Cantor-type theorem for locally constrained graph homomorphisms

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Dedicated to our children Jarmila Fialová (born December 25, 2002) and Ondřej Maxa (born May 4, 2003).

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Abstract

In the paper we show that the simultaneous existence of a locally surjective and of a locally injective graph homomorphisms between a connected graph G and a connected and finite graph H assures that all such homomorphisms are in fact locally bijective.

We give a short proof of this assertion which unifies previously known partial results of this form. We utilize the notion of universal cover, and relate its properties to the notion of degree refinement, which was used as a principal tool in other works.

1 Introduction

In this paper we consider so called locally constrained graph homomorphisms. For graphs G and H a (graph) homomorphism is a vertex mapping $f : V_G \to V_H$ satisfying the property that $(f(u), f(v))$ is an edge of H whenever vertices u and v are adjacent in G.

In addition, we consider three types of "local" restrictions. We may request that for every vertex u of G the set of its neighbors $N_G(u)$ is mapped

- *bijectively* onto $N_H(f(u))$, then the mapping f is called *locally bijective*, or also a *full covering projection*, or

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- *injectively* into $N_H(f(u))$, then f is called *locally injective*, or also a *partial covering projection*, or
- *surjectively* onto $N_H(f(u))$, then f is called *locally surjective*, or also a *surjective covering projection.*

(The symbol $N_H(f(u))$ denotes the set of all neighbors of $f(u)$ in the graph H.)

All these three sorts of mappings have interesting theoretical and practical aspects. Full covers were first considered in algebraic graph theory [3]. As special cases of covering spaces from algebraic topology [14], graph covers are used in many applications in topological graph theory [9].

Partial covers were applied in a hardness proof for the existence of distance constrained labelings of graphs [8], a notion stemming from a highly practical problem of interference-free frequency assignment for wireless networks.

Surjective covers have been introduced by Everett and Borgatti [6], who called them role colorings. They originated in the theory of social behavior. The graph H, i.e. the *role graph*, models roles and their relationships, and we ask whether roles (i.e. the vertices of the role graph) can be assigned to individuals of a given society such that the relationships are preserved: Each person playing a particular role has among its neighbors exactly all necessary roles as they are prescribed by the model.

Computational applications of graph covers were used by Angluin [2] who studied "local knowledge" in distributed computing environments, and by Courcelle and Métivier [5] who showed that nontrivial minor-closed graph classes cannot be recognized by local computations.

In [4] Bodlaender raised the question of computational complexity of H-cover problems. The H-(partial) cover problem asks if a given input graph G (partially) covers H (here H is considered a fixed parameter of the problem). Similarly we define the H-role assignment problem. In [1] Abello *et al.* showed that there are both polynomial-time solvable (*easy*) and NP-complete (*difficult*) versions of this problem depending on the parameter graph H. The complexity of the H-cover problem was further studied in [10, 11]. Several infinite classes of both polynomial and NP-complete instances were recognized, however, currently no good conjecture concerning a good characterization of graphs H, for which the H-cover problem is polynomially solvable, is at hand (assuming, of course, $P{\neq}\mathsf{NP}$).

At this point we would like to highlight two structural theorems that provide an important tool for proving NP-hardness reduction from the H-cover problem to the H-partial cover and the H-role assignment problems.

Theorem 1 ([7]) *Let G be a finite graph and let H be a finite connected graph. Suppose there is a locally bijective homomorphism from G to H. Then any locally injective homomorphism from G to H is locally bijective, i.e. a full cover.*

An similarly:

Theorem 2 ([12]) *Let G be a finite graph and let H be a finite connected graph. Suppose there is a locally bijective homomorphism from G to H. Then any locally surjective homomorphism from G to H is locally bijective, i.e. a full cover.*

Let us remark that both of these theorems can be strengthened such that instead of the existence of a full covering projection it suffices to assume that both graphs have the same degree refinement matrix (see [7, 12]) which is a weaker condition (see e.g. [10]). The notion of degree refinement and of its matrix originated in the graph isomorphism theory and was very useful in proving relationship between the existence of full and other covering projections. Since this structure can be computed in polynomial time (cf. Section 4), it frequently participated in NP-hardness constructions for various graph covering problems.

In our study we involve the notion of the universal cover instead of the degree refinement to link all these theorems together. This notion was defined by Angluin in [2] in a study on common covers. We show, that by use of this structure we can generalize these two structural theorems in a unified one, which is moreover valid also for infinite graphs G:

Theorem 3 *Let G be a graph and let H be a finite connected graph. Suppose there is a locally surjective homomorphism $g : G \to H$ and a locally injective homomorphism $h : G \to H$. Then both g and h are locally bijective, i.e. full covers.*

The paper is organized as follows: The next section provides definitions used later. Our main result, Theorem 3, is proved in the third section. Several relationships between universal covers and degree refinement matrices are discussed in the last section.

2 Preliminaries

If not stated otherwise we consider simple graphs $G = (V_G, E_G)$, i.e. with no loops or multiple edges. Here V_G is a vertex set and E_G is an edge set, i.e. a set of pairs of distinct vertices from V_G. If the set V_G is finite then we say that G is a finite graph.

A walk in G is a finite sequence of vertices in which any consecutive pair of vertices forms an edge of G. If a walk A ends by the same vertex as another walk B starts, we can concatenate these two walks and denote the resulting walk by $A \circ B$.

A path is a walk where each vertex appears at most once. A graph is called connected if for any pair of vertices exists a finite walk that connects them. It follows that such a finite path also exists. A forest is a possibly infinite graph in which any two vertices are connected by at most one finite path. A connected forest is called a tree.

The distance between two vertices in a connected graph is equal to the length (the number of edges) of the shortest walk connecting these two vertices. It is obvious that such a walk is always a path. The diameter of a finite connected graph G is equal to the maximum distance taken over all pairs of vertices of G.

We say that a walk A' is a *forth-and-back extension* of A, if A' is formed as $A_1 \circ (u, v, u) \circ A_2$ from the walk A of form $A_1 \circ A_2$. In other words walks A and A' are almost identical, only the edge (u, v) is traversed in two consecutive steps in the walk A' compared to the walk A.

We define an equivalence relation $\sim$ on the class of all finite walks as the symmetric and transitive closure of the forth-and-back extension relationship. Two walks are equivalent if both are obtained from the same walk by a serie of forth-and-back extensions. It is natural to represent each equivalence class by a walk in which no edge is traversed twice in two consecutive steps. We use notation $A = [B]_\sim$ to express the fact that the walk A represents the class containing the walk B.

With a connected graph G we associate its *universal cover* T_G constructed as follows: Fix an arbitrary vertex $u \in V_G$. The vertex set V_{T_G} of the universal cover consists of all finite walks in the graph G starting from the vertex u, factorized by the relation $\sim$. Two such walks A and B form an edge of T_G, if $A = [B \circ (u, v)]_\sim$ for some edge (u, v) of G, or equivalently, if one of these two walks extends the other by a single edge at the end.

It is an easy observation that T_G is in fact a tree. Moreover T_G fully covers G, because the mapping $f : T_G \to G$ such that $f((u, \dots, v)) = v$ is a locally bijective homomorphism. Unless G itself is a finite tree (in which case T_G is finite and $T_G \simeq G$), T_G is an infinite graph.

The definition of T_G is independent of the choice of the initial vertex u up to an isomorphism: If we initiate the construction of a universal cover T'_G in another vertex v we may define an isomorphism $f : T_G \to T'_G$ by $f(A) = [B \circ A]_\sim$ where B is an arbitrary walk from v to u in G.

3 Main result

In this section we give a short proof of Theorem 3. Before we prove the theorem we explore a relationship between covering projections on graphs and on their universal covers.

Let G and H be two simple connected graphs and let f be an arbitrary homomorphism $G \to H$. We derive a mapping $f' : V_{T_G} \to V_{T_H}$ by $f'(A) = [(f(u_1), f(u_2), \ldots, f(u_n))]_\sim$, for $A = (u_1, u_2, \ldots, u_n)$. (Without loss of generality we may assume that if vertices of T_G are walks starting in the vertex u_1, then the vertices of T_H are walks emanating from $f(u_1)$.)

Lemma 1 *If a mapping $f : V_G \to V_H$ is a full or partial or surjective, resp., covering projection, then the mapping $f' : V_{T_G} \to V_{T_H}$ is also covering projection with the same local constraint (i.e. full or partial or surjective, resp.).*

Proof: To validate this statement it is sufficient to observe that for an arbitrary walk $A = (u_1, \ldots, u_n)$, its neighborhood $N_{T_G}(A)$ is in one-to-one correspondence to the neighborhood $N_G(u_n)$ of the vertex u_n in G.

When $N_G(u_n)$ is mapped bijectively onto $N_H(f(u_n))$, and $N_{T_G}(A)$ is in one-to-one correspondence with $N_G(u_n)$. Similarly $N_{T_H}(f'(A))$ is in one-to one correspondence with $N_H(f(u_n))$. The composition of these three relations is a bijective mapping between $N_{T_G}(A)$ and $N_{T_H}(f'(A))$.

$$N_{T_G}(A) \longleftrightarrow N_G(u_n) \longrightarrow N_H(f(u_n)) \longleftrightarrow N_{T_H}(f'(A))$$

The proof for injective and surjective mappings f is analogous. $\square$

Lemma 2 *(i) Any locally surjective homomorphism f from a graph G to a connected graph H is globally surjective.*
(ii) Any locally injective homomorphism f from a connected graph G to a forest H is globally injective.

Proof: (i) Suppose a vertex v of H remains (globally) uncovered. Then v is connected by a path to some covered vertex of H and we get a contradiction with the local surjectivity of f along this path.
(ii) Suppose there are two vertices u, v in G such that $f(u) = f(v)$ in H. As G is connected, u and v are connected by a path in G. This path must be mapped by f to a cycle in H (which is impossible if H is a forest). $\square$

The next lemma contains the core argument of our paper.

Lemma 3 *Let G be a connected graph and let H be a finite connected graph. Suppose there is a locally surjective homomorphism $g : G \to H$ and a locally injective homomorphism $h : G \to H$. Further let g' and h' be defined as above. Then*

- *g and h are locally bijective homomorphisms*
- *g' and h' are isomorphisms between the corresponding universal covers T_G and T_H.*

Proof: Consider the universal covers T_G and T_H for G and H. Due to the previous lemma the derived mapping g' is a surjective covering projection from T_G to T_H and h' is a partial covering projection between the same graphs.

According to Lemma 2 g' is globally surjective and h' is globally injective.

Denote by d the diameter of the graph H. For a universal cover T and a vertex $A \in V_T$ we denote by $M(A)$ the set of all vertices that are at distance at most $d+1$ from A. For any A the set $M(A)$ induces a finite subtree of T.

Select a vertex $B \in V_{T_H}$ such that $|M(B)|$ is maximal. Due to the global surjectivity of g' there exists a vertex $A \in V_{T_G}$ such that $g'(A) = B$. Also denote $h'(A) = C$.

Now we get

$$|M(A)| \geq |M(g'(A))| = |M(B)| \geq |M(C)| = |M(h'(A))| \geq |M(A)| \quad (1)$$

The first inequality follows from the local surjectivity of g', the second from the choice of B and the third from the injectivity of h'.

Since both sides of the inequality are the same, we get in fact the equality.

From $|M(A)| = |M(g'(A))|$ follows that g' acts as an isomorphism between $M(A)$ and $M(g'(A))$. The same holds for h' as well. Observe that the set $M(A)$ was selected such that every vertex v of H appears as the last vertex of some walk $A_v \in M(g'(A))$. Moreover the particular A_v can be chosen such that all its neighbors are still inside $M(g'(A))$.

Since g' restricted to $M(A)$ was shown to be an isomorphism, we get that for some $A_u \in M(A)$ with $g'(A_u) = A_v$ the projection g' maps bijectively $N(A_u)$ onto $N(A_v)$. From this immediately follows that the mapping g acts as a bijection between $N(u)$ and $N(v) = N(g(u))$.

The last argument holds for any vertex u, so we have derived that g is a locally bijective homomorphism, i.e. a full covering projection between G and H. By a substitution of h instead of g the same result can be derived also for the mapping h.

By Lemma 2 any locally bijective homomorphism between two trees is an isomorhism. $\square$

To get the proof of Theorem 3 we apply Lemma 3 separately on each component of G.

By a similar argument we can get the following corollary.

Corollary 1 *Let G and H be connected graphs and let H be finite. If the two universal covers T_G and T_H of G and H are isomorphic then any locally injective or locally surjective homomorphism from G to H is locally bijective.*

Proof: The statement follows from the fact that any locally surjective or locally injective graph homomorphism between the two isomorphic trees T_G and T_H is locally bijective.

The proof of this assertion mimics the proof of Lemma 3. We select $B \in V_{T_H}$ which maximizes $|M(B)|$ and adjust equation (1) so it uses the tree isomorphism instead one of the two locally constrained homomorphisms. The remaining argumentation follows the same guidelines. $\square$

4 Concluding remarks

Remarks on finiteness. Observe that the finiteness of the graph H is necessary. A counterexample can be constructed as follows: Take $G = H = (\mathbb{N}, \{(i, i+1) : i \in \mathbb{N}\})$ and the mapping $h : V_G \to V_H$ such that $h(i) = i+1$. Then h is locally injective but not locally bijective.

It does not make any sense to consider the case when G is finite and H is infinite and connected since then no locally surjective homomorphism from G to H exists. This follows immediately from Lemma 2 part (i).

Universal cover versus degree refinement. We have already mentioned that in [7, 12] results were derived by use of so called degree refinement matrix. At this place we would like to relate this notion to our results. First let us briefly review the definition of *degree refinement.*

The *degree partition* of a (finite) graph G is a partition of its vertices into the minimum number of sets $S_1, \dots S_t$ for which there are constants $r_{i,j}$ such that for every $1 \leq i, j \leq t$, every vertex in S_i is adjacent to $r_{i,j}$ vertices in S_j. The *degree refinement* of G is the $t \times t$ matrix $R = (r_{i,j})$. Two degree refinements R_1, R_2 are considered as the same if there is a permutation matrix P such that $R_1 = P^T R_2 P$. The degree refinement matrix can be easily computed (see e.g. [13]).

Now let us quote a theorem of Leighton which relates our results to the notion of degree refinement:

Theorem 4 ([13]) *Given any two finite connected graphs G and H, the following conditions are equivalent:*
(i) G and H share a common finite full cover,
(ii) G and H have the same universal cover (i.e. $T_G \simeq T_H$)
(iii) G and H share a common (possibly infinite) full cover,
(iv) G and H have the same degree refinement.

Hence, in Corollary 1, our assumption $T_G \simeq T_H$ can be replaced by any of the equivalent conditions from Theorem 4. This implies that our approach and that of [7, 12] are in fact build on the same essence.

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