Noncrossing Hamiltonian Paths in Geometric Graphs

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Abstract

A geometric graph is a graph embedded in the plane in such a way that vertices correspond to points in general position and edges to segments connecting appropriate points. A noncrossing Hamiltonian path is a Hamiltonian path whose embedding in the plane is a simple curve in the plane. We study the problem asked by Micha Perles about the existence of a noncrossing Hamiltonian path for various classes of geometric graphs. Let $h_1(n)$ denote the largest k such that when we remove edges of an arbitrary complete subgraph of size at most k from a complete geometric graph on n vertices the resulting graph still has a noncrossing Hamiltonian path. We prove that there exist constants $0 < c_1 < c_2$ such that $c_1 \cdot \sqrt{n} < h_1(n) < c_2 \cdot \sqrt{n}$. Let $h_2(n)$ denote the largest k such that when we remove an arbitrary star with at most k edges from a complete geometric graph on n vertices the resulting graph still has a noncrossing Hamiltonian path. We show that $h_2(n) = \lceil n/2 \rceil - 1$. We also prove that $h_3(n) = \lceil n/2 \rceil - 1$ where $h_3(n)$ is the largest k such that when we remove at most k arbitrary edges from a complete geometric graph on n vertices whose vertices are in convex position then the graph still has a noncrossing Hamiltonian path. Further we prove that when we remove any matching from a complete geometric graph the resulting graph will have a noncrossing Hamiltonian path.

1 Introduction

A *geometric graph* is a graph drawn in the plane so that vertices are represented by points in general position (i.e., there are no three collinear points) and its edges are straight–line segments connecting the corresponding points. A *topological graph* is defined similarly, except that an edge is not drawn as a segment but as a simple (not–selfintersecting) Jordan arc which does not pass through any vertices except its endpoints. Further we assume for simplicity that if two edges of topological graph have a point in common then they intersect in this point. If any two edges of topological graph intersect in at most one point we say that the graph is *simple*. So any geometric graph is a special case of a simple topological graph.

Lately geometric and topological graphs are intensively studied topic. There are lots of papers studying which number of edges forces a geometrical or topological graph to contain some fixed subconfiguration (the trivial result of this type following from Euler's polyhedral formula is that any topological graph with at least $3n - 5$ edges must have two edges which intersect). In recent paper [6] Pach et. al. has proven that any topological graph which does not have three pairwise crossing edges must have at most $O(n)$ edges. This improves the previously know bound $O(n^{3/2})$ and also generalizes the result of Agarwal et. al. ([1]) who proved the bound $O(n)$ for geometric graphs. In another recent paper [4] has been established that for any k and l there exists a constant $C(k, l)$ such that if a topological graph has at least $C(k, l) \cdot n$ edges then it must have a $k \times l$–gridlike configuration (i.e., it contains $k + l$ edges such that each of the first k edges cross each of the last l edges). Further interesting results can be found in [8], [7], [5], [9] to name a few or in surveys on geometric and topological graphs ([3], [2]).

In our paper we study a problem when a given geometric graph has a noncrossing Hamiltonian path (i.e. Hamiltonian path which does not cross itself). The problem was presented by Micha Perles on DIMACS Workshop on Geometric Graph Theory in 2002. In our paper we focus on some classes of geometrical graphs. Let $h_1(n)$ denote the largest k such that when we remove edges of an arbitrary complete subgraph of size at most k from a complete geometric graph on n vertices the resulting graph still has a noncrossing Hamiltonian path. We prove that there exist constants $0 < c_1 < c_2$ such that $c_1 \cdot \sqrt{n} < h_1(n) < c_2 \cdot \sqrt{n}$ (Theorems 1 and 3). Let $h_2(n)$ denote the largest k such that when we remove an arbitrary star with at most k edges from a complete geometric graph on n vertices the resulting graph still has a noncrossing Hamiltonian path. In Theorem 6 we show that

$h_2(n) = \lceil n/2 \rceil - 1$. We also prove that $h_3(n) = \lceil n/2 \rceil - 1$ where $h_3(n)$ is the largest k such that when we remove at most k arbitrary edges from a complete geometric graph on n vertices whose vertices are in convex position then the graph still has a noncrossing Hamiltonian path (Theorems 4 and 5). Further we prove that when we remove any matching from a complete geometric graph the resulting graph will have a noncrossing Hamiltonian path (Theorem 2).

The organization of the paper is following: In the section 2 we introduce basic definitions and notation. In sections 3 and 4 we study complete geometric graphs with removed complete subgraph and we prove the asymptotically tight bounds on the size of complete subgraph removed. In section 5 we prove the tight bounds on the number of edges removed from convex geometric graph and in section 6 we prove the tight bounds on the size of a star removed from a complete geometric graph.

2 Definitions and Notation

In this section we introduce basic definitions and notation used throughout this paper. A *geometric graph* G is an ordered triple (V, E, f) where (V, E) is a graph and f is a projection of V in R^2 (the Euclidean plane). Moreover f satistifies that $f(V)$ is a set of points in general position. The edge $(u, v) \in E$ is represented by a straight-line segment connecting $f(u)$ and $f(v)$. In the following text we will usually consider graph and its concrete embedding and so we will use the term *vertex* also for the point in the plane corresponding to the vertex and the term *edge* for a segment in a the plane corresponding to an edge. A *Hamiltonian path* in graph G is a path such that it contains all the vertices of G. A *noncrossing Hamiltonian path* in geometric graph $G = (V, E, f)$ is a Hamiltonian path which does not intersect itself in the embedding f of the graph. A *convex hull* of a set of points $X \subset R^2, X = \{x_1, \ldots, x_n\}$ is a set of points $H = \{h \in R^2 : \exists a_1, \ldots a_n$ such that $\forall i \in \{1, \ldots, n\}$ $a_i \in R, a_i \geq 0, \sum_{i=1}^n a_i = 1$ and $h = \sum_{i=1}^n a_i x_i\}$. We say that a point p lies *below a line* l (line l must not be parallel to y-axis) if it lies in the half-plane defined by l which contains $-\infty$ on the y axis. Similarly use the term *above a line.* A point u lies *to the left from* v if x-coordinate of u is less then or equal to x-coordinate of v. Analogously we define that u is *to the right from* v. Let s be a segment in R^2 defined by two points $x, y \in R^2$, x is to the left from y. $p \in R^2$ lies *below the segment* s if it lies below a line defined by x and y, to the right from x and to the left from y. Let $Z \subset R^2, Z = \{z_1, \ldots, z_n\}$ be a set of points in the plane. An *x-monotone order* of Z is such ordering of Z in which x-coordinates of the points form a monotone sequence. Analogously we can define a *y-monotone order* of Z.

3 The Lower Bound for Complements of Cliques

In the two following sections we will work with the special class $\mathcal{C}$ of geometric graphs — complements of complete subgraphs graphs. A geometric graph $G = (V, E, f)$ is in $\mathcal{C}$ iff there exist $X, Y \subseteq V$ such that $V = X \cup Y$, $X \cap Y = \emptyset$ and $E = \binom{V}{2} \setminus \binom{X}{2}$ (i.e., G is some complete graph without edges of some complete subgraph). We will prove that there exists constant $c_1 > 0$ such that for any geometric graph from $\mathcal{C}$ where $|X| \leq c_1 \cdot \sqrt{|V|}$ there exists a noncrossing Hamiltonian path.

Lemma 1 *Let $G = (V, E, f)$ be a geometric graph, $G \in \mathcal{C}$. If there exists a line l such that all the vertices from X are in one half-plane defined by l and at least $|X|$ vertices from Y are in the other half-plane then there exists a noncrossing Hamiltonian path in G.*

Proof: We may WLOG assume that line l is parallel to the y-axis and that all the vertices from the set X are in the left half-plane. We will use the procedure from figure 1 to create a noncrossing Hamiltonian path in G. The procedure takes an upper segment from the convex hull of vertices not yet added to the path (the whole set V in the beginning) which crosses l (see figure 2 on the left). If the last vertex on the path or the left end of the segment is from Y then we add to the path the left end of the segment (figure 2 on the right). Otherwise we add to the path the right end of the segment (figure 3 on the left). If l doesn't cross the convex hull then we simply add remaining vertices to the path in an x-monotone order (see figure 3 on the right for the example of created path).

It is clear that the algorithm finishes when it adds all the vertices to the path. It is also easy to see that there are no two consecutive vertices from X on the constructed path and hence it is really a path in G. The only place in the algorithm where two vertices from X could be added consecutively to the path is when it is adding vertices in an x-monotone order. But at that time there is at most one point from X not on the path (the other vertices from X were added in the previous steps because there are at least $|X|$ vertices from Y in the right half-plane). What remains to prove is that the path is noncrossing. We will check that after we add a new vertex to the path the path does not intersect the convex hull of the remaining vertices including the vertex just added to the path. From this it is obvious that the path does not intersect itself (at each vertex each of the following edges of the path must lie in the convex hull and the previous edges lie outside of it). When the path contains only one vertex (the one from the convex hull of all vertices) the claim is obviously true. When we add the new vertex to the path the edge connecting the new vertex with the previous vertex on the path cannot intersect the convex hull of the remaining vertices — if the edge intersected the convex hull then the previous vertex on the path would have to lie in the half-plane as shown in the figure 4. But then we get contradiction with the choice of the vertex in the previous step of the algorithm (the vertex cannot be an endpoint of the segment of the convex hull intersecting l). From the induction we know that any other edge of the path cannot intersect the convex hull and so we have proven that the path does not intersect itself.

■

Now we will prove a similar result for more general choice of sets X and Y:

Theorem 1 *Let $G = (V, E, f)$ be a geometric graph, $G \in \mathcal{C}$. If $|Y| \geq 2 \cdot |X| \cdot (|X| + 1)$ then there exists a noncrossing Hamiltonian path in G.*

Proof: We can WLOG assume that there are no two vertices from V with the same x coordinate. Now consider partitioning of the plane into $|X| + 1$ strips such that each strip contains a vertex from X on its boundary (see figure 5). From the pigeonhole principle we know that there is a strip S with at least $2 \cdot |X|$ vertices from $|Y|$ in it. Let x_l denote the number of vertices from X to the left from S (we also count the vertex on the left boundary of S). Similarly we define x_r. Now we can certainly choose a vertex z from S such that there are at least $2 \cdot x_l$ vertices in S to the left from z and at least $2 \cdot x_r$ vertices in S to the right from z. Now we apply following procedure on the vertices to the left from z (and then the analogous procedure on the vertices to the right from z). We find lines l_1, l_2 such that $z \in l_1 \cap l_2$, both l_1 and l_2 contain some vertex lying to the left from S and there is no vertex lying to the left from S which would lie above l_1 or below l_2 (see figure 5). It is clear that there are at least x_l vertices in S lying either below l_1 or above l_2. Lets WLOG assume that there are at least x_l vertices in S lying

Input: Two sets of points in the plane X, Y and line l
Output: Noncrossing Hamiltonian path

```
begin
    PathLen := 0;
    V := X ∪ Y;
    Left := False;
    LastLeft := True;
    forever do begin
        Seg := CrossSegment(V, l);
        if Seg = ∅ then
            break;
        if Left then begin
            if (Path[PathLen-1] ∈ Y) or (Seg.Left ∈ Y) then begin
                Path[PathLen] := Seg.Left;
                PathLen := PathLen + 1;
                V := V \ Seg.Left;
                LastLeft := True;
            end
            else
                Left := False;
        end
        else begin
            Path[PathLen] := Seg.Right;
            PathLen := PathLen + 1;
            V := V \ Seg.Right;
            Left := True;
            LastLeft := False;
        end;
    end;
    while V ≠ ∅ do begin
        if LastLeft then
            P := Rightmost(V)
        else
            P := Leftmost(V);
        Path[PathLen] := P;
        PathLen := PathLen + 1;
        V := V \ P;
    end;
    Output(Path)
end.
```

Figure 1: An algorithm for finding a noncrossing Hamiltonian path. Function `CrossSegment` returns a segment of convex hull of given set which intersects l (actually there are two such segments so the function returns the one which intersects l at position with bigger y coordinate — we suppose l is parallel to y axis). Functions `Leftmost` and `Rightmost` return leftmost respectively rightmost point from the given set.

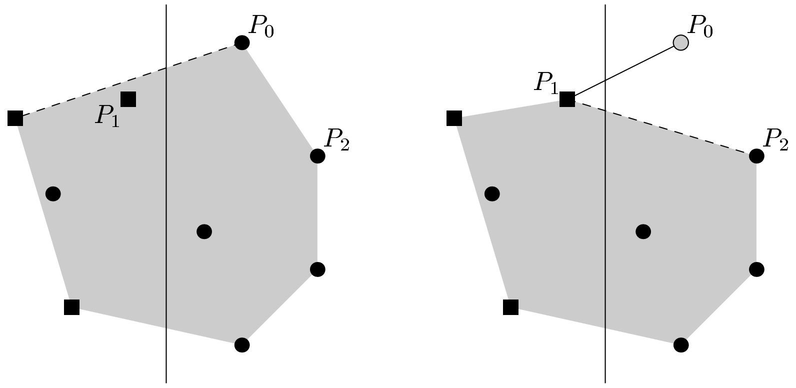

Figure 2: On the left is the first step of the algorithm. On the right is the second step of the algorithm — add left end of the segment to the path.

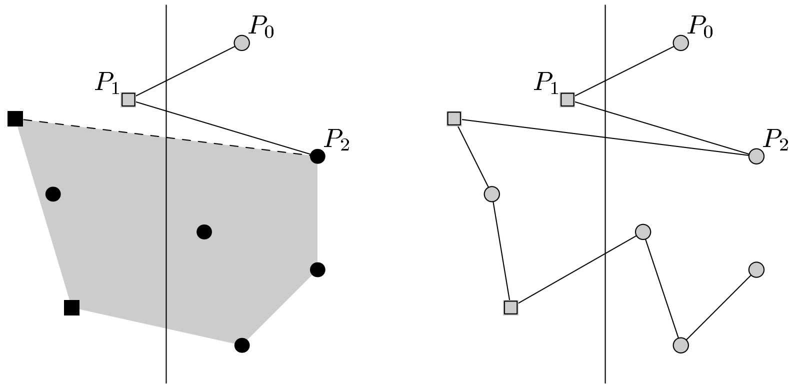

Figure 3: On the left is the third step of the algorithm — add right end of the segment to the path. On the right is the whole noncrossing Hamiltonian path.

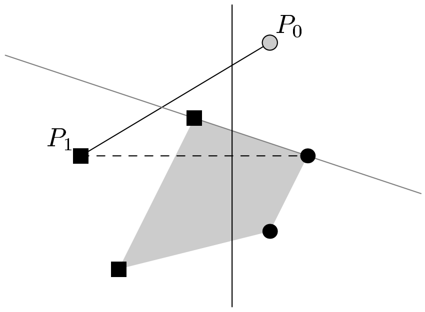

Figure 4: If the segment intersected convex hull the previous vertex on the path would have to lie in the lower half-plane.

below l_1. Let Z denote the set of vertices lying to the left from z and below l_1. Now we can use Lemma 1 on the set Z (line l from the statement of the lemma is the boundary of the strip S). From the lemma we get noncrossing path containing all the vertices from Z. Because there are no vertices from Z above the first segment of the path (all the vertices must lie below a line defined by the segment of the convex hull from the second step of the procedure from Lemma 1 and the whole area above the first segment of the path lies above this line — see figure 6) we can replace the first segment of the path by the path going through all the vertices above the line l_1 in an x-monotone order (see figure 7). By the replacing we got noncrossing path using all the vertices of V to the left from z ending in z. Analogically we can get similar path for the vertices of V to the right from z, then join these two paths in z and get noncrossing Hamiltonian path for G.

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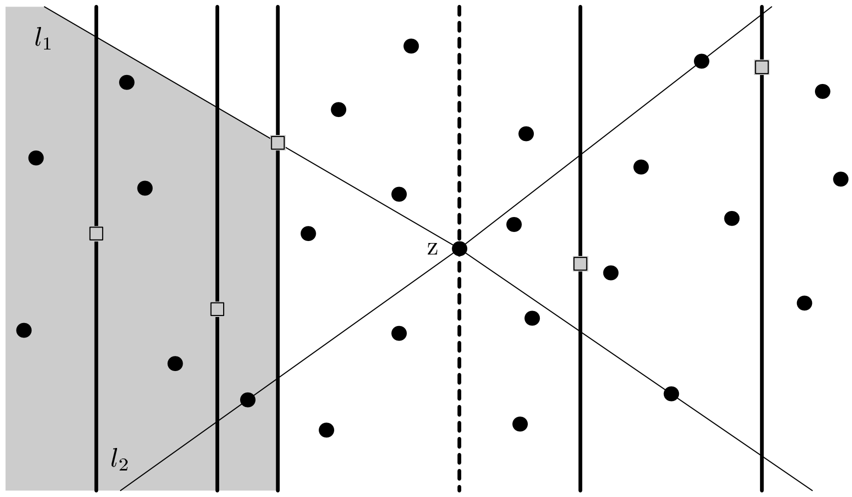

Figure 5: Partitioning of the plane into the strips and choice of the vertices on which Lemma 1 should be used.

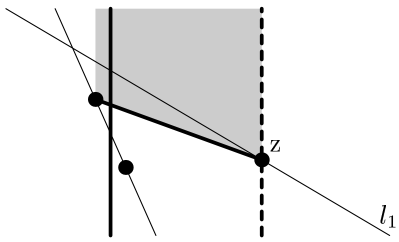

Figure 6: There are no vertices lying above the first segment of the path and below the line l_1

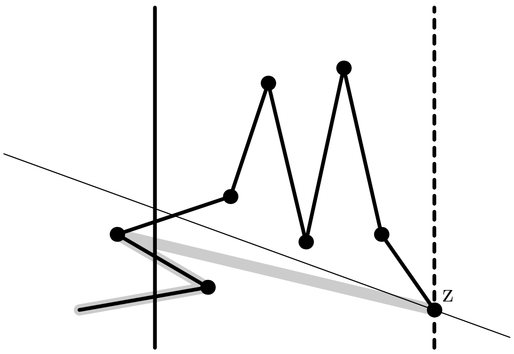

Figure 7: Replacing the first segment of the path with an x-monotone path

Now using the Theorem 1 we can prove the following corollary:

Corollary 1 *Let $G = (V, E, f)$ be a geometric graph and let $k = 2 \cdot \left(\binom{|V|}{2} - |E|\right)$ (i.e., k is twice the number of edges not present in G). If $2 \cdot k \cdot (k+1) \leq |V| - k$ then there exists a noncrossing Hamiltonian path in G.*

Proof: The idea of this proof is easy. We will just use Theorem 1 on the original graph with removed edges of the complete subgraph containing all the edges missing in G.

More formally let X be the set of all vertices from V whose degree is less than $|V|-1$. The size of X is clearly less than or equal to k. Let $Y = V \setminus X$. From the statement of the corollary we know that $|Y| \geq 2 \cdot |X| \cdot (|X|+1)$ and hence the assumptions from the statement of Theorem 1 are satisfied and we can conclude that G has a noncrossing Hamiltonian path.

■

Using the algorithm with the similar idea as the algorithm in Lemma 1 we can also prove the following result for the complements of matchings:

Theorem 2 *Let $G = (V, E, f)$ be a geometric graph such that $F = \binom{V}{2} \setminus E$ is a matching (graph with maximum degree one) and $|V| \geq 3$. Then G has a noncrossing Hamiltonian path.*

Proof: The case when $|V| = 3$ is trivial and so we can assume $|V| \geq 4$. We will use the following algorithm for a construction of a noncrossing

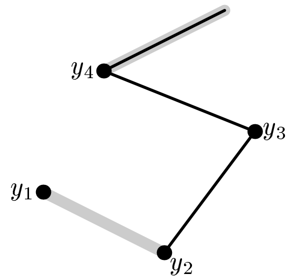

Figure 8: The situation when we cannot add last point to the noncrossing path.

Hamiltonian path: first take any point on the convex hull of V. Let x be the last vertex on the path constructed so far. Then at each step we take the vertex y on the convex hull of the remaining points such that $\{x, y\}$ is an edge in G and it does not intersect the convex hull of the remaining points and we add y to the path. The vertex y with desired properties will always exist if there are at least two remaining vertices. Let H' denote the convex hull in the previous step of the algorithm and H the convex hull in the last step. In the previous step x was on the convex hull H' and so in the current situation there must be some segment y_1y_2 of the convex hull H such that both segments xy_1 and xy_2 do not intersect the convex hull H. Because at least one pair must be an edge in G (F was a matching) we have just proven the existence of y. If there is only one vertex and it is not connected by an edge to the last point on the path we cannot add it to the path. Let y_1 be the last remaining vertex and y_2, y_3, and y_4 be the last vertices on the path in the reverse order (remember that $|V| \geq 4$). In this situation we remove the vertices y_2 and y_3 from the path so we get a situation as drawn in figure 8. Because F is a matching y_4 must be connected by an edge to both y_1 and y_2 and one of these edges does not intersect the convex hull of the remaining points. So we can WLOG add y_1 to the path and then finish the path by adding y_3 and y_2.

■

4 The Upper Bound

In this section we will prove that there exist geometric graphs in $\mathcal{C}$ such that the size of X is $O(\sqrt{|V|})$ and the graphs do not have a noncrossing Hamiltonian path. By proving this we get two asymptotically tight bounds on the number of vertices in X.

Lemma 2 *Let $G = (V, E, f)$ be a geometric graph such that all the vertices of V are in the convex position. Let $P = (v_{i_1}, \ldots, v_{i_n})$ be a noncrossing Hamiltonian path in G. Then for any $j \in \{1, \ldots, n-2\}$ holds that from the three vertices $v_{i_j}, v_{i_{j+1}}, v_{i_{j+2}}$ at least two are neighbors on the convex hull of V.*

Proof: Assume that for some j no two vertices are neighbors on the convex hull of V. Then the vertices on the convex hull are split into three nonempty parts — WLOG the parts are the segments of convex hull from v_{i_j} to $v_{i_{j+1}}$, from $v_{i_{j+1}}$ to $v_{i_{j+2}}$ and from $v_{i_{j+2}}$ to v_{i_j}. But noncrossing path can enter only two of these three parts (once the path enters some part it cannot leave it without crossing itself) and hence such path cannot be Hamiltonian.

■

Theorem 3 *For each $n_0 \in N$ there exists $n \in N, n_0 \leq n$ such that there exists a geometric graph $G = (V, E, f), G \in \mathcal{C}, |V| = n$ satisfying $|X| < 3 \cdot \sqrt{n}$ without a noncrossing Hamiltonian path.*

Proof: Lets have some n_0. As n we choose the smallest natural number greater than n_0 such that n is the second power of some natural number. Now we create a geometric graph on n vertices. We place n vertices of the graph on the circle with equal distances. Then we split the vertices into $\sqrt{n}$ groups (each of size $\sqrt{n}$) in such way that each group will form a continuous segment on the circle. Now we define partitioning of V into X and Y (and by this we define edges in the graph). We will choose arbitrarily one group (let's call it g) and put all its vertices to X. In the other groups we put two vertices on the borders to X and remaining points to Y (see figure 9). Clearly $|X| = (\sqrt{n} - 1) \cdot 2 + \sqrt{n} < 3 \cdot \sqrt{n}$ so the only thing remaining to prove is that the graph does not have a noncrossing Hamiltonian path.

Lets consider first vertex u in g. Because $u \in X$ it must be connected to some vertex v from Y by the path. We will write g' for the group containing the vertex v. Because both neighbors of u on the convex hull are also from

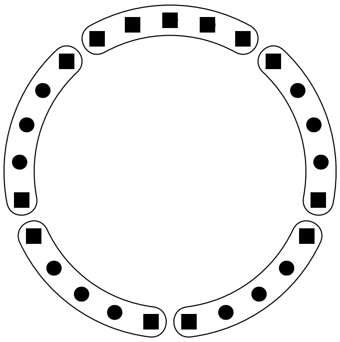

Figure 9: Construction of a graph on n vertices with $|X| = O(\sqrt{n})$ without a noncrossing Hamiltonian path

X, v cannot be a neighbor of u. From this we can also trivially conclude that neither u nor v can be endpoints of the Hamiltonian path. Now we focus our attention on the part of the path which should contain all the remaining vertices of g. From Lemma 2 we know that the path must go from v either to some other vertex of g' or to the neighbor of u in g. From any vertex of g the path must return back to g' to the neighbor of the vertex last used in g. From this we get that on the noncrossing Hamiltonian path must be alternating vertices from g and from g' in such a way that between any two vertices from g there must be a vertex w, $w \in g' \cap Y$. But we have $\sqrt{n}$ vertices from g and only $\sqrt{n} - 2$ vertices from $g' \cap Y$ and so this is impossible.

■

5 Vertices in Convex Position

In the following section we will consider a class $\mathcal{D}$ of convex geometric graphs. A geometric graph $G = (V, E, f)$ is in $\mathcal{D}$ iff f projects the vertices of V on the set of $|V|$ points in a convex position. We will show that if we remove $\lceil |V|/2 \rceil - 1$ edges from the complete geometric graph then the noncrossing Hamiltonian path still exists (Theorem 4) but if we remove

$\lceil |V|/2 \rceil$ edges it need not exist (Theorem 5). It is interesting that the bounds are tight.

Theorem 4 *Let $G = (V, E, f)$ be a geometric graph, $G \in \mathcal{D}$, $n = |V|$. Let $F = \binom{V}{2} \setminus E$. If $|F| \leq \lceil \frac{n}{2} \rceil - 1$ then there exists noncrossing Hamiltonian path in G.*

Proof: Let $v_0, v_1, \ldots, v_{n-1}$ be the vertices of G in clockwise order, starting with arbitrary one. Consider a geometric graph $G' = (V, \binom{V}{2}, f)$. Let P_i be the path v_i, v_{i+1}, v_{i-1}, v_{i+2}, v_{i-2}, $\ldots$ (counting the indices modulo n) in G'. We observe that the paths $P_0, \ldots, P_{\lfloor n/2 \rfloor}$ are pairwise disjoint noncrossing Hamiltonian paths in G'. Since $|F| \leq \lceil \frac{n}{2} \rceil - 1$ we are done for n even — at least one of them must avoid F and hence it is a noncrossing Hamiltonian path in G. If n is odd we observe that there are $\lfloor n/2 \rfloor$ edges $\{v_0, v_n\}, \{v_1, v_{n-1}\}, \ldots, \{v_{\lfloor n/2 \rfloor - 1}, v_{\lfloor n/2 \rfloor + 1}\}$ which are not in any P_i. Let A denote this set of edges. Because the differences between the vertex numbers in edges of A cover all the numbers from the interval $1 \ldots \lfloor n/2 \rfloor$ (counting modulo $\lfloor n/2 \rfloor + 1$) we can WLOG assume that at least one of the edges of F is in A (and hence is not in any P_i). Now we can conclude using the same argument as for n even that one of the P_i's is a noncrossing Hamiltonian path.

■

Theorem 5 *For each $n, n \geq 2$ there exists a geometric graph $G_n = (V_n, E_n, f_n)$ such that $G_n \in \mathcal{D}$, $|V_n| = n$, $F_n = \binom{V_n}{2} \setminus E_n$, $|F_n| = \lceil \frac{n}{2} \rceil$ and G_n does not have a noncrossing Hamiltonian path.*

Proof: Let v_0, v_1, $\ldots$, v_{n-1} be the vertices of G_n in clockwise order, starting with arbitrary one. At first we make an easy observation: The first (and the last) edge of a noncrossing Hamiltonian path P joins some two vertices of V that are adjacent on the convex hull of V. Consequently, if v_i and v_j are not adjacent and $\{v_i, v_j\}$ is an edge of such path, P contains at least one edge of a convex hull from each of the intervals $v_i \ldots v_j$ and $v_j \ldots v_i$.

Let $k = \lceil \frac{n}{2} \rceil$. We choose $F_n = \{\{v_0, v_1\}, \{v_1, v_2\}, \ldots, \{v_{k-1}, v_k\}\}$ and E_n as a complement of F_n. Let $B = \{v_0, \ldots, v_k\}$. Suppose there exists noncrossing Hamiltonian path P avoiding F_n. No edge in P may join two points of B, as then by the observation above it would have to contain an edge from F_n. Therefore B is independent set of P which is impossible as the greatest independent set of P is of size k.

■

6 Complement of Star

In this section we will consider class of geometric graphs $\mathcal{S}$. A geometric graph $G = (V, E, f)$ is in $\mathcal{S}$ iff $E = \binom{V}{2} \setminus F$ where F is a star $K_{1,k}$ (i.e., $G \in \mathcal{S}$ iff G is a complement of a star). We prove that for $k \leq \lceil |V|/2 \rceil - 1$ there always exists a noncrossing Hamiltonian path but for $k \geq \lceil |V|/2 \rceil$ it need not exist.

Theorem 6 *For any geometric graph $G = (V, E, f)$ on n vertices, $G \in \mathcal{S}$ such that $|F| \leq \lceil \frac{n}{2} \rceil - 1$ there exists a noncrossing Hamiltonian path in G. For any $n, n \geq 2$ there exists a geometric graph $G_n, G_n \in \mathcal{S}$ with $|F_n| = \lceil \frac{n}{2} \rceil$ such that there is no noncrossing Hamiltonian path.*

Proof: Let C be a center of star F. Lets split the plane into cones by extensions of edges of F. If there is a cone that contains at least 2 vertices that are not covered by F, we use the construction from figure 10 (left). Otherwise there must be exactly one vertex in each of the cones. At most one of the cones spans angle greater than straight, let x be a vertex inside it if such cone exists, any if it does not. x splits its cone into two, so at least one of them spans angle smaller than straight. Now we use the construction from figure 10 (right).

Let V_n and F_n look as on figure 11 (the important property is that all vertices not covered by F_n are on the convex hull of V_n while none of F_n vertices except for its center C and two border vertices do). If n is odd we place the one remaining vertex of the star anywhere inside the convex hull.

Suppose there is a path P with required properties. If C is the endpoint of P, then the second vertex is one of uncovered ones, the third one belongs to one of half-planes determined by the first edge and P cannot get to the other half-plane without intersecting its first edge and so the path cannot be Hamiltonian. Analogously if the C is in the middle of P, the edges of P adjacent to it split the plane into three parts, each of them contains at least one vertex and P cannot cover more than two of them without intersecting itself.

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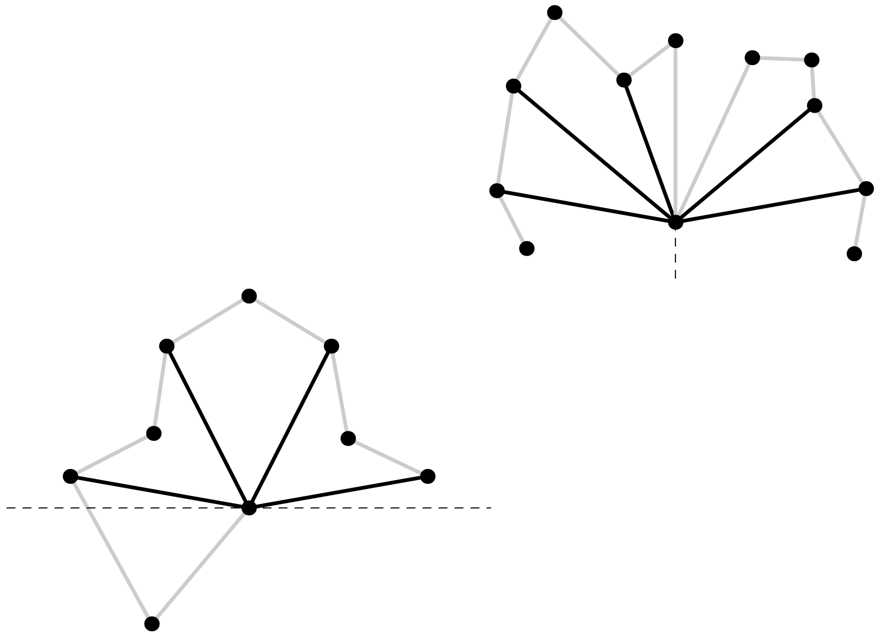

Figure 10: Construction of noncrossing Hamiltonian path. There are either two uncovered vertices in one cone (on the left) or they are not (on the right).

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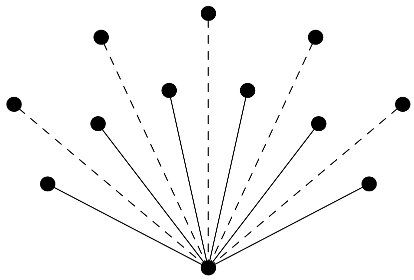

Figure 11: Complete geometric graph without a star without a noncrossing Hamiltonian path.

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