Measuring of Aesthetic Invariant of Images

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Abstract

We define four new aesthetics — Combinatorial Entropies — for graph drawings, sketches and images and show their relevance for picture processing by means of results of integral geometry.We include an analysis of two pictures.

1 Introduction

Global characteristics (sometimes called aesthetics [3], [4]) of graph drawings, schemata and visual information in general are difficult to find. These combinatorial characteristics (such as crossing-, bend-, area-, angle- or length- minimization) are mostly NP-complete (even to approximate) and have other drawbacks from the practical point of view: in some cases these parameters are not invariant to scale and rotation and, perhaps most importantly, the determination of these parameters assumes an analytic description of the drawing.

Papers [1], [10] suggested a different "aesthetics" which is based on the integral geometry and the geometric probability ([11], [5]). The

*Supported by a Grant LN00A56 of the Czech Ministry of Education

defined parameter *Fractional Length* [6] or *Combinatorial Entropy* [1] (two terms with the same meaning) has several advantages:

- the drawing (picture, visual data) need not be given analytically, the input may be a scanned picture. This allows us analyze, to sort and to compare real pictures, scenes, photos and visual art in general;
- Combinatorial Entropy is scale and rotation invariant;
- Combinatorial Entropy is easy to determine and it is "robust".

Combinatorial Entropy of a drawing was linked in [12] to the crossing number for VLSI and 2-layer drawing models.

A serious drawback of Combinatorial Entropy is that it is defined for drawings and curves only. The shaded and 2-dimensional areas have to be replaced and preprocessed for real computation. This is the weakest point of Combinatorial Entropy because results are strongly dependant on the way how a preprocessing is done.

In this note we suggest several enrichments of the global analysis of pictures which allows us to compare more realistic images with black and white areas (or ink blots) and also shaded parts (i.e. gray levels). The resulting parameters are still called *Combinatorial Entropy* (we sometimes add subscript or adjective to indicate which measure we have in mind).

The method, which is an adaptation of techniques of integral geometry (as in detail e.g. in [11]), is briefly explained in Section 2. We derive basic formulas (3), (6) and Propositions 1, 2, which are used in the definition of various forms of the Combinatorial Entropy. In Section 3 we define combinatorial entropy for pictures with shaded areas and gray tones (weighted areas) and we give some interpretation of these parameters. It is also here where we formulate the *Hereditary Thesis* (as a refinement of [9]) which in the present setting finds yet another interpretation (and this suggested to study "calculus of partitions" which we are presently developing jointly with V. Douchová).

In Section 4 we give an analysis of various forms of two pictures [7], [8]. The note concludes with a brief summary and outline for future research.

2 An outline of the method

2.1 Schemata, sketches and drawings

Respecting the group of isometries of plane a line P is defined by its distance p ($p \geq 0$) from the origin and by the angle ϕ ($0 \leq \phi \leq 2\pi$) that the direction perpendicular to P makes with fixed direction (Poincarré). The coordinates (p, ϕ) are therefore the polar coordinates of foot of the perpendicular from the origin onto the line P.

The measure of a set of lines X is then defined by the following integral

$$m(X) = \int_X dp\ d\phi. \tag{1}$$

The measure of a set of lines is counted as the integral of differential form $dP = dp\ d\phi$ over given set (dP is also called the density of the set P).

The measure of a set of lines that intersects a given bounded convex set K (with the support function $p = p(\phi)$) is then given by

$$m(P; P \cap K \neq \emptyset) = \int_{P \cap K \neq \emptyset} dp\ d\phi = \int_0^{2\pi} p\ d\phi = C, \tag{2}$$

where C is the length of ∂K (the perimeter of K).

Let Γ be a piecewise differentiable curve parametrized by the arc length ($x = x(s)$ and $y = y(s)$). Let the length of Γ be L. Let line P intersects the curve Γ at a point (x, y) and forms with the tangent at this point an angle θ. Coordinates (s, θ) determine line P uniquely. In this parametrization the density dP of lines has the form $dP = |\sin\theta|\ ds\ d\theta$. Integration over all lines that intersect Γ we get on right side $2L$. On the left side we count line P so many times as it has intersections with Γ. Let this count be n. Then we get $\int n\ dP = 2L$. Let C be the perimeter of convex hull of Γ (i.e. the measure of the set of lines which intersects Γ), then average number of intersections of curve Γ with a random line is

$$E(n) = \frac{2L}{C}. \tag{3}$$

The above derivation can be easily changed to the weighted case (which can be interpreted as the grayness or intensity of the line). Let $w(s)$ be a weight function on Γ. We now count every line according to the weight of its intersect points with Γ. We have an analogy to (3):

Proposition 1: Let Γ and C be as above. Let $w(s)$ be the piecewise differentiable function, parametrized by the arc length of Γ and let $\tilde{n}$ be the sum of $w(s)$ in intersection points of Γ with random line. Then the mean value of $\tilde{n}$ is

$$E(\tilde{n}) = \frac{2}{C} \int_0^L w(s) ds. \tag{4}$$

Remark: The linear drawing can be well approximated by curves. In fact every (connected) line drawing $\mathcal{D}$ can be considered as a closed curve $\mathcal{C}$ by traversing every arc-segment (between crossings) twice by two "perturbated" ε-close arcs (we have then an Eulerian graph so such traversing is possible for every connected picture). It is clear that the number of crossings of most lines P with $\mathcal{C}$ will be twice the number of crossings of P with $\mathcal{P}$. On the other hand the length L of $\mathcal{C}$ is twice the length of picture $\mathcal{P}$. As the result of this we can use formulas (3), (4) (which are derived for curves) for line drawing and interpret them accordingly (of course without doubling arcs).

2.2 Blots and spots

Let D be an open connected set in plane with total area F and let P be a line. Denote σ the total length of the intersection $P \cap D$. Formula $\sigma\ dp$ is the differential of area and so we have $F = \int_0^\infty \sigma\, dp$. Integration over all directions then yields

$$\int\limits_{D \cap P \neq \emptyset} \sigma\, dP = \int\limits_{D \cap P \neq \emptyset} \sigma\, dp\ d\phi = \int\limits_0^\pi F\, d\phi = \pi F. \tag{5}$$

Let C be now perimeter of the convex hull of D. The formula for mean length of a chord is then

$$E(\sigma) = \frac{\pi F}{C}. \tag{6}$$

The weighted formula for the 2-dimensional case can be done similarly. Let w be a weight function on D then $w(D)$ defines a surface in $\mathbb{R}^3$. Denote $\tilde{\sigma}$ the length of $w(P \cap D)$ (i.e. the length of curve defined on a surface $w(D)$). Then $\tilde{\sigma}\ dp$ is the differential of area of the surface.

Proposition 2: Let D be an open connected set in plane and C perimeter of its convex hull. Let w be a piecewise differentiable function from D to $\mathbb{R}$. Let $\tilde{F}$ be total area of surface $w(D)$ and $\tilde{\sigma}$ be the length of a curve obtained as image of intersection of D with a random line. Then the mean value of $\tilde{\sigma}$ is

$$E(\tilde{\sigma}) = \frac{\pi \tilde{F}}{C}. \tag{7}$$

3 Interpretation of Combinatorial Entropies

Suppose a picture drawing $\mathcal{P}$ be given. Define the (unweighted, 1-dimensional) Combinatorial Entropy $EC_1(\mathcal{P})$ as the average number of intersections of the picture $\mathcal{P}$ with a random line.

This is discussed in detail in [1] and the formula (3) allows us to interpret this number as the average line density of $\mathcal{P}$ — hence the alternative name Fractional Length.

Suppose now drawing $\mathcal{P}$ be given with various line intensities measured by the weight function $w : \mathcal{P} \to \mathbb{R}$. Define the (weighted, 1-dimensional) Combinatorial Entropy $EC_1^w(\mathcal{P})$ as the average weight of intersections of the picture $\mathcal{P}$ with a random line.

Proposition 1 allows us to interpret this number as the density of weighted length of curve (EC_1^w could be also called weighted Fractional Length).

Now suppose a picture $\mathcal{P}$ with black and white areas given. Consider lines P intersecting $\mathcal{P}$ and denote EC_2 the average 1-dimensional measure of intersection of P with $\mathcal{P}$. Formula (6) allows us to interpret EC_2 as the average length of chord in $\mathcal{P}$.

In two examples in Section 4 we modified a gray picture to a black and white variant by simple thresholding.

Finally, suppose a picture $\mathcal{P}$ with shaded areas given. (This most general case simply corresponds to a differentiable function $w : \mathcal{P} \to \mathbb{R}$ which defines a surface in $\mathbb{R}^3$). Then we can define the 2-dimensional weighted Combinatorial Entropy EC_2^w as the average length of the crosscut curves on the surface. These measures are illustrated by examples in Section 4.

Let us remark that 2-dimensional entropies EC_2 and EC_2^w are not invariant under scale (as expected from the dimension point of view). But we are mostly not interested in the absolute value of 2-dimensional entropies. The main purpose of these parameters is to enhance comparison of combinatorial entropies of a particular visual data. We are especially interested in the comparison of values of EC_2, EC_2^w in various parts of the picture. We formulate this as an updated form of

Hereditary Thesis: A picture is harmonious if any two of its similarly meaningful parts have similar combinatorial entropies.

Of course "similarity of meaning" is neither defined nor a primitive notion. However in concrete instances this is often not a handicap. Besides if no preferences in a picture are given then the uniform partitions serve as a good model. This is illustrated on the second example (Figure 2 of the following section).

4 Two examples

There is no space in this short note to include more examples which we tested. We include an analysis of two concrete pictures taken from [7], [8] and illustrate some of the features of Combinatorial Entropies.

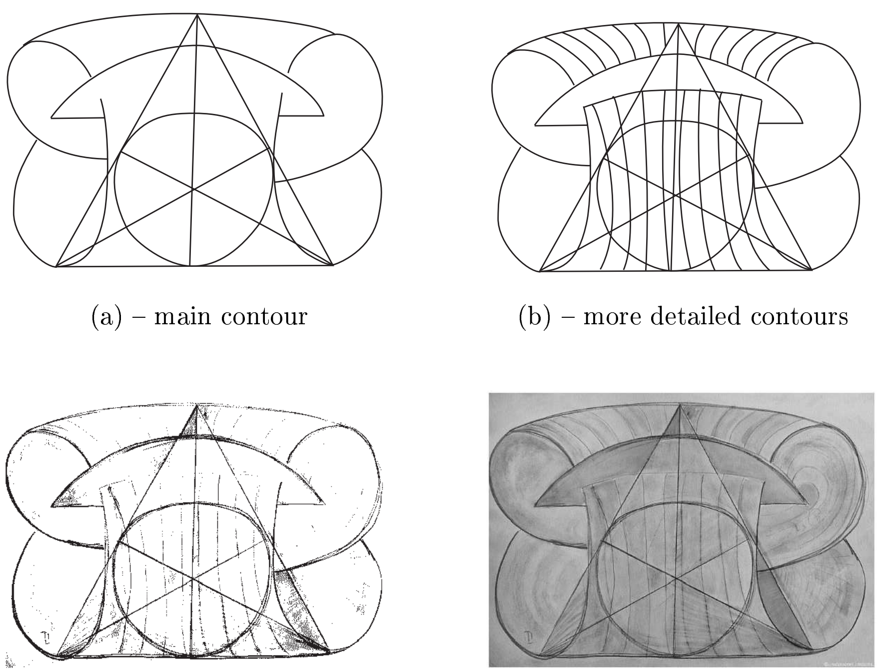
The painting [7] was scanned and processed in four different ways: first are redrawn main contours $\mathcal{P}_a^1$ and enriched contours $\mathcal{P}_b^1$, then the black and white version $\mathcal{P}_c^1$ is obtained by thresholding from the grayscale (scanned) image $\mathcal{P}_d^1$.

All our measurements (see Table 1) are based on large number — cca 10^4 — of experiments. For black and white images (a)–(c) we can use all four methods whereas grayscale image (d) can be measured only by 2-dimensional methods. 1-dimensional weighted method EC_1^w gives in this case same results as EC_1 (because all lines have weight 1).

	$\mathcal{P}_a^1$	$\mathcal{P}_b^1$	$\mathcal{P}_c^1$	$\mathcal{P}_d^1$
EC_1	7.01	10.71	≈ 10.84	(≈ 10.84)
EC_2	10.58	16.71	34.63	(34.63)
EC_2^w	12.94	20.54	40.12	366.25

Table 1: Combinatorial Entropies for pictures of Figure 1.

EC_1 is defined only for line drawings $\mathcal{P}_a^1$ and $\mathcal{P}_b^1$. For black and white image $\mathcal{P}_c^1$ we can approximate EC_1 by counting every interval of intersection $\mathcal{P}_c^1 \cap P$ as one intersection with a curve. By this approximation we obtain larger value than for $\mathcal{P}_b^1$. This is due to noisy parts of image which are counted as lines. These anomalies, which arise from thresholding, cause that we cannot distinguish between images $\mathcal{P}_b^1$ and $\mathcal{P}_c^1$. Moreover 1-dimensional methods cannot be used for grayscale image $\mathcal{P}_d^1$ (we can only define $EC_1(\mathcal{P}_d^1)$ as the EC_1 of corresponding black and white image $\mathcal{P}_c^1$). These are main reasons why we are developing 2-dimensional methods.



(a) – main contour

(b) – more detailed contours

(c) – black and white

(d) – grayscale

Figure 1: EUROCOMB'03 poster, [7].

2-dimensional entropies depend on image size (for this case 560×400) and are in general larger then corresponding 1-dimensional entropies (because of nonzero width of lines). In fact $EC_2(\mathcal{P}_d^1)$ cannot be measured (because $\mathcal{P}_d^1$ is the grayscale image) but it is natural to define it as EC_2 of corresponding black and white image P_c^1.

EC_2 well distinguishes three black and white images and EC_2^w does the same for all four images. This is the main feature of 2-dimensional Combinatorial Entropies, which we see as a refinement of our earlier method ([1], [9], [10]). This is the case in all examples we tested (further computional results and their interpretation can be found in [2]). From 1- and 2-dimensional measurements we can additionally obtain for example average widths of lines (which for given images are 1.5, 1.56 and 3.19).

Neither EC_2^w nor EC_2 is scale invariant but we can consider their ratio. EC_2^w/EC_2 is scale invariant if we change number of quantizing levels accordingly to scale of image. These ratios are 1.22, 1.23, 1.16 and 10.58 in the case of Figure 1. We can interpret these numbers as measure of change of intensity in picture.

The second picture $\mathcal{P}^2$ (Figure 2) which we analyze here is [8]. We consider four versions: $\mathcal{P}_a^2$ is scanned gray version, the picture $\mathcal{P}_b^2$ is $\mathcal{P}_a^2$ processed for contours (following [1]), the picture $\mathcal{P}_c^2$ is black and white version of $\mathcal{P}_a^2$ (with threshold at 81 on a scale 0–255) and the picture $\mathcal{P}_d^2$ is uniform partition of $\mathcal{P}_a^2$ illustrating values for Hereditary Thesis.

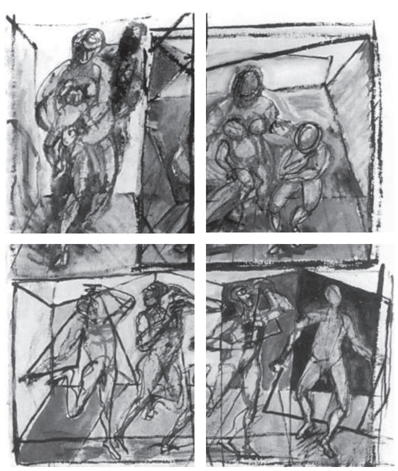
From 1-dimensional entropies EC_1 (Table 2a, 2b) we can see that bottom of the picture has larger density of curves. But 2-dimensional case (Table 2c) shows that more changes in intensity are in bottom-left and top-right parts of image. Therefore the most complex part of image is its bottom-left. From the small value of ratio (Table 2d) in bottom-right rectangle we can conclude that in this part dominate one color areas.

From differences between parts in Table 2d we can also see that uniform division is not sensitive to content of this picture. It means (following Hereditary Thesis) that this division is not harmonious with given picture. This is the inverse use of Hereditory Thesis: it may be used either to test wheather an image is harmonious or it can be used for finding meaningful parts (via Combinatorial Entropies).

(a) – grayscale (scanned) version

(b) – line contours

(c) – black and white version



(d) – uniform division

Figure 2: Antropogeometry, [8].

10.81		11.44
	24.49	
13.85		13.47

(a) – $EC_1(\mathcal{P}_b^2)$

6.94		7.43
	16.41	
10.01		9.23

(b) – $EC_1(\mathcal{P}_c^2)$

196.67		203.53
	425.95	
222.47		197.50

(c) – $EC_2^w(\mathcal{P}_a^2)$

1.88		2.13
	1.95	
2.28		1.70

(d) – $EC_2^w/EC_2(\mathcal{P}_a^2)$

Table 2: Selected Combinatorial Entropies for parts of $\mathcal{P}^2$.

5 Summary, concluding remarks

We defined four (increasingly more sensitive) statistical measures of (line and 2-dimensional) drawing and painting. These measures are called Combinatorial Entropies and we showed their visual relevance using results from integral geometry (so called kinematic formulae).

Numerous examples were tested and they suggest that these parameters capture well some of the features of visual data. Particularly the Hereditary Thesis (as an evidence for a balanced of harmonious drawing) was tested. Of these experiments two examples are included in this note.

It remains to be seen whether higher order combinatorial entropies are related to other well known characteristic of pictures. Particularly the balance of shade and light areas may have a meaning for design and analysis of scenes.

We want to introduce color Combinatorial Entropy and comment on the interpretation of vector weights after a better understanding of the analysis of gray pictures.

Further details are going to appear in the full version of this paper [2].

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