal{B} &
\end{array}$$

Suppose there is a B -block between the two G -blocks. Consider the last such B -block and denote it by \mathcal{B}' . The lemma immediately follows, since we can take p and r to be the last element of \mathcal{B}' and the first element of \mathcal{B} .

Now, suppose there is no B -block between any the two G -blocks.

If \mathcal{B} is followed by an R -block, then the lemma clearly follows. So, let \mathcal{B} be followed by a G -block. The same reasoning as above, combined with the assumption that R appears next to B , implies that one of the following two scenarios happens:

$$\begin{array}{ccccccc}
\dots & G \dots G & R \dots R & G \dots G & B \dots B & R \dots R & \\
& & \uparrow & & \uparrow & & \\
& & p & & r & &
\end{array}$$

or

$$\begin{array}{ccccccc}
\dots & G \dots G & B \dots B & G \dots G & R \dots R & B \dots B & \\
& & \uparrow & & \uparrow & & \\
& & p & & r & &
\end{array}$$

which completes the proof. □

Lemma 2 *Let $c : [n] \rightarrow \{B, G, R\}$ be a 3-coloring of $[n]$ such that*

$$\min\{|c^{-1}(B)|, |c^{-1}(G)|, |c^{-1}(R)|\} \geq 5$$

and every two different colors appear next to each other. Then there is a rainbow AP(3).

Proof: By Lemma 1, we can assume that there are p, q, r , such that

$$\begin{array}{cccccccc}
\dots & G & R & \dots & R & B \dots B & G & \dots \\
& \uparrow & & \uparrow & \uparrow & \uparrow & \uparrow & \\
& p & & R/B & q & B & r & \\
& & & \text{only} & & \text{only} & &
\end{array}$$

So, we assume that $p+2 \leq q < r-2$, $c(p) = G$, $c(p+1) = R$, $c(q) = R$, $c(r) = G$, $c(i) = B$ for all $q < i < r$ and $c(j) \neq G$ for all $p+1 < j < q$. Since $|c^{-1}(G)| \geq 5$, then $p \geq 3$ or $r \leq n-2$. Without loss of generality, assume $p \geq 3$.¹ If $r+q$ is even, then $q, \frac{r+q}{2}, r$ is a rainbow $AP(3)$. So, let $r+q$ be odd. It follows that an even number of elements between q and r are colored by B . If $c(q-1) = R$ then $q-1, \frac{r+q-1}{2}, r$ is a rainbow $AP(3)$. Let $c(q-1) = B$ and let $s = \min\{i \in [p, q] | c(i) = B\}$. Then,

$$\begin{array}{cccccccccccc}
\dots & G & R \dots R & B & \dots & B & R & B \dots B & G & \dots \\
& \uparrow & \uparrow & \uparrow & \uparrow & & \uparrow & \uparrow & \uparrow & & \uparrow \\
& p & \text{only } R & s & R/B & & q & \text{only } B & r & & \\
& & & & & & & \text{even\#} & & &
\end{array}$$

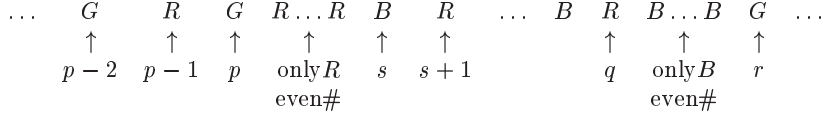
Notice that s could be equal to $q-1$. Now, if $p+s$ is even, then $p, \frac{p+s}{2}, s$ is a rainbow $AP(3)$. Otherwise, an even number of elements between p and s are colored by R . If $c(p-1) = G$, then $p-1, \frac{p+s-1}{2}, s$ is a rainbow $AP(3)$. So, let $c(p-1) = R$. If $c(s+1) = B$ (here, $s \neq q-1$), then $p, \frac{p+s+1}{2}, s+1$ is a rainbow $AP(3)$. Hence, let $c(s+1) = R$. Then, the interval $[p-1, r]$ is colored as follows.

$$\begin{array}{cccccccccccc}
\dots & R & G & R \dots R & B & R & \dots & B & R & B \dots B & G & \dots \\
& \uparrow & \uparrow & \uparrow & \uparrow & \uparrow & & & \uparrow & \uparrow & \uparrow & \\
& p-1 & p & \text{only } R & s & s+1 & & & q & \text{only } B & r & \\
& & & \text{even\#} & & & & & & \text{even\#} & &
\end{array}$$

Suppose that $p+r-1$ is even. If $c(\frac{p+r-1}{2}) = R$, then $p, \frac{p+r-1}{2}, r-1$ is a rainbow $AP(3)$. If $c(\frac{p+r-1}{2}) = B$, then $p-1, \frac{p+r-1}{2}, r$ is a rainbow $AP(3)$.

So, let $p+r-1$ be odd. If $c(p-2) = B$ then $p-2, p-1, p$ is a rainbow $AP(3)$. Suppose $c(p-2) = R$. If $c(\frac{p+r-2}{2}) = R$, then $p, \frac{p+r-2}{2}, r-2$ is a rainbow $AP(3)$. If $c(\frac{p+r-2}{2}) = B$, then $p-2, \frac{p+r-2}{2}, r$ is a rainbow $AP(3)$. Hence, the only remaining case is when $c(p-2) = G$.

¹Otherwise, consider the coloring $c' : [n] \rightarrow \{B, G, R\}$, defined by $c'(i) = c(n-i+1)$ for every $i \in [n]$.



Since an even number of elements between q and r are colored by B , there exists k , $2 \leq k < \frac{r-p}{2}$, such that $c(p+2k) = B$. Suppose that k is the smallest number with this property. Now, consider elements $\frac{p+p+2k}{2} = p+k$ and $\frac{p-2+p+2k}{2} = p+k-1$. By the property of k and the fact that one of the elements $k, k-1$ is even, we conclude that either $c(p+k) = R$ or $c(p+k-1) = R$. Then, either $p-2, p+k-1, p+2k$ or $p, p+k, p+2k$ is a rainbow $AP(3)$. \square

Lemma 2 immediately implies

Corollary 1 *Let $c : [n] \rightarrow \{B, G, R\}$ be a rainbow-free 3-coloring of $[n]$ such that*

$$\min\{|c^{-1}(B)|, |c^{-1}(G)|, |c^{-1}(R)|\} \geq 5.$$

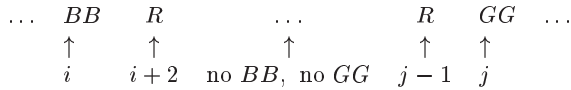
Then there exists a dominant color.

Clearly, there can be only one dominant color.

The following lemma will be instrumental in showing that in every rainbow-free 3-coloring of $[n]$ there exists a *recessive color*, that is, a color $w \in \{B, G, R\}$ such that no two consecutive numbers are colored by w .

Lemma 3 *Let $c : [n] \rightarrow \{B, G, R\}$ be a 3-coloring of $[n]$ such that R is the dominant color. Suppose there exist i and j , so that $i < j$, $c(i) = c(i+1) = B$, $c(i+2) = c(j-1) = R$, $c(j) = c(j+1) = G$. Suppose also that the interval $[i+2, j-1]$ does not contain two consecutive elements that are both colored by B or by G . Then, there is a rainbow $AP(3)$.*

Proof: Suppose there exists a rainbow-free coloring c with the properties above. Consider the interval $[i, j+1]$.



Suppose no element of $[i + 3, j - 2]$ is colored by R . Since R is the dominant color, $[i + 3, j - 2]$ is either a B -block or a G -block. Then, either $i + 1, i + 2, i + 3$ or $j - 2, j - 1, j$ is a rainbow $AP(3)$, contradicting the assumption that c is rainbow-free.

Suppose no element of $[i + 3, j - 2]$ is colored by G . Since $c(j) = G$ and $c(j - 1) = R$, then $c(j - 2) = R$. Since $c(j + 1) = G$ and $c(j - 1) = R$, then $c(j - 3) = R$. Iterating this reasoning from right to left, we conclude that $[i + 2, j - 1]$ is an R -block. Then, clearly, there exists a rainbow $AP(3)$, and, thus, we arrive at a contradiction. A symmetric argument shows that at least one of the elements of $[i + 3, j - 2]$ is colored by B .

Therefore, there is at least one element of each color in $[i + 3, j - 2]$.

Suppose $i + j + 1$ is odd. Since R is the dominant color and there are no consecutive elements in $[i + 3, j - 2]$ both colored by B or by G , at least one of the elements $\lfloor \frac{i+j+1}{2} \rfloor, \lfloor \frac{i+j+1}{2} \rfloor + 1$ is colored by R . This implies that one of the arithmetic progressions

$$i, \quad \frac{i+j}{2} = \lfloor \frac{i+j+1}{2} \rfloor, \quad j$$

$$i + 1, \quad \frac{i+j+2}{2} = \lfloor \frac{i+j+1}{2} \rfloor + 1, \quad j + 1$$

is a rainbow $AP(3)$.

Therefore, if c is rainbow-free then $i + j + 1$ is even. It follows that $c(\frac{i+j+1}{2}) \neq R$, otherwise, $i, \frac{i+j+1}{2}, j$ is a rainbow $AP(3)$.

If there exists $i + 2 < k < \frac{i+j+1}{2}$, with $c(k) = G$, then $c(2k - i) \neq R$ and $c(2k - i - 1) \neq R$, otherwise $i, k, 2k - i$ or $i + 1, k, 2k - i - 1$ is a rainbow $AP(3)$. Since R is the dominant color, it follows that $2k - i$ and $2k - i - 1$ are both colored by B or by G , which contradicts the assumed property of i and j . Therefore, if $c(k) = G$, where $i + 2 < k < j - 1$, then $k \geq \frac{i+j+1}{2}$.

A symmetric argument implies that if $c(k) = B, i + 2 < k < j - 1$, then $k \leq \frac{i+j+1}{2}$.

Without loss of generality, assume that $c(\frac{i+j+1}{2}) = G$, the other case being symmetric. Then, interval $[i, j + 1]$ is colored as follows.

$$\begin{array}{cccccccc} \dots & BB & R & \dots & RGR & \dots & R & GG & \dots \\ & \uparrow & \uparrow & & \uparrow & \uparrow & \uparrow & \uparrow & \\ & i & i + 2 & \text{no } BB & \frac{i+j+1}{2} & \text{no } GG & j - 1 & j & \\ & & & \text{no } G & & \text{no } B & & & \end{array}$$

Hence, there are no elements colored by G in $[i, \frac{i+j-1}{2}]$ and there are no elements colored by B in $[\frac{i+j+1}{2}, j + 1]$. Since an element of each color appears in $[i + 2, j - 1]$, there exists $p \geq 2$ such that $c(\frac{i+j+1}{2} - l) = R$ for

all $l \in [1, p]$ and $c\left(\frac{i+j+1}{2} - p - 1\right) = B$. Moreover, since c is rainbow-free, p must be even. It follows that $c\left(\frac{i+j+1}{2} + p + 1\right) = G$.

Let $x := \frac{i+j+1}{2} - p - 1$ and $y := \frac{i+j+1}{2} + p + 1$. Define intervals $\mathcal{I}_s = [a_s, b_s]$, $\mathcal{J}_s = [c_s, d_s]$, $s \in [0, r-1]$, where $a_s = i + 2(sp + s + 1)$, $b_s = i + 2((s+1)p + s) + 1$, $c_s = j - 2((s+1)p + s)$, $d_{s+1} = j - 2(sp + s + 1) + 1$, and r is the smallest index such that $1 \leq c_{r-1} - y = x - b_{r-1} \leq 2p$. Notice that the number of elements in each of these intervals is $2p$.

For each $l \in [p]$, the following two arithmetic progressions

$$\begin{aligned} i, \quad \frac{i+j+1}{2} - l, \quad i + j + 1 - 2l - i = j - 2l + 1 \\ i + 1, \quad \frac{i+j+1}{2} - l, \quad i + j + 1 - 2l - i - 1 = j - 2l \end{aligned}$$

are not rainbow only if both $j - 2l$ and $j - 2l + 1$ are red. Hence, \mathcal{J}_0 is an R -block. This implies that \mathcal{I}_0 is an R -block, since $c\left(\frac{i+j+1}{2}\right) = G$ and c is rainbow-free. Since \mathcal{I}_0 is an R -block and $c(x) = B$, we conclude that \mathcal{J}_1 is an R -block. This, in turn, implies that \mathcal{I}_1 is an R -block, since $c\left(\frac{i+j+1}{2}\right) = G$. Iterating this argument, we conclude that all the intervals $\mathcal{I}_s, \mathcal{J}_s, s \in [0, r-1]$, are R -blocks. We have the following two cases.

- **Case 1.** $2 < c_{r-1} - y = x - b_{r-1} \leq 2p$.

In this case, $y + 2p + 2 \in \mathcal{J}_{r-1}$ and $x, y, y + 2p + 2$ is a rainbow $AP(3)$, contradicting our assumption that c is rainbow-free.

- **Case 2.** $c_{r-1} - y = x - b_{r-1} = 1$ or $c_{r-1} - y = x - b_{r-1} = 2$.

In this case, $x - p - 1 \in \mathcal{I}_{r-1}$ and $x - p - 1, x, \frac{i+j+1}{2}$ is a rainbow $AP(3)$, thus, arriving at a contradiction.

Therefore, c cannot be rainbow-free. \square

Corollary 2 *Let $c : [n] \rightarrow \{B, G, R\}$ be a rainbow-free 3-coloring of $[n]$ such that R is the dominant color. Then, either B or G is a recessive color.*

Proof: Suppose that neither B nor G is a recessive color. Then, among all pairs of elements (i, j) , such that $c(i) = c(i+1) = B$ and $c(j) = c(j+1) = G$, choose the one where $|j - i|$ is minimal. Without loss of generality, assume that $i < j$. Then, by the choice of i and j , $c(i+2) = R, c(j-1) = R$ and interval $[i+2, j-1]$ does not contain two consecutive elements both colored by B or by G . Lemma 3 implies that c contains a rainbow $AP(3)$, which is a contradiction. \square

Finally, we are in a position to prove Conjecture 2.

Theorem 1 *Let $n \equiv 0 \pmod{3}$. For every equinumerous 3-coloring of $[n]$ there exists a rainbow AP(3).*

Proof: The claim is true for $n \leq 15$ [JLMNR]. Let $n \geq 15$, $n \equiv 0 \pmod{3}$, and let $c : [1, n] \rightarrow \{B, G, R\}$ be an equinumerous 3-coloring. Suppose that c is rainbow-free. By Corollary 1, there is a dominant color, say R . By Corollary 2, one of the remaining colors, say G , is recessive. It follows that every element colored by G is followed² by an element colored by R . Since there are elements of $[n]$ colored by B , there exists at least one pair $i, j \in [n]$, such that $c(i) = B$, $c(j) = G$, and all the elements between i and j are colored with R . Since the number of elements between i and j must be greater than or equal to two, at least one of these elements is such that both of its neighbors are not colored by G . It follows that $|c^{-1}(G)| < |c^{-1}(R)|$, which contradicts our assumption that c is equinumerous. Therefore, c is not rainbow-free. \square

3 Concluding remarks

This note settles the question of the existence of a rainbow arithmetic progression in equinumerous 3-colorings of $[n]$. However, Conjecture 1 is still open. We hope that our lemmas in Section 2 with some additional ideas could prove that conjecture as well.

There are many directions and generalizations one might consider. For a discussion on this topic, as well as similar results for \mathbb{Z}_n , consult [JLMNR]. One natural direction is imitating the well known Rado's theorem for the monochromatic analogue [GRS90] and generalizing the problems above for rainbow solutions of other linear equations, under appropriate conditions on the cardinality of the color classes. The equation $x + y = z$ has already been studied. Alekseev and Savchev [AS87, G94] proved that every equinumerous 3-coloring of $[3n]$ contains a rainbow solution of this equation. Schönheim [Sch90, S95], answering the question of E. and G. Szekeres, proved that for every 3-coloring of $[n]$, such that every color class has cardinality greater than $n/4$, the equation $x + y = z$ has rainbow solutions. Here, $n/4$ is optimal.

Finally, it would be very interesting to prove similar rainbow-type results for longer arithmetic progressions and larger numbers of colors. For

²or preceded, if $c(n) = G$

example, it is not known whether every equinumerous 4-coloring of $[4n]$ contains a rainbow $AP(4)$. However, note that for every n and $k > 3$, there exists a k -coloring of $[n]$ with no rainbow $AP(k)$ and with each color class of size at least $\lfloor \frac{n+2}{3 \lfloor (k+4)/3 \rfloor} \rfloor$ [JLMNR].

References

- [AS87] V. E. Alekseev, S. Savchev: Problem M. 1040, *Kvant* **4** (1978), 23.
- [G94] R. K. Guy: Unsolved Problems in Number Theory, *Springer-Verlag* 1994. (E11–12)
- [GRS90] R. L. Graham, B. L. Rothschild, J. H. Spencer: Ramsey Theory, *John Wiley and Sons* 1990.
- [JLMNR] V. Jungić, J. Licht, M. Mahdian, J. Nešetřil, R. Radoičić: Rainbow arithmetic progressions and anti-Ramsey results, *Combinatorics, Probability, and Computing*, to appear.
- [Sch90] J. Schönheim: On partitions of the positive integers with no x , y , z belonging to distinct classes satisfying $x + y = z$, in *R. A. Mollin (ed.) Number theory (Proceedings of the First Conference of the Canadian Number Theory Association, Banff 1988)*, de Gruyter (1990), 515–528.
- [S16] I. Schur: Über die Kongruenz $x^m + y^m \equiv z^m \pmod{p}$, *Jahresb. Deutsche Math. Verein* **25** (1916), 114–117.
- [S95] G. Székely, ed.: Contests in Higher Mathematics, Miklós Schweitzer Competition 1962-1991, Problem Books in Mathematics, *Springer* 1995. (C.22)
- [Sz75] E. Szemerédi: On sets of integers containing no k elements in arithmetic progression, *Acta Arithmetica* **27** (1975), 199–245.
- [vW27] B. L. van der Waerden: Beweis einer Baudetschen Vermutung, *Nieuw Archief voor Wiskunde* **15** (1927), 212–216.