Coloring Powers of Chordal Graphs

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Abstract

We prove that the k-th power G^k of a chordal graph G with maximum degree Δ is $O(\sqrt{k}\Delta^{(k+1)/2})$-degenerated for even values of k and $O(\Delta^{(k+1)/2})$-degenerated for odd ones. In particular, this bounds the chromatic number $\chi(G^k)$. The bound proven for odd values of k is the best possible. Another consequence is the bound $\lambda_{p,q}(G) \le \left\lfloor \frac{(\Delta+1)^{3/2}}{\sqrt{6}} \right\rfloor (2q-1) + \Delta(2p-1)$ on the least possible span $\lambda_{p,q}(G)$ of an $L(p,q)$-labeling for chordal graphs G with maximum degree Δ. On the other hand, a construction of such graphs with $\lambda_{p,q}(G) \ge \Omega(\Delta^{3/2}q + \Delta p)$ is presented.

1 Introduction

The concept of an $L(p,q)$-labeling of graphs (and in particular that of an $L(2,1)$-labeling) is an important graph-theoretical model for assignment of radio frequencies. An $L(p,q)$-*labeling* is an assignment of numbers $0,\dots,K$ to the vertices of an input graph G such that each two adjacent vertices receive numbers which differ by at least p and each two vertices at distance two receive numbers which differ by at least q. The number K is called the *span* and the minimum span for which a proper labeling exists is denoted by $\lambda_{p,q}(G)$. $L(p,q)$-labelings have also been intensively studied from the algorithmic point of view [5, 6].

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A study of relation between the minimum span of an $L(2,1)$-labeling and the maximum degree Δ of a graph G started in the paper of Griggs and Yeh [8]. They conjectured that each graph G has an $L(2,1)$-labeling with the span at most Δ^2 and proved the upper bound $\Delta^2 + 2\Delta$. This bound was later improved to $\Delta^2 + \Delta$ by Chang et. al. [3]. The best currently known upper bound $\Delta^2 + \Delta - 1$ is a consequence of a recent result for the channel assignment problem of the author and Škrekovski [10]. The conjecture of Griggs and Yeh remains unsettled but it is known to be true for several special classes of graphs, among them chordal graphs. The upper $(\Delta+3)^2/4$ for chordal graphs was proved by Sakai [11]. The general bound $(\Delta+2p-1)^2/4$ for the case of $L(p,1)$-labelings of chordal graphs was shown in [4]. In the present paper, we prove asymptotically optimal bounds of $O(\Delta^{3/2})$ on the minimum span of an $L(2,1)$-labeling and $O(\Delta^{3/2}q+\Delta p)$ on the minimum span of an $L(p,q)$-labeling for chordal graphs with maximum degree Δ.

1.1 Results

We are actually concerned with a more general problem, namely coloring arbitrary powers of chordal graphs. A graph is said to be *chordal* if it contains no induced cycle of length four or more. It is well-known that a graph is chordal if and only if it has a perfect elimination sequence. A *perfect elimination sequence* is an ordering $v_1, \ldots, v_n$ of vertices of G such that for each i, the neighbors of the vertex v_i preceding it in the sequence form a clique in G. The *k-th power* G^k of a graph G is the graph on the same vertex set such that two vertices in G^k are joined by an edge if their distance in G is at most k. Note that $\chi(G^2) = \lambda_{1,1}(G) + 1$. It is known [1] that if the girth of G is at least 7, then $\chi(G^2) \le O(\Delta^2/\log\Delta)$ and otherwise the chromatic number of G may reach $\Theta(\Delta^2)$ (this is witnessed by bipartite incidence graphs of projective planes). Our results establish that $\chi(G^2) \le O(\Delta^{3/2})$ if G is chordal.

More generally, we show that if G is a chordal graph with maximum degree Δ, then G^k is $O(\sqrt{k}\Delta^{(k+1)/2})$-degenerated (Theorem 2), i.e., each subgraph H of G^k contains a vertex of degree at most $O(\sqrt{k}\Delta^{(k+1)/2})$. If k is odd, we even prove that G^k is $O(\Delta^{(k+1)/2})$-degenerated (Theorem 4). The bound proven for odd values of k in Theorem 4 is the best possible.

Theorems 2 and 4 have several interesting corollaries: First, if G is a chordal graph with maximum degree Δ, then the chromatic number of its k-th power G^k is at most $O(\sqrt{k}\Delta^{(k+1)/2})$ if k is even and at most $\Delta^{(k+1)/2}/4+$

$O(\Delta^{(k-1)/2})$ if k is odd. The minimum span of an $L(2,1)$-labeling of G is at most $O(\Delta^{3/2})$ and the minimum span of an $L(p,q)$-labeling of G is at most $O(\Delta^{3/2}q+\Delta p)$. All our results are asymptotically tight (for a fixed k) and the presented upper bound for odd powers of chordal graphs is even the best possible as remarked above (cf. Theorems 4 and 5): We show a construction of chordal graphs G on n vertices with maximum degree Δ such that G^k is a clique for $n=\Omega(\Delta^{(k+1)/2})$ in Section 5. The lower bound $\Omega(\Delta^{3/2}q+\Delta p)$ for the minimum span of an $L(p,q)$-labeling of chordal graphs is then presented (Corollary 3).

The shown upper and lower bounds match for odd powers of chordal graphs, but they do not match for even powers. Though in the case of the second power and $L(2,1)$-labeling, the proven leading coefficients are quite close. Namely, we establish in Corollary 1 for a chordal graph G with maximum degree Δ that:

$$\chi(G^2) \leq \left\lfloor \frac{(\Delta+1)^{3/2}}{\sqrt{6}} \right\rfloor + \Delta + 1 \approx 0.4082\Delta^{3/2} + O(\Delta)$$

And in Theorem 3:

$$\lambda_{2,1}(G) \leq \frac{\Delta^{3/2}}{\sqrt{6}} + O(\Delta) \approx 0.4082\Delta^{3/2} + O(\Delta)$$

On the other hand, there exists a chordal graph G with maximum degree Δ such that (Theorem 6):

$$\lambda_{2,1}(G) \geq \chi(G^2) - 1 \geq \frac{2\Delta^{3/2}}{3\sqrt{3}} - O(\Delta^{23/21}) \approx 0.3849\Delta^{3/2} - O(\Delta^{23/21})$$

We conjecture that the lower bound is tight:

Conjecture 1 *If G is a chordal graph with maximum degree Δ, then:*

$$\chi(G^2) \leq \lambda_{2,1}(G) + 1 \leq \frac{2\Delta^{3/2}}{3\sqrt{3}} + O(\Delta)$$

1.2 Separators in Chordal Graphs

The proof of one of our upper bound theorems (Theorem 2) is based on a partial separator lemma which we prove in Section 2 and is of its own interest. We call Lemma 1 a *partial separator* lemma because we cut only a part of the set U (of size between κ and 2κ) from the rest of the graph. We remark that the following separator theorem for chordal graphs was proved by Gilbert, Rose and Edenbrandt in [7]: For each chordal graph G and each set $U \subseteq V(G)$, there is a clique C in G such that every component of $G \setminus C$ contains at most $|U|/2$ vertices of U. However, the proof of this result does not seem to be possible to be adopted to our case. The main obstacle is that the technique used for proving the above result does not seem to provide some lower bounds on the size of the cut part of the graph.

2 A Partial Separator Lemma

It is well-known [2] that the class of chordal graphs is precisely the class of intersection graphs of subtrees of a tree, i.e., for each chordal graph G, there exists a tree $\mathcal{T}$ such that for each vertex v of G there is a subtree T_v of $\mathcal{T}$ with the following property: The vertices v and v' of G are joined by an edge if and only if $T_v \cap T_{v'} \neq \emptyset$. Such a representation of a chordal graph can be polished to a little more restricted intersection representation which we call a *nice tree representation* of a chordal graph. We leave details of a straightforward proof of the next proposition to the reader ($A \div B$ denotes the symmetric difference of sets A and B):

Proposition 1 *Let G be a chordal graph. Then, there exists a rooted tree $\mathcal{T}$ with subtrees T_v for each vertex $v \in V(G)$ representing the graph G. Let $\mathcal{T}_u = \{v \in V(G) \,|\, u \in T_v\}$ for $u \in V(\mathcal{T})$. In addition, the representation $\mathcal{T}$ satisfies the following:*

- *$T_v \cap T_w \neq \emptyset$ for $v, w \in V(G)$ if and only if $vw \in E(G)$.*
- *Each vertex of $\mathcal{T}$ has at most two children.*
- *If u is a leaf or the root of $\mathcal{T}$, then $\mathcal{T}_u = \emptyset$.*
- *If u is a vertex of $\mathcal{T}$ with a single child u', then $|\mathcal{T}_u \div \mathcal{T}_{u'}| \leq 1$.*
- *If u is a vertex of $\mathcal{T}$ with two children u' and u'' and a parent u_0, then $\mathcal{T}_u = \mathcal{T}_{u_0} = \mathcal{T}_{u'} = \mathcal{T}_{u''}$.*

Sketch of proof: Let $\mathcal{T}_0$ be a tree representation of the chordal graph G. Replace each vertex of $\mathcal{T}_0$ of degree three or more by a 3-regular subtree. This assures the properties the second and the last property of the statement. Next, replace each edge of the tree $\mathcal{T}_0$ by a sufficiently long path ending (or starting) each subtree T_v, $v \in V(G)$, on the path one at time. Similarly add sufficiently long paths to the leaves of $\mathcal{T}_0$ such that subtrees T_v, $v \in V(G)$, end one at time on them. The obtained tree representation of G can be rooted at any of its leaves.

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Proposition 1 makes the proof of the following lemma quite simple:

Lemma 1 *Let G be a chordal graph, U a subset of $V(G)$ and $1 \le \kappa \le |U|$ a real number. Then G contains a clique C such that there is subgraph G_0 of G consisting of a disjoint union of some of the components of $G \setminus C$ and $\kappa \le |U \cap V(G_0)| \le 2\kappa$.*

Proof: Let $\mathcal{T}$ be a nice tree representation of G as described in Proposition 1. Let $\mathcal{T}(v)$ be a subtree of $\mathcal{T}$ rooted at the vertex v of $\mathcal{T}$ and let $\sigma(v)$, $v \in V(\mathcal{T})$, be the number of subtrees T_u with $u \in U$ which are fully contained in $\mathcal{T}(v)$ and $v \notin T_u$, i.e., $\sigma(v) = |\{u \in U \,|\, T_u \subseteq \mathcal{T}(v) \setminus \{v\}\}|$. Notice the following:

- If v is the root of $\mathcal{T}$, then $\sigma(v) = |U|$.
- If v is a leaf of $\mathcal{T}$, then $\sigma(v) = 0$.
- If v has a single child v', then $\sigma(v)$ is either $\sigma(v')$ or $\sigma(v') + 1$.
- If v has two children v' and v'', then $\sigma(v) = \sigma(v') + \sigma(v'')$.

Choose a vertex v of $\mathcal{T}$ such that $\sigma(v) \ge \kappa$ but $\sigma(v') < \kappa$ for each child v' of v in $\mathcal{T}$. Let C be the clique consisting of the vertices w whose tree contains v, i.e., $v \in T_w$. Let further G_0 be the subgraph of G induced by the vertices w such that $T_w \subseteq \mathcal{T}(v) \setminus \{v\}$. Note that G_0 is a union of components of $G \setminus C$. In addition, $|U \cap V(G_0)| = \sigma(v) \ge \kappa$. The vertex v is not a leaf of $\mathcal{T}$ because $\kappa > 0$. If v has a single child v', then $|U \cap V(G_0)| < \kappa + 1 \le 2\kappa$ because $|\mathcal{T}_v \div \mathcal{T}_{v'}| \le 1$ and $\sigma(v') < \kappa$. If v has two children v' and v'', then $|U \cap V(G_0)| = \sigma(v) = \sigma(v') + \sigma(v'') < 2\kappa$ by the choice of the vertex v.

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The proved factor 2 in Lemma 1 is the best possible: Consider three paths consisting of m vertices and identify the ends of the paths to get a tree with a single vertex v of degree three. The obtained graph G is a tree and hence it is chordal. Let U be all the vertices of G except for v. The tightness is witnessed for the choice $\kappa = m + 1$ as m tends to infinity.

3 Even Powers and $L(p,q)$-labelings

The result of the following theorem is improved for odd values of k in the next section, but we state the theorem in its general form:

Theorem 2 *Let G be a chordal graph with maximum degree $\Delta \geq 2$ and let $k \geq 2$. Then each subgraph H of G^k contains a vertex of degree at most:*

$$\left\lfloor \sqrt{\frac{91k-118}{384}}(\Delta+1)^{(k+1)/2} \right\rfloor + \Delta$$

Proof: We can suppose that G is connected. Assume that the statement of the theorem is false and let H be a subgraph of G^k with minimum degree at least $d_0 := \left\lfloor \sqrt{\frac{91k-118}{384}}(\Delta+1)^{(k+1)/2} \right\rfloor + \Delta + 1$. Apply Lemma 1 for $\kappa = d_0/3 \geq 1$ and $U = V(H)$. Let C be the clique and G_0 the union of some of components of $G \setminus C$ as it is described in the statement of the lemma. Let $G' = G \setminus (G_0 \cup C)$, $U_0 = U \cap V(G_0)$ and $U' = U \cap V(G')$. Set κ_0 to be $|U_0|$. We count the number of induced paths in G of length at most k from a vertex $u \in U_0$ to a vertex $w \in U'$ through a vertex $v \in C$. The length of a path is the distance between its end-vertices. Since the minimum degree of H in G^k is at least d_0 and $|U_0| = \kappa_0$, there must be at least $\kappa_0(d_0 - |C| - \kappa_0) \geq \frac{2(d_0-\Delta)^2}{9}$ such paths (observe that $|C| \leq \Delta$ because $\kappa_0 > 0$ and G is connected). Next, the number of such paths is bounded from above and then we obtain a contradiction by combining both the bounds.

Let $\deg_0(v)$, $\deg_C(v)$ and $\deg'(v)$ be the number of neighbors of $v \in C$ among $V(G_0)$, C and $V(G')$, respectively. Note that $\deg_0(v) + \deg_C(v) + \deg'(v) \leq \Delta$. The number of induced paths of length l between a vertex of U_0 and of U' which contain exactly one vertex of C is at most $\sum_{v \in C} \deg_0(v)\deg'(v)(l-1)(\Delta-1)^{l-2}$. This is because a vertex v of the clique C can be one of $l-1$ inner vertices of the path, there are $\deg_0(v)$ choices for its neighbor in G_0 and $\deg'(v)$ choices for its neighbor in G' and there are at most $\Delta-1$ choices at each vertex in G_0 and in G' to continue the path. Note that even not all such paths join a vertex of U_0 and U'. Hence the number of such induced paths of length at most k with one vertex of C is bounded by the following sum:

$$\sum_{l=2}^{k} \sum_{v \in C} \deg_0(v)\deg'(v)(l-1)(\Delta-1)^{l-2} \leq \sum_{v \in C} \deg_0(v)\deg'(v)(k-1)\Delta^{k-2} \leq$$

$$\sum_{v\in C}\deg_0(v)(\Delta-\deg_C(v)-\deg_0(v))(k-1)\Delta^{k-2}\le$$
$$\sum_{v\in C}\frac{(\Delta-\deg_C(v))^2}{4}(k-1)\Delta^{k-2}=\sum_{v\in C}\frac{(\Delta+1-|C|)^2}{4}(k-1)\Delta^{k-2}\le$$
$$\frac{|C|(\Delta-|C|+1)^2}{4}(k-1)\Delta^{k-2}\le\frac{(\Delta+1)^3\Delta^{k-2}(k-1)}{27}\tag{1}$$

Let $D_0=\sum_{v\in C}\deg_0(v)$ and $D'=\sum_{v\in C}\deg'(v)$ be the number of edges joining the clique C to G_0 and to G' respectively. Note that $D_0+D'+2|C|(|C|-1)\le|C|\Delta$. An induced path between a vertex of U_0 and of U' cannot contain three or more vertices of C and if it contains two vertices, such two vertices must be consecutive. The number of the induced paths of length l containing an edge of C is at most $D_0D'(l-2)(\Delta-1)^{l-3}$. This is because an edge of C can be one of $l-2$ inner edges of the path, there are at most D_0 vertices of G_0 adjacent to C and at most D' vertices of G' adjacent to C and there are at most $\Delta-1$ choices at each vertex in G_0 and $G\setminus(C\cup G_0)$ how to continue the path. The following sum hence bounds the number of such induced paths of length at most k:

$$\sum_{l=2}^{k}D_0D'(l-2)(\Delta-1)^{l-3}\le D_0D'(k-2)\Delta^{k-3}\le$$
$$\frac{(|C|(\Delta+2-2|C|))^2}{4}(k-2)\Delta^{k-3}\le\frac{(\Delta+1)^4\Delta^{k-3}(k-2)}{64}\tag{2}$$

Summing (1) and (2), we get the desired upper bound on the number of the induced paths of length at most k:

$$\frac{(\Delta+1)^3\Delta^{k-2}(k-1)}{27}+\frac{(\Delta+1)^4\Delta^{k-3}(k-2)}{64}\le\frac{91k-118}{1728}(\Delta+1)^{k+1}$$

Comparing the upper and the lower bounds yields a contradiction with the inequality $d_0>\sqrt{\frac{91k-118}{384}}(\Delta+1)^{(k+1)/2}+\Delta$ following from the choice of d_0:

$$\begin{aligned}\frac{2(d_0-\Delta)^2}{9}&\le\frac{91k-118}{1728}(\Delta+1)^{k+1}\\(d_0-\Delta)^2&\le\frac{91k-118}{384}(\Delta+1)^{k+1}\\d_0&\le\sqrt{\frac{91k-118}{384}(\Delta+1)^{k+1}}+\Delta\end{aligned}$$

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An immediate consequence of Theorem 2 is the following bound for chromatic number of powers of chordal graphs:

Corollary 1 *Let G be a chordal graph with maximum degree $\Delta \geq 1$. Then:*

$$\chi(G^k) \leq \left\lfloor \sqrt{\frac{91k - 118}{384}}(\Delta + 1)^{(k+1)/2} \right\rfloor + \Delta + 1 = O(\sqrt{k}\Delta^{(k+1)/2})$$

In particular:

$$\chi(G^2) \leq \left\lfloor \frac{(\Delta + 1)^{3/2}}{\sqrt{6}} \right\rfloor + \Delta + 1 \approx 0.4082\Delta^{3/2} + O(\Delta)$$

Another corollary of Theorem 2 is a $O(\Delta^{3/2})$-bound on the minimum span of a $L(2,1)$-labeling and a $L(p,q)$-labeling:

Theorem 3 *Let G be a chordal graph with maximum degree $\Delta \geq 1$. Then:*

$$\lambda_{p,q}(G) \leq \left\lfloor \frac{(\Delta + 1)^{3/2}}{\sqrt{6}} \right\rfloor (2q - 1) + \Delta(2p - 1) = \frac{\Delta^{3/2}}{\sqrt{6}}(2q - 1) + O(\Delta p)$$

In particular:

$$\lambda_{2,1}(G) \leq \left\lfloor \frac{(\Delta + 1)^{3/2}}{\sqrt{6}} \right\rfloor + 3\Delta = \frac{\Delta^{3/2}}{\sqrt{6}} + O(\Delta) \approx 0.4082\Delta^{3/2} + O(\Delta)$$

Proof: Let $v_1, \ldots, v_n$ be an ordering of the vertices of G such that v_i has at most $\left\lfloor \frac{(\Delta+1)^{3/2}}{\sqrt{6}} \right\rfloor + \Delta$ neighbors among the vertices $v_1, \ldots, v_{i-1}$ in G^2. Such an ordering exists by Theorem 2. Label the vertices of G in this order by labels from 0 to $\left\lfloor \frac{(\Delta+1)^{3/2}}{\sqrt{6}} \right\rfloor (2q - 1) + \Delta(2p - 1)$ in a greedy fashion. Consider a step when a label to the vertex v_i is to be assigned. Each of the neighbors of v_i in G^2 among $v_1, \ldots, v_{i-1}$ prevents assigning at most $2q - 1$ different labels to v_i. In addition, each of (at most Δ) neighbors of v_i in G among $v_1, \ldots, v_{i-1}$ prevents assigning at most $2p - 1 - (2q - 1) = 2p - 2q$ additional labels to v_i. Altogether at most $\left\lfloor \frac{(\Delta+1)^{3/2}}{\sqrt{6}} \right\rfloor (2q - 1) + \Delta(2p - 1)$ labels are forbidden and thus there is at least a single label which can be assigned to v_i.

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4 Odd Powers

The proof of degeneracy in the case of odd powers of chordal graphs is based on a simple proof of the well-known fact that odd powers of chordal graphs are chordal. As noted in the introduction, the proven upper bound is the best possible (cf. Theorem 5).

Theorem 4 *Let G be a chordal graph with maximum degree $\Delta \geq 2$ and let $k \geq 3$ be an odd integer. Then, G^k is d-degenerated for the following choice of d:*

$$d = \begin{cases} k & \text{if } \Delta = 2, \\ \left\lfloor \frac{(\Delta+2)^2}{4} \right\rfloor - 1 & \text{if } \Delta \geq 3 \text{ and } k = 3, \\ \left\lceil \frac{\Delta+1}{2} \right\rceil + \left\lfloor \frac{(\Delta+1)^2}{4} \right\rfloor \frac{(\Delta-1)^{(k-1)/2}-1}{\Delta-2} - 1 & \text{otherwise.} \end{cases}$$

Proof: If $\Delta = 2$, then G is a union of disjoint paths and hence G^k is k-degenerated. Let us assume that $\Delta \geq 3$ in the rest of the proof. Let $\mathcal{T}$ be a (not necessarily nice) tree representation of G where T_v is the subtree representing a vertex v of G. Let $\mathcal{T}'$ be the following tree representation of G^k: The underlying tree of $\mathcal{T}'$ is the same one as of $\mathcal{T}$. A vertex v of G^k is represented by a tree $T'_v = \bigcup_{w \in N_{(k-1)/2}(v)} T_w$ where $N_i(v)$ is the set of all the vertices at distance at most i from v in G. It is easy to check that each of T'_v is connected, i.e., a subtree of $\mathcal{T}'$. In addition, $T'_u \cap T'_v \neq \emptyset$ if (and only if) u and v are at distance at most $2 \cdot \frac{k-1}{2} + 1 = k$ in G. Hence $\mathcal{T}'$ is a tree representation of G^k and in particular G^k is chordal.

If $\omega(G^k)$ is the size of the largest clique of G^k, then the graph G^k is d-degenerated with $d = \omega(G^k) - 1$ (consider a perfect elimination sequence for G^k). We prove the following inequality:

$$\omega(G^k) \leq \begin{cases} \left\lfloor \frac{(\Delta+2)^2}{4} \right\rfloor & \text{if } k = 3, \\ \left\lceil \frac{\Delta+1}{2} \right\rceil + \left\lfloor \frac{(\Delta+1)^2}{4} \right\rfloor \frac{(\Delta-1)^{(k-1)/2}-1}{\Delta-2} & \text{otherwise.} \end{cases}$$

Let C_0 be a largest clique of G^k. Then there exists a vertex u_0 of $\mathcal{T}'$ such that $u_0 \in T'_v$ for each $v \in C_0$. Let C be the clique of G comprised by all the vertices v such that $u_0 \in T_v$ and p its size. The distance of a vertex v from C (in G) is the smallest distance of the vertex v from some vertex of C. The clique C_0 consists (precisely) of all the vertices which are at distance at most $(k-1)/2$ from C (recall the definition of $\mathcal{T}'$). There are at most $p(\Delta - (p-1))$ vertices at distance one and additional at most $p(\Delta - (p-1))(\Delta - 1)$ vertices at distance two from C. In general, the number of vertices at distance i from C is at most $p(\Delta - (p-1))(\Delta-1)^{i-1}$. Hence the size of C_0 is bounded from above by the following sum:

$$p + p(\Delta+1-p) + p(\Delta+1-p)(\Delta-1) + \ldots + p(\Delta+1-p)(\Delta-1)^{(k-3)/2}$$

If $k = 3$, then the size of C_0 is at most:

$$p + p(\Delta+1-p) = p(\Delta+2-p) \leq \left\lfloor \frac{\Delta+2}{2} \right\rfloor \left\lceil \frac{\Delta+2}{2} \right\rceil = \left\lfloor \frac{(\Delta+2)^2}{4} \right\rfloor$$

In the general case $k \geq 5$, the size of C_0 is at most:

$$p + p(\Delta+1-p)\frac{(\Delta-1)^{(k-1)/2}-1}{\Delta-2} \leq$$

$$\left\lceil \frac{\Delta+1}{2} \right\rceil + \left\lfloor \frac{\Delta+1}{2} \right\rfloor \left\lceil \frac{\Delta+1}{2} \right\rceil \frac{(\Delta-1)^{(k-1)/2}-1}{\Delta-2} =$$

$$\left\lceil \frac{\Delta+1}{2} \right\rceil + \left\lfloor \frac{(\Delta+1)^2}{4} \right\rfloor \frac{(\Delta-1)^{(k-1)/2}-1}{\Delta-2}$$

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An immediate corollary of Theorem 2 is the following bound for the chromatic number of odd powers of chordal graphs:

Corollary 2 *Let G be a chordal graph with maximum degree Δ and let $k \geq 3$ be an odd integer. Then:*

$$\chi(G^k) \leq \begin{cases} 2 & \textit{if } \Delta = 1, \\ k+1 & \textit{if } \Delta = 2, \\ \left\lfloor \frac{(\Delta+2)^2}{4} \right\rfloor & \textit{if } \Delta \geq 3 \textit{ and } k = 3, \\ \left\lceil \frac{\Delta+1}{2} \right\rceil + \left\lfloor \frac{(\Delta+1)^2}{4} \right\rfloor \frac{(\Delta-1)^{(k-1)/2}-1}{\Delta-2} & \textit{otherwise.} \end{cases}$$

5 Lower bounds

We first prove the tightness of Theorem 4:

Theorem 5 *Let $k \geq 3$ be an odd integer and let $\Delta \geq 3$ be an integer. There is a chordal graph G on n vertices such that G^k is a clique for the following choice of n:*

$$n = \begin{cases} 2 & \textit{if } \Delta = 1, \\ k+1 & \textit{if } \Delta = 2, \\ \left\lfloor \frac{(\Delta+2)^2}{4} \right\rfloor & \textit{if } \Delta \geq 3 \textit{ and } k = 3, \\ \left\lceil \frac{\Delta+1}{2} \right\rceil + \left\lfloor \frac{(\Delta+1)^2}{4} \right\rfloor \frac{(\Delta-1)^{(k-1)/2}-1}{\Delta-2} & \textit{otherwise.} \end{cases}$$

Proof: If $\Delta = 1$, then set G to be K_2. If $\Delta = 2$, set G to be a path on $k+1$ vertices. If $\Delta \geq 3$ and $k = 3$, then it is enough to choose G as a clique of size $\lceil \frac{\Delta+2}{2} \rceil$ such that each vertex of it is adjacent to $\lfloor \frac{\Delta}{2} \rfloor$ vertices of degree one. Assume in the rest that $\Delta \geq 3$ and $k \geq 5$.

Let G consist of a clique of size $\lceil \frac{\Delta+1}{2} \rceil$ such that each of the vertices of the clique has $\lfloor \frac{\Delta+1}{2} \rfloor$ neighbors each of degree one. These vertices of degree one form the first level. Each vertex of the first level has $\Delta - 1$ neighbors which form the second level, each vertex of the second level has $\Delta - 1$ neighbors which form the third level, etc. The graph G has altogether $(k-1)/2$ levels. It is easy to verify that the maximum degree of G is Δ and G^k is a clique. The number of vertices of G is equal to the following sum:

$$\left\lceil \frac{\Delta+1}{2} \right\rceil + \left\lfloor \frac{\Delta+1}{2} \right\rfloor \left\lceil \frac{\Delta+1}{2} \right\rceil + \left\lfloor \frac{\Delta+1}{2} \right\rfloor \left\lceil \frac{\Delta+1}{2} \right\rceil (\Delta-1) + \ldots +$$

$$\left\lfloor \frac{\Delta+1}{2} \right\rfloor \left\lceil \frac{\Delta+1}{2} \right\rceil (\Delta-1)^{(k-3)/2} =$$

$$\left\lceil \frac{\Delta+1}{2} \right\rceil + \left\lfloor \frac{(\Delta+1)^2}{4} \right\rfloor \frac{(\Delta-1)^{(k-1)/2}-1}{\Delta-2}$$

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We use the following result of Iwaniec and Pintz [9] in our lower bound construction in the case of even powers:

Proposition 2 *For each $n \geq 2$, there is a prime p such that $n - n^{23/42} \leq p \leq n$.*

Theorem 6 *Let $k \geq 2$ be an even integer and let $\Delta \geq 12$. If $k \geq 4$, then there exists a chordal graph G on n vertices such that G^k is a clique for the following choice of n:*

$$n = \begin{cases} \frac{2\Delta^{3/2}}{3\sqrt{3}} - O(\Delta^{23/21}) \approx 0.3849\Delta^{3/2} - O(\Delta^{23/21}) & \text{if } k = 2, \\ \frac{18}{25\sqrt{15}}\Delta^{(k+1)/2} - O(\Delta^{k/2+2/21}) & \text{if } k \geq 4. \end{cases}$$

In particular, there is a chordal graph G with maximum degree Δ such that:

$$\lambda_{2,1}(G) \geq \chi(G^2) - 1 \geq \frac{2\Delta^{3/2}}{3\sqrt{3}} - O(\Delta^{23/21}) \approx 0.3849\Delta^{3/2} - O(\Delta^{23/21})$$

Proof: Consider first the case that $k = 2$. Let $q_0 = \left\lfloor\sqrt{\frac{\Delta}{3}}\right\rfloor - 1 \geq 1$. If $q_0 \geq 2$, let p be a prime such that $q_0 - q_0^{23/42} \leq p \leq q_0$ (it exists by Proposition 2) and (X, Π) be a projective plane of order p (thus $|X| = p^2 + p + 1$). If $q_0 = 1$, let $p = 1$ and (X, Π) be a pair of sets such that $|X| = 3$ and Π contains all three pairs of elements of X. In the rest of the proof, it does not matter whether (X, Π) is a projective plane or a "triangle". Let $m = \left\lfloor 2\sqrt{\frac{\Delta}{3}}\right\rfloor \geq 4$. The desired graph G consists of vertices v_x for each $x \in X$ and vertices $v_\pi^1, \ldots, v_\pi^m$ for each $\pi \in P$. Thus G has altogether $(p^2 + p + 1)(m + 1) = \frac{2\Delta^{3/2}}{3\sqrt{3}} - O(\Delta^{23/21})$ vertices.

The vertices v_x, $x \in X$, form a clique of size $|X| = p^2 + p + 1$. Each vertex v_π^i, $\pi \in \Pi$ and $1 \leq i \leq m$, is joined by an edge to each vertex v_x, $x \in X$, such that $x \in \pi$. The sequence of vertices containing the vertices v_x first and then the vertices v_π^i is a perfect elimination sequence and hence G is chordal. In order to show that G^2 is a clique, we need to verify that each pair of non-adjacent vertices has a common neighbor. This is clear for a pair consisting of a vertex v_x and a vertex v_π^i. Consider next a pair of vertices v_π^i and $v_{\pi'}^{i'}$. The sought common neighbor of the two vertices is any vertex v_x such that $x \in \pi \cap \pi'$ (such an $x \in X$ exists because (X, Π) is a projective plane).

It remains to verify that the maximum degree of G is at most Δ. The degree of a vertex v_x contained in the central clique is the following:

$$p^2 + p + (p+1)m \leq (q_0 + 1)(q_0 + m) \leq \sqrt{\frac{\Delta}{3}} \cdot \left(\sqrt{\frac{\Delta}{3}} + 2\sqrt{\frac{\Delta}{3}}\right) = \Delta$$

The degree of a vertex v_π^i is even smaller, namely $(p+1)m$.

If $k \geq 4$, we proceed as follows. The graph G is as in the case $k = 2$ for the choice of parameters $q_0 = \left\lfloor\sqrt{\frac{3\Delta}{5}}\right\rfloor - 1$ and $m = \left\lfloor 2\sqrt{\frac{\Delta}{15}}\right\rfloor$. The number p is again a prime between $q_0 - q_0^{23/42}$ and q_0. If $q_0 = 1$, then $p = 1$ and (X, Π) is a triangle. We consider the vertices v_π^i to form the first level (similarly as in the proof of Theorem 5), each of these vertices has $\Delta - m(p+1)$ neighbors which form the second level. Additional $(k-4)/2$ levels are formed as in the beginning in the case of odd k by adding $\Delta - 1$ new vertices to each vertex of the preceding level. The maximum degree of G is Δ and G^k is a clique. The number of vertices of G forming the last level is equal to the following product:

$$(p^2 + p + 1)m(\Delta - m(p+1))(\Delta - 1)^{(k-4)/2} =$$

$$\frac{3\Delta}{5} \cdot 2\sqrt{\frac{\Delta}{15}} \cdot \left(\Delta - \frac{2\Delta}{5}\right) \cdot (\Delta - 1)^{(k-4)/2} - O(\Delta^{k/2+2/21}) =$$

$$\frac{18}{25\sqrt{15}}\Delta^{(k+1)/2} - O(\Delta^{k/2+2/21})$$

Hence the number of vertices of G is $\frac{18}{25\sqrt{15}}\Delta^{(k+1)/2} - O(\Delta^{k/2+2/21})$.

■

The construction presented in Theorem 6 gives also a good lower bound on the minimum span of an $L(p, q)$-labeling of a chordal graph:

Corollary 3 *There exists a chordal graph G with maximum degree Δ such that $\lambda_{p,q}(G) \geq \Omega(\Delta^{3/2}q + \Delta p)$.*

Proof: Consider the chordal graph G from Theorem 6 on $n = \frac{2\Delta^{3/2}}{3\sqrt{3}} - O(\Delta^{23/21})$ vertices such that G^2 is a clique. Hence, $\lambda_{p,q}(G) \geq \left(\frac{2\Delta^{3/2}}{3\sqrt{3}} - 1\right) q - O(\Delta^{23/21}q)$. On the other hand, since G contains a clique of size $q_0^2 + q_0 + 1 = \Theta(\Delta)$ (for the choice of q_0 from the proof of Theorem 6), we have $\lambda_{p,q}(G) \geq \Omega(\Delta p)$. Combining both the bounds on $\lambda_{p,q}(G)$ give the claimed bound.

■

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