Generalization of the polygon-crossing problem

Jakub Černý Zdeněk Dvořák Vít Jelínek
Pavel Podbrdský

Abstract

We study the maximum possible number of intersections of a simple k–gon with a simple l–gon for $k, l \geq 3$ odd. We generalize this problem to a larger class of objects. We prove that the number of intersections cannot exceed $kl - k - (l-3)/2$. This improves the best known upper bound and gives alternative solution of the case $l = 5$.

1 Introduction

Determining the maximum complexity of the union of two or more geometric objects in the plane is among basic extremal geometric problems, see e.g [1], [2], [3], [4], [6], [7] and [8]. Let $k, l \geq 3$ be given integer numbers. We are interested in the problem of determining the maximum possible number $f(k,l)$ of intersections of a simple k–gon and a simple l–gon. This problem was studied in [3] – the cases when k or l is even are solved there, but an unrecoverable error appears in the case of k and l being both odd. For k and l both even is $f(k,l) = kl$ and for k even and l odd is $f(k,l) = kl - k$. For k and l both odd easy bounds $kl - k - l + 3 \leq f(k,l) \leq kl - k$ are proved in [3]. The same problem was studied in [2] for odd k and l. The problem was solved for $l = 5$ when $f(k,5) = 4k - 2$ and for $k, l \geq 7$ the upper bound from [3] was improved to $f(k,l) \leq kl - k - \lceil l/6 \rceil$. The conjecture that the lower bound is tight was formulated in [2].

A similar problem was also studied in [5].

We extend the definition of polygon to a larger class of objects and generalize the polygon–crossing problem. This defines the function $g(k,l) \geq f(k,l)$. The main result of this paper about the $f(k,l)$ is the following theorem

Theorem 1 *Let $k, l \geq 3$ be odd integers. Then*

$$f(k,l) \leq kl - k - (l-3)/2.$$

This theorem is an easy consequence of theorem 8 (which proves the same upper bound for the function $g(k,l)$) proved in Section 6. Note that the obtained upper bound is exactly in the middle between the easy upper and lower bounds proved in [3].

Theorem 1 improves the upper bound proved in [2] and together with the fact that $f(k,l)$ is even gives also an alternative proof of the equality $f(k,5) = 4k - 2$ also proved in [2].

We introduce basic definitions and notation in Section 2. We recall known results about the function $f(k,l)$ proved in [2] and [3] in Section 3. We prove several auxiliary lemmas in Section 4. We prove the statement of theorem 8 for a special configuration in Section 5 (theorem 7). Finally we prove theorem 8 in general case in Section 6.

The general problem of determining the exact value of $f(k,l)$ (and $g(k,l)$) for $k, l \geq 7$ odd remains open. We also do not know if $f(k,l) = g(k,l)$ in general and if the function $g(k,l)$ is symmetric.

2 Definitions and Notation

We mean by a *segment* a closed line segment. If A and B are points of the plane, then we write AB for the segment connecting them. We assume throughout the whole paper that all the end–points of all the segments are in general position. Two segments *intersect* if they share an interior point and we call that point their *intersection*. Two segments *miss* each other if they do not intersect. For a segment $s = AB$ we denote by l_s the line given by the points A and B. Points A and B are *separated* by a line l if the segment AB is intersected by l (in other words if the points A and B are lying in the different halfplanes given by l). We also say that the pair (A, B) of points is separated by l if the points A and B are separated by l.

The sequence of non–intersecting segments $A_1A_2, A_2A_3, \dots, A_kA_{k+1}$ where $A_i \neq A_j$ for $i \neq j$ is called a *path*; we write shortly $A_1A_2 \dots A_kA_{k+1}$ instead of $A_1A_2, A_2A_3, \dots, A_kA_{k+1}$ in the paper. The *length* of the path is the number of segments which it contains. If $A_{k+1} = A_1$, then the sequence forms a *k–gon*; if we do not want to emphasize the length of the sequence, we say a *polygon* instead of a *k–gon*. The points $A_1, A_2, \dots, A_k$ are called *vertices* of the k–gon. *Insersections of two polygons* are the intersections of segments which form their boundaries. For given $k, l \geq 3$ we denote $f(k,l)$ the maximum number of intersections of some k–gon and l–gon.

The sequence of non–intersecting segments $A_1B_1, A_2B_2, \ldots, A_kB_k$ ($k \geq 1$) such that every two of the points $A_1, A_2, \ldots, A_k, B_1, B_2, \ldots, B_k$ except for the pairs (B_1, A_2), (B_2, A_3),..., (B_k, A_1) are different is called a *generalized k–gon* (or a *generalized polygon*). The pairs of points (B_1, A_2), (B_2, A_3), ..., (B_k, A_1) (some of which can possibly degenerate to a single point) are called *vertex pairs* of the generalized polygon. The *length* of the generalized polygon is the number of segments which it contains. It is clear that every polygon is also a generalized polygon (with all vertex pairs degenerated to a single point). Let K be a generalized polygon and let L be a (normal) polygon. We say that a vertex pair (A, B) of K *fits* with L if for every segment s of L the points A and B are not separated by the line l_s. We call a pair (K, L) *fitting* if every vertex pair (A, B) of K fits with L. In the other words when lines given by the segments of L separates the plane to some cells, every two points of every vertex pair of K are lying in the same cell (see figure 1).

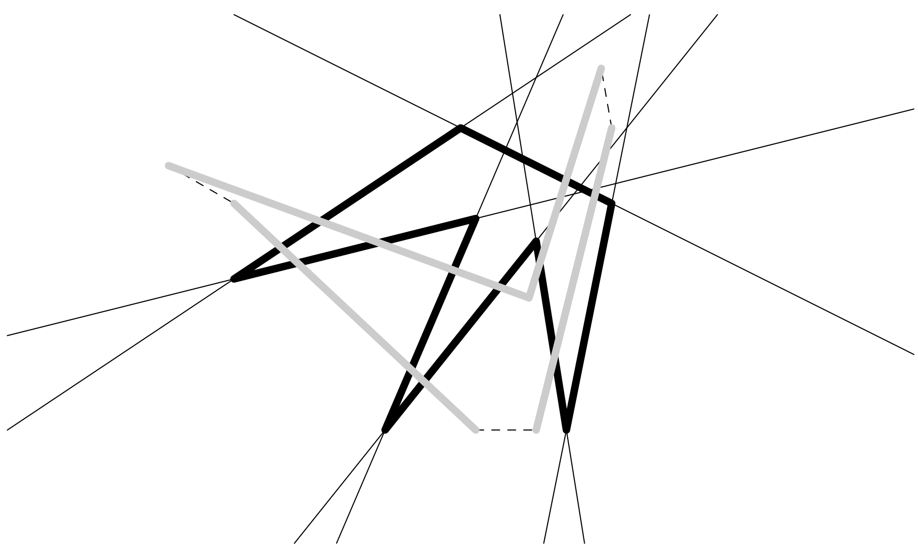

Figure 1: A fitting pair of a generalized k–gon and a (normal) l–gon.

By a (k,l)–*fitting pair* we mean every fitting pair (K, L) of a generalized k–gon and a (normal) l–gon.

Now we can define for given $k \geq 1$ and $l \geq 3$ the number $g(k,l)$ as the maximum number of intersections of some fitting pair of generalized k–gon and (normal) l–gon. Suppose that we have a k–gon K and an l–gon L. Every k–gon is also a generalized k–gon and the pair (K,L) is fitting, because every vertex pair of K is degenerated to a single point and thus not separated by any line. Hence it is clear that $f(k,l) \leq g(k,l)$ for $k \geq 3$. Note that it is no more clear from the definition of $g(k,l)$ that the function g is symmetric.

3 Known results

In this section we will recall some known results about the function $f(k,l)$. The following theorems and lemmas are based on very elementary arguments and are proved (using a different notation) in [3]. They solve the easy case when at least one of the numbers k and l is even. However in case of k,l being both odd the lower and upper bounds in theorem 4 are equal only if $l = 3$. Note that the upper bounds in theorem 2, 3 and 4 and also the fact that $f(k,l)$ is even for every k and l easily follows from lemma 1.

Lemma 1 *Let K be a k–gon and let p be a line. The number of intersections of p with the segments of K is even and at most k. Thus no line intersects all the segments of a polygon with odd number of vertices.*

Theorem 2 *Let $k,l \geq 3$ be two even integers. Then $f(k,l) = kl$ (the lower bound is shown by figure 2).*

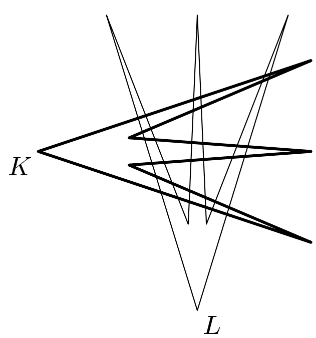

Figure 2: A k–gon K and an l–gon L with kl intersections for k and l both even.

Theorem 3 *Let $k \geq 3$ be even and let $l \geq 3$ be odd. Then $f(k,l) = k(l-1)$ (the lower bound is shown by figure 3).*

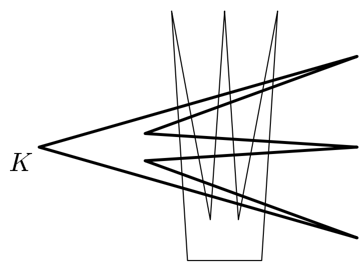

Figure 3: A k–gon K and an l–gon L with $k(l-1)$ intersections for even k and odd l.

Theorem 4 *Let $k, l \geq 3$ be two odd integers. Then $kl - k \geq f(k,l) \geq kl - k - l + 3$ (the lower bound is shown by figure 4).*

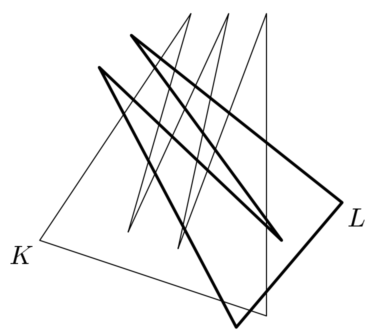

Figure 4: A k–gon K and an l–gon L with $kl - k - l + 3$ intersections for both k and l odd.

Note: Figures 2, 3 and 4 can be generalized for arbitrary k, l by an easy trick. We can substitute one segment of the polygon K (or L) which looks like I by three segments which look like a narrow N. This can obviously be done in such a way that we obtain sufficiently many new intersections.

The following two theorems proved in [2] determine the exact value of $f(k,l)$ for $l = 5$ and improve the upper bound in theorem 4 for $k, l \geq 7$ odd.

Theorem 5 *Let $k \geq 5$ be odd. Then $f(k,5) = 4k - 2$.*

Theorem 6 *Let $k, l \geq 7$ be odd. Then $f(k,l) \leq kl - k - \lceil l/6 \rceil$.*

4 Auxiliary results

In this section we will prove few easy lemmas which will be needed later for the investigation of the upper bound on the function $g(k,l)$.

Lemma 2 *Let $k \geq 1$ and $l \geq 3$ be odd integers and let (K, L) be a (k,l)–fitting pair. Then every segment of K intersects at most $l-1$ segments of L and every segment of L intersects at most $k-1$ segments of K.*

Proof: The lemma follows from an easy parity argument. Let s be a segment of K. Suppose for contradiction that every segment of L intersects s. Hence it is clear that every pair of neighbouring vertices of L is separated by the line l_s. But the number of the vertices of L is odd so there has to be at least one pair of neighbouring vertices which is not separated by the line l_s. This leads us to a contradiction. We see that s intersects at most $l-1$ segments of L.

The proof of the second part of this lemma can be done in the same way. We just have to realize that the points of no vertex pair of K are separated by a line given by a segment of L. □

Note: We have just proved something slightly stronger. For every segment s of K the line l_s intersects even number of the segments of L and for every segment s of L the line l_s intersects even number of the segments of K. This result holds also without the assumtions on parity of k and l.

Lemma 3 *Suppose that there is a $k \times l$ chessboard with some checkers satisfying the following properties:*

(1) *In every row and in every column there is at least one checker.*

(2) *For every checker there is some other checker in the same row or in the same column. In the other words there is no checker being the only in its row and column at the same time.*

(3) *There is no pair of two identical rows with only one checker (such pairs of columns are allowed).*

Then the number of checkers at the chessboard is at least $k + l/2$.

Proof: Let us denote c the number of checkers at the chessboard. Let m ($m \leq k$) be the number of rows with the only one checker. Due to property (1) there are at least two checkers in every other row. By simple counting checkers in every row we see that $c \geq m + 2(k - m) = 2k - m$. Now let us count checkers in every column. All the m rows with only one checker has this checker in different columns due to the property (3). In all this m columns there have to be at least two checkers, otherwise the property (2) would be violated. In every other column there is at least one checker due to property (1). We conclude that $c \geq 2m + (l - m) = l + m$. By adding both obtained estimates we have $2c \geq 2k + l$. □

Lemma 4 *Let $P = A_1A_2 \ldots A_n$ ($n \geq 2$) be a path and let B_1B_2 and C_1C_2 be two segments intersecting all segments of a path P. Then either the pairs (B_1, C_1) and (B_2, C_2) or the pairs (B_1, C_2) and (B_2, C_1) are both not separated by any line $l_{A_kA_{k+1}}$, $k = 1, 2, \ldots, n-1$.*

Proof: For $n = 2$ the statement of this lemma is trivial. Let us consider the case $n = 3$ first. When the segment s intersects both segments A_1A_2 and A_2A_3 we see that one of the endpoints of s has to lie in the region α and the other in the region β (see figure 5). Points in the region α (or β) are not separated by the lines $l_{A_1A_2}$ and $l_{A_2A_3}$. When we have two segments B_1B_2 and C_1C_2 intersecting both A_1A_2 and A_2A_3, one endpoint of both segments lies in the region α and the other endpoint of both segments lies in the region β. The statement of lemma easily follows from this.

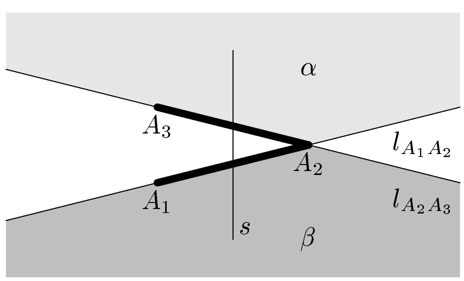

Figure 5: A segment s intersecting a path $A_1A_2A_3$.

The case $n > 3$ is an easy consequence of the case $n = 3$. Let us have the segments B_1B_2 and C_1C_2 both intersecting all segments of the path

$A_1A_2\ldots A_n$. We can WLOG suppose that the points B_1 and C_1 are not separated by the line $l_{A_1A_2}$. The case $n=3$ shows us that the points B_1 and C_1 are not separated by the line $l_{A_2A_3}$. Using the case $n=3$ for the path $A_2A_3A_4$ we see that the points B_1 and C_1 are not separated by the line $l_{A_3A_4}$. By repeating this argument we see that the points B_1 and C_1 are not separated by any line $l_{A_kA_{k+1}}$, $k=1,2,\ldots,n-1$. The same arguments hold for the points B_2 and C_2. $\Box$

5 Special case

The purpose of this section is to prove the following claim:

Theorem 7 *Let $k,l \geq 3$ be odd integers and let (K,L) be a (k,l)–fitting pair. Assume that there is a segment AB of K and a segment XY of L such that AB intersects every segment of L except for XY. Let WX and YZ be the two segments of L adjacent to XY. Assume, that the points W and Z are not separated by the line l_{XY}. Then the number of intersections of K and L is at most $kl-k-(l-3)/2$.*

Proof: First we will prove that there exists a set S of at most $(l+3)/2$ segments of L, such that no segment of K intersects all the segments in S. We will consider two cases separately (see figure 6):

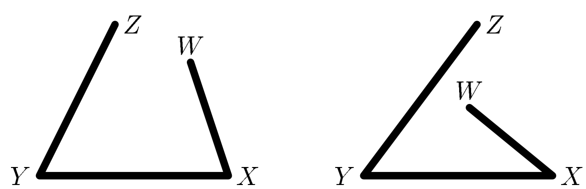

Figure 6: Two possible mutual positions of the points W, X, Y and Z.

- The points W, X, Y and Z are in convex position or possibly $W = Z$ in the case $l=3$ (the left part of figure 6). Then no segment may intersect WX, XY and YZ simultaneously. We can choose $S = \{WX, XY, YZ\}$.

- The points W, X, Y and Z are not in convex position (the right part of figure 6). We can WLOG suppose that the point W lies inside the triangle XYZ. Let $s_1, s_2, s_3, \ldots, s_{l-3}$ be the consecutive segments of L on that path from W to Z which do not pass through the points X and Y. We prove that the set $S = \{WX, XY, YZ\} \cup \{s_1, s_3, s_5, \ldots, s_{2i+1}, \ldots, s_{l-4}\}$ fulfills our requirements. Assume for contradiction that there is a segment CD of K which intersects all the segments of the set S (see figure 7).

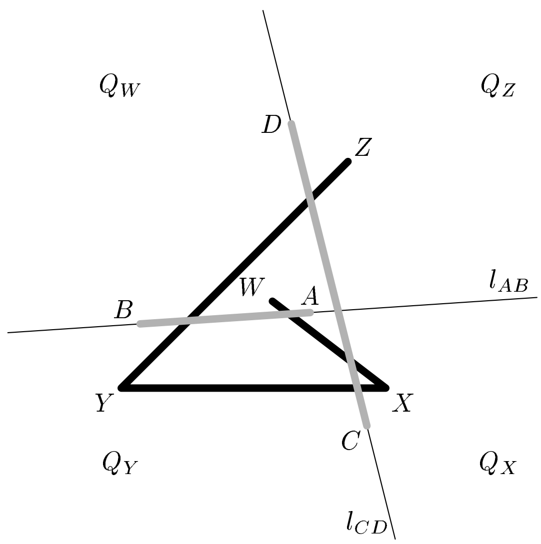

Figure 7: The segment CD interects all segments of the set S.

The line l_{AB} cannot intersect the segment XY due to lemma 2, because it is intersecting all other segments of L. Hence it follows that the line l_{AB} separates X and Y from W and Z. The line l_{CD} intersects all the segments WX, XY and YZ and hence separates X and Z from W and Y. These two lines split the plane into four parts ("quadrants"), each containing one of the points W, X, Y and Z. We denote by Q_W, Q_X, Q_Y and Q_Z the quadrants containing the points W, X, Y and Z respectively. Observe that the line l_{AB} has to intersect the segment CD and the line l_{CD} does not intersect the segment

AB. Hence the segment AB is separated from the point X by the line l_{CD}.

We denote by V_{i-1} and V_i the endpoints of the segments s_i ($V_0 = W$ and $V_{l-3} = Z$).

Now we prove by induction that $V_i \in Q_W$ for every i even and $V_i \in Q_X$ for every i odd. The statement is obvious for $i = 0$. If $i > 0$ and i is even then by induction hypothesis $V_{i-1} \in Q_X$. However the segment s_i must intersect the segment AB. The segment AB is separated from Q_X by the line l_{CD} so s_i must also intersect the line l_{CD}. It follows that the two endpoints of s_i lie in the opposite quadrants, so $V_i \in Q_W$. If i is odd, then $s_i \in S$, so s_i intersects both AB and CD. By induction hypothesis $V_{i-1} \in Q_W$, so V_i must lie in the opposite quadrant Q_X. This completes the induction.

It follows that $V_{l-3} \in Q_W$, which yields a contradiction, because $V_{l-3} = Z$ and $Z \in Q_Z$. We found the set S of the required properties.

Now we are ready to finish the proof of theorem 7. Let m be a cardinality of the set S ($m \leq (l+3)/2$). Every segment of K intersects at most $m-1$ segments of S. Hence there are at most $k(m-1)$ intersections between the segments of K and the segments of S. Moreover, each of the $l-m$ segments of L not belonging to S intersect at most $k-1$ segments of K due to lemma 2. Hence there are at most $(k-1)(l-m)$ intersections between the segments of K and the segments of L which do not belong to S. Thus the total number of intersections of K and L is at most $k(m-1)+(l-m)(k-1) = kl - k - l + m \leq kl - k - (l-3)/2$. $\square$

6 Proof of the upper bound

In this section we prove theorem 8 which states the upper bound on number of intersections from theorem 1 for a (k,l)-fitting pairs. Theorem 1 is an easy consequence of theorem 8 due to the fact $f(k,l) \leq g(k,l)$.

Theorem 8 *Let $k \geq 1$ and $l \geq 3$ be odd integers. Let K be a generalized k–gon and let l be a (normal) l–gon. Let (K,L) be a fitting pair. Then the number of intersections of K and L is at most $kl - k - (l-3)/2$. In the other words*

$$g(k,l) \leq kl - k - (l-3)/2.$$

Proof: We will proceed by induction on k. For $k = 1$ the generalized polygon K consist of one segment A_1B_1 only. Moreover (B_1, A_1) is its vertex pair and hence is not separated by any line given by the segment of L. There cannot be any intersections between the segment of K and the segments of L and hence $g(k,l) = 0 \leq kl - k - (l-3)/2$. For $k = 3$ and l odd we see from lemma 2 that every segment of L intersect at most two segments of K and hence $g(k,l) \leq 2l \leq kl - k - (l-3)/2$.

Now suppose that we have $k > 3$ odd and the statement of theorem 8 holds for all $k' < k$ odd. Let (K, L) be a (k,l)–fitting pair. Consider a $k \times l$ chessboard $\mathcal{C}$ with some checkers constructed as follows. The rows of $\mathcal{C}$ will correspond to the segments of K and the columns of $\mathcal{C}$ will correspond to the segments of L. The field in the intersection of some row and column of $\mathcal{C}$ will be occupied by a checker if and only if the corresponding segments of K and L do not intersect. The empty fields of $\mathcal{C}$ correspond to intersecting pairs of segments of K and L. We have to prove that $\mathcal{C}$ contains at most $kl - k - (l-3)/2$ empty fields i.e. at least $k + (l-3)/2$ checkers. When $\mathcal{C}$ satisfies the assumptions of lemma 3 we can find at least $k + l/2$ chechers and our theorem holds. Hence we only need to consider cases when at least one of the conditions of lemma 3 is violated.

- The condition (1) is violated. There exists some empty row or some empty column of $\mathcal{C}$. This means that the corresponding segment of K (or L) intersects every segment of L (or K). This contradicts lemma 2. This case is impossible.

- The condition (2) is violated. There exists some row and some column such that their only checker lies in their intersection. This means that the corresponding segment AB of K and the corresponding segment XY of L have the following property: The segment AB intersects every segment of L except for the segment XY and vice versa. We easily see that the line l_{XY} does not intersect the segment AB, otherwise lemma 2 would be contradicted. Hence the points A and B lie in the same halfplane given by the line l_{XY}. Let WX and YZ be the two segments of L adjacent to Y. As both of them intersect the segment AB, the points W and Z have to lie in the same halfplane given by the line l_{XY} and are not separated by this line. It is clear that the assumptions of theorem 7 are satisfied and thus the number of intersections of K and L is at most $kl - k - (l-3)/2$ in this case.

- The condition (3) is violated. There exist two identical rows with only one field occupied with a checker in $\mathcal{C}$. This means that there exist two segments AB and $A'B'$ of K and a segment XY of L such that both AB and $A'B'$ intersect all the segments of L except for the segment XY.

Let WX and YZ be the segments of L adjacent to the segment XY. Consider first the case that points W and Z are not separated by the line l_{XY}. We see that the assumptions of the theorem 7 are satisfied and thus the number of intersecions of K and L is at most $kl - k - (l-3)/2$. Now we will deal with the case that the points W and Z are separated by the line l_{XY} (see figure 8).

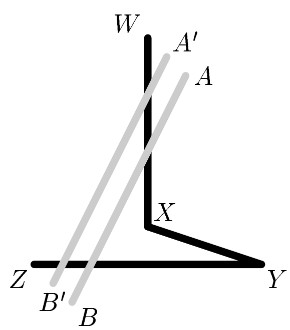

Figure 8: The points W and Z are separated by the line l_{XY}.

As the segments AB and $A'B'$ intersects the segments WX and YZ in the different halfplanes given by the line l_{XY} it is clear that the pairs (A, B) and (A', B') are separated by the line l_{XY}. We can WLOG suppose that the points A and A' are not separated by the line l_{XY} (as drawn in figure 8). It is also clear that the points A and A' are not separated by the line WX. As the segments AB and $A'B'$ intersect all the segments of L except for XY we conclude from lemma 4 used for the path P consisting of all segments of L except for XY that the points A and A' are not separated by any line given by a segment of L. The same arguments apply for the points B and B'.

Now WLOG suppose that (A, A') is not a vertex pair of K. (If (A, A') is a vertex pair of K then (B, B') is not, because $k \geq 3$.) The points A and A' split the of segments of K into two parts K_1 and K_2 which are also generalized polygons. We will show that both of them form a fitting pair with L. There are two possibilities (see figure 9).

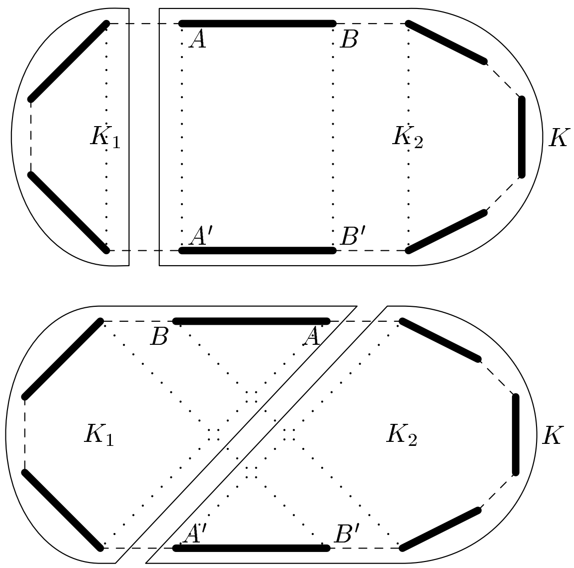

Figure 9: The parts K_1 and K_2 of a generalized polygon K.

- The points A and A' are endpoints of two segments of the same part K_i, $i = 1$ or $i = 2$ (the upper part of figure 9). Then the points A and A' form a vertex pair in the part K_i and as shown above they are not separated by any line given by a segment of L and therefore the vertex pair (A, A') fits with L. Other vertex pairs of K_i are also vertex pairs of K and therefore fit with L. It is clear that the pair (K_i, L) is fitting. Let $j = 3 - i$. Denote C and C' the second vertex belonging to A and A' in a vertex pair in K respectively. Then (C, C') is a vertex pair of K_j. We see that the pairs (C, A), (A, A') and (A', C') are not separated by any line given by a segment of L and thus the vertex pair (C, C') fits with L. The other vertex pairs of K_j are also vertex pairs of K and therefore fit with L. It is clear that also the pair (K_j, L) is fitting.
- The point A is endpoint of some segment of K_1 and the point A' is endpoint of some segment of K_2 (the lower part of figure 9). The arguments in this case are similar to the previous ones.

Lengths of both K_1 and K_2 are at least 1 and together give k. We can WLOG suppose that the length of K_1 is odd (call it o) and the length of K_2 is even (call it e). We use the induction hypothesis for K_1 – the number of intersections of K_1 and L is at most $ol - o - (l-3)/2$. For the K_2 we use the fact that every segment of K_2 intersects at most $l-1$ segments of L due to lemma 2. Therefore the number of intersections of K_2 and L is at most $el - e$. Summing together we get at most $(e+o)l - o - e - (l-3)/2 = kl - k - (l-3)/2$ intersections of K and L.

The proof of theorem 8 is finished. □

7 Conclusion

We have generalized the polygon–crossing problem to a class of fitting pairs of a generalized k–gon and a (normal) l–gon. For $k \geq 1$ and $l \geq 3$ both odd we have proved the upper bound $g(k,l) \leq kl - k - (l-3)/2$ which also improves best known upper bound for the function $f(k,l)$ and gives alternative solution of the case $l = 5$. The original conjecture that $f(k,l) = kl - k - l + 3$ remains open for $k, l \geq 7$. The gap between the upper and lower bound is still linear in the number of vertices. We do not know if $f(k,l) = g(k,l)$ in general. We also do not know if the function $g(k,l)$ is symmetric.

Acknowledgement

Our attention was attracted to the polygon–crossing problem by Jan Kratochvíl and Pavel Valtr when we attended “Problem seminar” at Charles University. The research was done as a part of Research Experience for Undergraduates 2002 programme partly organized by DIMACS at Rutgers University and DIMATIA at Charles University. The REU advisors were János Komlós from Rutgers University and Jan Kratochvíl and Jaroslav Nešetřil from Charles University.

References

[1] B. Aronov, M. Sharir: The common exterior of convex polygons in the plane, Comput. Geom. 8 (3), 1997 pp. 139–149.

[2] J. Černý, J. Kára, D. Král', P. Podbrdský, M. Sotáková, R. Šámal: On the Number of Intersections of Two Polygons, Comment. Math. Univ. Carolin., to appear.

[3] M. B. Dillencourt, D. M. Mount, A. J. Saalfeld: On The Maximum number of Intersections of Two Polyhedra in 2 and 3 Dimensions, Proceedings of the Fifth Canadian Conference on Computational Geometry, Waterloo, Ontario, August, 1993, pp. 49–54.

[4] A. Efrat, M. Sharir: On the complexity of the union of fat convex objects in the plane, Discrete Comput. Geom. 23, 2000, pp. 171–189.

[5] B. Grünbaum: Selfintersections of polygons, Geombinatorics 8, 1998, no. 2, pp. 37–45.

[6] M. van Kreveld: On fat partitioning, fat covering and the union size of polygons, Comput. Geom. 9, 1994, no. 4, pp. 197–210.

[7] J. Matoušek, J. Pach, M. Sharir, S. Sifrony, E. Welzl: Fat triangles determine linearly many holes, SIAM J. Comput. 23, 1994, no. 1, pp. 154–169.

[8] J. Pach, M. Sharir: On the boundary of the union of planar convex sets, Discrete Comput. Geom. 3, 1999, pp. 321–328.