No Helly theorem for stabbing translates by lines in $\mathbb{R}^3$

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Abstract

For each $n > 2$ we construct a convex body $K \subset \mathbb{R}^3$ and a finite family $\mathcal{F}$ of disjoint translates of K such that any $n-1$ members $\mathcal{F}$ admit a line transversal, but $\mathcal{F}$ has no line transversal.

1 Introduction

For a wider background of the questions considered in this note we refer to Wenger [10] or other surveys on Helly-type results and geometric transversal theory.

Let $\mathcal{F}$ be a family of convex sets in $\mathbb{R}^d$. A line that intersects every member of $\mathcal{F}$ is called a *line transversal* of $\mathcal{F}$. In 1989 Tverberg [9] proved the following: If $\mathcal{F}$ is a family of disjoint translates of a compact convex set in $\mathbb{R}^2$ such that any 5 members of $\mathcal{F}$ admit a line transversal, then $\mathcal{F}$ admits a line transversal.

This statement was first conjectured by Grünbaum [3] in a paper where he proved a special case, namely when K is a parallelogram. Prior to this, Danzer [1] proved the case where K is a disc. In 1986 Katchalski [5] proved the weaker statement in which the number 5 is replaced by 128 (this was the first Helly-type theorem for line transversals for general translates). It should also be noted that Tverberg's transversal theorem is the best possible, meaning if any of the assumptions are weakened the conclusion no longer holds.

What can be said about line transversals of families of disjoint translates in $\mathbb{R}^3$? A special case of a theorem by Santaló [8] asserts that if $\mathcal{F}$ is a family of disjoint translates of a parallelotope in $\mathbb{R}^3$ and any 40 members admit a line transversal, then $\mathcal{F}$ admits a line transversal. A recent result by Holmsen, Katchalski, and Lewis [4] shows that there is a bounded Helly number for line transversals to families of disjoint unit balls in $\mathbb{R}^3$. But for translates of a general compact convex set in $\mathbb{R}^3$ the Helly number is no longer bounded, as the following theorem shows.

Theorem 1 *For any integer $n > 2$ there exists a compact convex set $K \subset \mathbb{R}^3$ and a finite family $\mathcal{F}$ of disjoint translates of K in $\mathbb{R}^3$ such that any $n-1$ members of $\mathcal{F}$ admit a line transversal, and yet $\mathcal{F}$ admits no line transversal.*

2 Proof of Theorem 1

Similar to almost all previous examples with lines in $\mathbb{R}^3$, our construction uses the hyperbolic paraboloid $\Sigma \subset \mathbb{R}^3$ with equation $z = xy$ (we imagine the z-axis being vertical). For $a \in \{0, 1, \ldots, 6\}$ let $h_a \subset \mathbb{R}^3$ be the plane $\{x = a\}$. Let l_a be the line $h_a \cap \Sigma$ and let s_a be the segment on l_a with $y \in [0, 1]$. Note that each line transversal of $\{s_0, s_1, \ldots, s_6\}$ lies on Σ and is one of the lines $\{y = y_0, z = xy_0, y_0 \in [0, 1]\}$.

We choose a very small $\varepsilon > 0$ and we let π_α be the vertical plane $\{x = \alpha + \varepsilon y\}$, $0 < \alpha < 1$. The intersection $\Sigma \cap \pi_\alpha$ is the graph of a convex function. Further let $t_j \subset \Sigma$ be the line $\{y = j/n, z = xy\}$, $j = 1, 2, \ldots, n$; see Fig. 1. For each i we define $(n-1)$-point set $P_i = \{p_{ij} : j \in \{1, 2, \ldots, n\} \setminus \{i\}\}$, where $\{p_{ij}\} = t_j \cap \pi_{i/2n}$.

We now introduce conditions on the convex body K and the family $\mathcal{F}$ of translates of K. First of all we want $\mathcal{F}$ to contain translates K_a for $a \in \{0, 1, \ldots, 6\}$ such that $K_a \cap \Sigma = s_a$ and K_a lies above Σ for a even and below Σ for a odd.

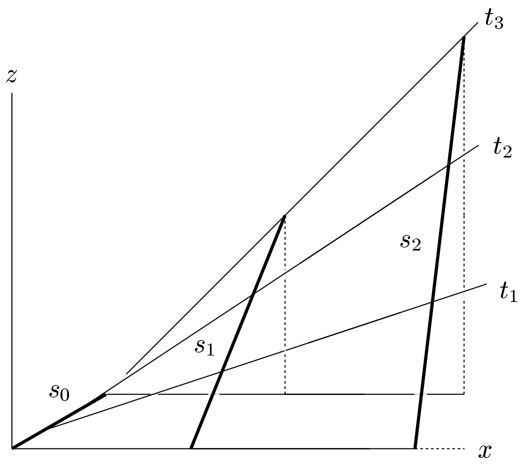

Figure 1: Some of the objects in the construction (for $n = 3$).

We claim that every line transversal of $\{K_0, K_1, \ldots, K_6\}$ is also transversal of $\{s_0, s_1, \ldots, s_6\}$. To see this, note that any line transversal t of $\{K_a : a = 0, 2, 4, 6\}$ has to go above Σ (non-strictly) at every section with the plane h_a, $a = 0, 2, 4, 6$, and since t intersects Σ at most twice, it goes completely above Σ in at least 2 of the 3 intervals $[0, 2]$, $[2, 4]$, and $[4, 6]$. Similarly, if t is a transversal of $\{K_a : a = 1, 3, 5\}$, it has to go completely below Σ in at least 1 of the intervals $[1, 3]$ and $[3, 5]$. Therefore, every line transversal of $\{K_0, K_1, \ldots, K_6\}$ shares as segment with Σ, and so it lies completely on Σ.

Next, we want $\mathcal{F}$ to contain translates L_i, for $i = 1, 2, \ldots, n$, such that L_i lies above Σ and $L_i \cap \Sigma = P_i$.

Let us assume, for the moment, that there exists a convex body K that satisfies the conditions above such that $\mathcal{F}$ is a family of disjoint translates of K. Then each $n - 1$ members of $\mathcal{F}$ admit a line transversal, but not the entire family. To see this, note that t_i is the unique transversal to $\mathcal{F} \setminus \{L_i\}$, and that t_i misses L_i.

The construction. Now we give an explicit construction of K that satisfies the conditions above. First we construct a set $X \subset \mathbb{R}^3$ consisting of segments $\sigma_0, \sigma_1, \ldots, \sigma_6$ and points q_{ij}, $i = 1, 2, \ldots, n$, $j \in \{1, 2, \ldots, n\} \setminus \{i\}$, all lying in the narrow vertical strip $\{0 \le x \le \delta\}$, with $\delta > 0$ small. Then we define $K = \mathrm{conv}(X)$.

First we let X_0 be the segment $\sigma_0 = s_0$, lying within the vertical plane $h_0 = \{x = 0\}$. Having constructed X_{i-1}, $i = 1, 2, \ldots, n$, we obtain X_i by appending a suitably translated copy $Q_i = \{q_{ij} : j \in \{1, 2, \ldots, n\} \setminus \{i\}\}$ of the set P_i. This means that we let m_{i-1} be the point of X_{i-1} with the largest y-coordinate (e.g., $m_0 = (0, 1, 0)$), we define v_i as the translation vector that moves m_{i-1} to the point of P_i with the smallest y-coordinate (i.e., $v_1 = p_{12} - m_0$ for $i = 1$ and $v_i = p_{i1} - m_i$ for $i = 2, 3, \ldots, n$), and we set $Q_i = P_i - v_i$ and $X_i = X_{i-1} \cup Q_i$. We note that $Q_1 \cup \ldots \cup Q_n$ all lie in the vertical plane $\{y = \varepsilon x + 1\}$.

Having constructed X_n, we similarly append to it translated copies $\sigma_2, \sigma_4, \sigma_6$ of the segments s_2, s_4, s_6, respectively, in this order. These lie in the vertical plane $h_\delta = \{x = \delta\}$, where δ is the maximum x-coordinate in X_n; that is, the x-coordinate of the last point $q_{n(n-1)} \in Q_n$. See Fig. 2(a).

Let X_L denote the set constructed so far. For later use, we note that X_L is monotone, in the following sense: If $p, q \in X_L$ are points, with $y(p) < y(q)$ ($y(p)$ is the y-coordinate of p), then $x(p) \le x(q)$, with equality only if p, q lie on a common segment σ_a.

Next, we let $X_U = \sigma_1 \cup \sigma_3 \cup \sigma_5$, where σ_a is a translate of s_a, $a = 1, 3, 5$, such that X_U lies within the vertical strip $\{x = 0, 0 \le y \le 3\}$, very high above X_L, and forms the graph of a concave function. Explicitly, $\sigma_5 = \{x = 0, z = 5y + M, 0 \le y \le 1\}$, $\sigma_3 = \{x = 0, z = 3y + M + 2, 1 \le y \le 2\}$, $\sigma_1 = \{x = 0, z = y + M + 6, 2 \le y \le 3\}$, where M is a very large number. Finally, we set $X = X_L \cup X_U$.

Let $\mathcal{F}_X$ be the family of translates of X consisting of:

- $X + v_i$, for $i \in \{1, 2, \ldots, n\}$, and
- $X + w_a$, for $a \in \{1, 2, \ldots, 6\}$, where w_a is the translation vector with $\sigma_a + w_a = s_a$.

The family $\mathcal{F}$ is obtained by replacing each member of $\mathcal{F}_X$ by its convex hull, i.e. a translate of $K = \mathrm{conv}(X)$.

Checking the properties of $\mathcal{F}$. We now show that the family $\mathcal{F}$ has the desired properties when $\varepsilon > 0$ is chosen sufficiently small. For this we first consider the family $\mathcal{F}_X$ rather than $\mathcal{F}$. The set X is constructed so that the members of $\mathcal{F}_X$ intersect Σ exactly the way we want; that is, $(X + v_i) \cap \Sigma = P_i$ and $(X + w_a) \cap \Sigma = s_a$.

To see this, note that the slope (in the h_0 plane) of the segments $q_{ij}q_{i(j+1)}$ is very close to $\frac{i}{2n}$. If follows that each $X + v_i$ lies above Σ and intersects

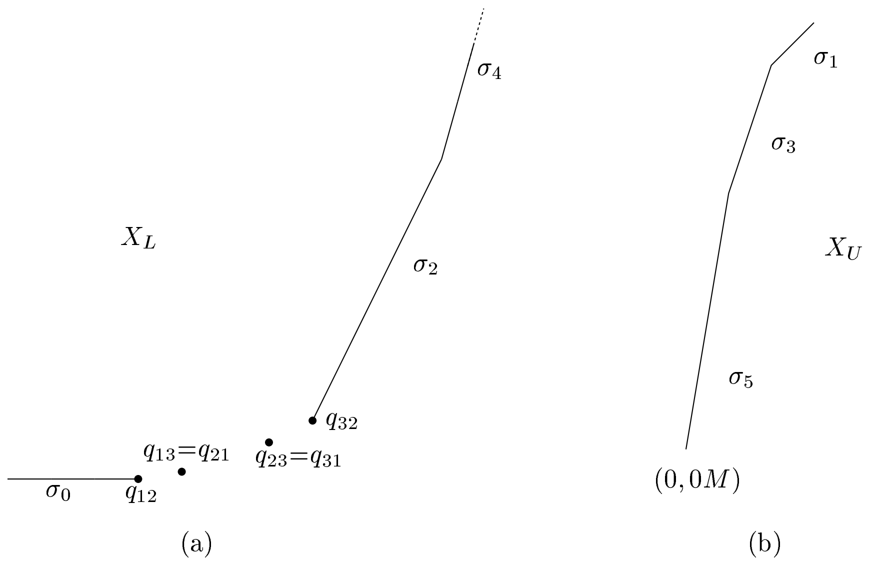

Figure 2: (a) The set X_L for $n = 3$ (projection to the yz plane); (b) the set X_U (shown in a smaller scale than X_L).

it only in P_i. Similarly, we check that for $a = 0, 2, 4, 6$, $X + w_a$ lies above Σ and intersects it exactly in s_a, and for $a = 1, 3, 5$, $X + w_a$ lies below Σ and intersects it exactly in s_a.

Clearly, the translates in $\mathcal{F}$ are disjoint, since they lie in disjoint parallel vertical strips (parallel to the yz plane and of width δ). It remains that the intersections of the members of $\mathcal{F}_X$ with Σ are not enlarged by taking convex hulls. We begin with a simple observation:

Observation 2 *Let p, q be points lying above Σ (not necessarily strictly), such that $x(p) \le x(q)$ and $y(p) \le y(q)$. Then all points of the segment pq lie above Σ (non-strictly). Moreover, an interior point of the segment pq can lie on Σ only if $x(p) = x(q)$ and all of the segment pq lies on Σ.*

To see this, consider the vertical plane ρ passing through p and q (if $x(p) = x(q)$ and $y(p) = y(q)$ then the claim clearly holds, so we may assume that ρ is unique). The intersection $\Sigma \cap \rho$ is the graph of a convex function, so the segment pq lies (non-strictly) above it. Moreover, that function is strictly convex unless $x(p) = x(q)$ or $y(p) = y(q)$, and in these cases it is a line. $\square$

Now let us consider a translate $X + v_i \in \mathcal{F}_X$. First we check that $\mathrm{conv}(X_L + v_i) \cap \Sigma = P_i$; this is immediate from Observation 2 (if we connect two points in $X_L + v_i$ by a segment, all interior points of that segment lie strictly above Σ, and similarly if we connect these "first-generation" points by segments).

The set $\mathrm{conv}(X_U + v_i)$ lies high above Σ. Any segment connecting a point of $\mathrm{conv}(X_L + v_i)$ to $\mathrm{conv}(X_U + v_i)$ is almost vertical, and so such segments do not create any new intersections with Σ. The union of all such segments already equals $\mathrm{conv}(X + v_i)$ (if $A, B \subseteq \mathbb{R}^3$ are convex sets, then $\bigcup\{ab : a \in A, b \in B\}$ is convex).

The argument for $\mathrm{conv}(X+w_a) \cap \Sigma = s_a$, $a = 0, 2, 4, 6$, is almost identical and we omit it. Finally, $\mathrm{conv}(X+w_a) \cap \Sigma = s_a$ for $a = 1, 3, 5$ is easy as well: The situation with $\mathrm{conv}(X_U + w_a)$ is clear, as it lives within the vertical plane h_a, which is intersected by Σ in a line, and the contribution of $X_L + w_a$ is discussed as above. This completes the proof of Theorem 1. $\square$

3 Remarks

We conclude with some remarks concerning the construction in Section 2.

1. Although the set K is very flat, we note that it must indeed be 3-dimensional. If not, and $\mathcal{F}$ is a finite family of translates of K such that every 3 members have a line transversal, then either $\mathcal{F}$ lies in a common plane, or all members lie in distinct parallel planes. In both cases the Helly number is bounded: in the first case by Tverberg's transversal theorem, and in the second case by a theorem of Grünbaum (see Theorem 5.6 of [2]).

2. It has already been noted that the members of $\mathcal{F}$ lie in disjoint parallel strips. Using this observation, labeling the strips going from left to right, say, gives an ordering of the members of $\mathcal{F}$. And by the disjointness of the strips it follows that any $n-1$ members of $\mathcal{F}$ are met by a line transversal consistently with the ordering. Thus the construction in Section 2 actually proves the following statement (which is stronger than Theorem 1).

Theorem 3 *There is no Hadwiger-type theorem for line transversals to families of disjoint translates in* $\mathbb{R}^3$.

3. Our example can also be easily modified to refute a theorem of the following type: "There exist c and k such that if every k members of a family of disjoint translates of a given convex body in $\mathbb{R}^3$ have a line transversal, then all but at most c members of that family have a line transversal."

   The modification goes as follows. First, instead of the 7 segments $s_0, s_1, \ldots, s_6$, we use sufficiently many segments s_a so that if any at most c of them are removed, a 7-tuple still remains for which the argument in the proof (that any transversal of the corresponding copies K_a of K lies on Σ) still works. And second, we choose the parameter n sufficiently large depending on k and c, and instead of the simple set system $\{\{1, 2, \ldots, n\} \setminus \{i\} : i = 1, 2, \ldots, n\}$ used for defining the sets P_i, we use another set system in which every k sets intersect but no $n - c$ do.

4. On the other hand, we do not have an example disproving a *fractional Helly theorem* of the following type:

   > There exists a natural number k such that for every $\alpha > 0$ there is a $\beta > 0$ such that whenever K is a convex body in $\mathbb{R}^3$ and $\mathcal{F} = \{K_1, K_2, \ldots, K_n\}$ is a family of disjoint translates of K such that at least $\alpha\binom{n}{k}$ among the $\binom{n}{k}$ subfamilies of $\mathcal{F}$ of size k admit a line transversal, then some at least βn sets of $\mathcal{F}$ have a line transversal.

   This, if true, would be an analogue of the result of Katchalski and Liu [6]; see, e.g., [7] for recent examples of fractional Helly theorems in situations where Helly-type theorems fail.

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