A lower bound on the size of Lipschitz subsets in dimension 3

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Abstract

A set $S \subset \mathbf{R}^d$ is *C-Lipschitz in the x_i-coordinate*, where $C > 0$ is a real number, if for every two points $a, b \in S$, we have $|a_i - b_i| \leq C \max\{|a_j - b_j| : j = 1, 2, \ldots, d, j \neq i\}$. Motivated by a problem of Laczkovich, the author asked whether every n-point set in $\mathbf{R}^d$ contains a subset of size at least $cn^{1-1/d}$ that is C-Lipschitz in one of the coordinates, for suitable constants C and $c > 0$ (depending on d). This was answered negatively by Alberti, Csörnyei, and Preiss. Here it is observed that a combinatorial result of Ruzsa and Szemerédi implies the existence of a 2-Lipschitz subset of size $n^{1/2}\varphi(n)$ in every n-point set in $\mathbf{R}^3$, where $\varphi(n) \to \infty$ as $n \to \infty$.

1 Introduction

Let us call a set $S \subset \mathbf{R}^d$ *C-Lipschitz in the x_i-coordinate*, where $C > 0$ is a real number, if for every two points $a, b \in S$, we have $|a_i - b_i| \leq C \max\{|a_j - b_j| : j \in [d], j \neq i\}$ (where $[d] = \{1, 2, \ldots, d\}$). Equivalently, S is C-Lipschitz in the x_d-coordinate, say, if there is a C-Lipschitz function $\mathbf{R}^{d-1} \to \mathbf{R}$ whose graph contains S. (Here the distance in $\mathbf{R}^{d-1}$ is measured in the ℓ_∞ norm, but very similar considerations apply with respect to the Euclidean norm, up to the values of the constants.)

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Let $\mathrm{LS}_{d,C}(n)$ denote the maximum integer k such that for every n-point set $P \subset \mathbf{R}^d$ there exists $i \in [d]$ and a k-point subset $S \subseteq P$ that is C-Lipschitz in the x_i-coordinate. It is not difficult to show that $\mathrm{LS}_{2,1}(n) \geq n^{1/2}$ (a proof is sketched below), and this implies $\mathrm{LS}_{d,1}(n) \geq n^{1/2}$ (project on the first two coordinates).

The following question was raised in [5]: given $d \geq 3$, are there constants $C > 0$ and $c > 0$ such that $\mathrm{LS}_{d,C}(n) \geq cn^{1-1/d}$ for all n? The grid $[m]^d$ shows that $\mathrm{LS}_{d,C}(m^d) \leq m^{d-1}$ (for all C), and it seems natural to suspect that the grid could be an extremal example. Moreover, for any n-point set in $\mathbf{R}^d$, at least one of the "shadows", i.e., projections to coordinate hyperplanes, has at least $n^{1-1/d}$ points [8], and so there is a C-Lipschitz subset in some coordinate with that many points for *some* C (depending on the set).

The problem of C-Lipschitz subsets was motivated by a beautiful question of Laczkovich [4]: is it true that for any set $E \subseteq \mathbf{R}^d$ of positive Lebesgue measure there exists a Lipschitz mapping $f\colon \mathbf{R}^d \to Q = [0,1]^d$ with $f(E) = Q$? A positive answer for $d = 2$ was obtained by Preiss [6]. An alternative proof, based on the fact that $\mathrm{LS}_{2,1}(n) \geq n^{1/2}$, was given in [5]. It was also noted that $\mathrm{LS}_{d,1}(n) \geq \mathit{const} \cdot n^{1-1/d}$ would imply a positive answer to Laczkovich's question in the same way. However, the suggested bound $\mathrm{LS}_{d,1}(n) \geq \mathit{const} \cdot n^{1-1/d}$ turned out to be false in a very strong sense.

First of all, Tardos observed that $\mathrm{LS}_{3,1}(n) = O(n^{1/2})$, as is witnessed by the set $\{(i,j,i+j);\ i,j \in [m]\}$. This example still leaves the possibility that $\mathrm{LS}_{3,C}(n)$ could be of the order $n^{2/3}$ for $C = 2$, say. However, this was recently disproved by Alberti, Csörnyei, and Preiss [1]. Their example shows that for any C, we have $\mathrm{LS}_{3,C}(n) \leq n^{2/3-\delta}$ for some $\delta = \delta(C) > 0$. (The problem was also considered by Szabó and Tardos [9], who suggested some combinatorial generalizations and provided negative examples for these, which do not seem to transfer to the original problem, however).

In this note, we show that $\mathrm{LS}_{3,2}(n)$ exceeds the trivial lower bound of $n^{1/2}$ by a factor tending to infinity:

Theorem 1 *We have* $\mathrm{LS}_{3,2}(n) \geq n^{1/2}\varphi(n)$*, where* $\varphi(n)$ *is a function tending to* ∞ *as* $n \to \infty$*.*

The lower bound for $\varphi(n)$ obtained from our proof has order of magnitude approximately $\log^* n$ (where the iterated logarithm function $\log^* x$ is defined by $\log^* x = 0$ for $x \leq 1$ and $\log^* x = 1 + \log^*(\log_2 x)$ for $x > 1$).

This particular approach cannot yield anything better than about $e^{\sqrt{\log n}}$ for $\varphi(n)$. This follows from a lower bound for a lemma of Ruzsa and Szemerédi used in the proof below (the lower bound is based on a well known construction, due to Behrend [2], of a dense set of integers without 3-term arithmetic progressions).

2 Proof

Let $C \geq 1$ be fixed. For points $a, b \in \mathbf{R}^d$, we define $a \prec_i b$ if $b_i - a_i > C \cdot \max\{|a_j - b_j| : j \in [d], j \neq i\}$. That is, if we think of the x_i-axis as being vertical, b lies "steep above" a. Further we write $a \preceq_i b$ if $a = b$ or $a \prec_i b$. Clearly, each $\preceq_i$ is a partial ordering on $\mathbf{R}^d$. We note that $S \subseteq \mathbf{R}^d$ is C-Lipschitz in the x_i-coordinate if and only if it is an antichain in the ordering $\preceq_i$ (no two elements of S are comparable under $\preceq_i$).

Since $C \geq 1$, we have $\prec_i \cap \prec_j = \emptyset$ whenever $i \neq j$. In particular, for $d = 2$, every chain in $\preceq_1$ is an antichain in $\preceq_2$. By a simple consequence of a well-known theorem of Dilworth, every n-point partially ordered set contains an antichain of at least $n^{1/2}$ elements or a chain of at least $n^{1/2}$ elements. Applying this to $\preceq_1$, we have the proof of $\mathrm{LS}_{2,1}(n) \geq n^{1/2}$ promised above.

Now we prove Theorem 1. Let $d = 3$ and consider the partial orderings $\preceq_1, \preceq_2, \preceq_3$ as above with $C = 2$. All the required combinatorial information about them, besides the fact $\prec_i \cap \prec_j = \emptyset$ mentioned above, is contained in the following:

Lemma 2 *Let $S_i = \{\{a, b\} : a \preceq_i b\}$ be the symmetrization of the relation $\preceq_i$. Then there are no three points $a, b, c \in \mathbf{R}^3$ with $\{a, b\} \in S_1$, $\{b, c\} \in S_2$, and $\{a, c\} \in S_3$.*

Proof. Informally, if we go from a almost parallel to the x_1-axis and then continue almost parallel to the x_2-axis, we cannot get back to a in a direction almost parallel to the x_3-axis. Less informally, suppose that a, b, c contradict the claim in the lemma. By symmetry, we may assume $|a_1 - b_1| \geq |b_2 - c_2|$. Then $|a_3 - b_3| < \frac{1}{2}|a_1 - b_1|$ and $|b_3 - c_3| < \frac{1}{2}|b_2 - c_2| \leq \frac{1}{2}|a_1 - b_1|$, which gives $|a_3 - c_3| < |a_1 - b_1|$. But $|a_1 - c_1| \geq |a_1 - b_1| - |b_1 - c_1| > |a_1 - b_1| - \frac{1}{2}|b_2 - c_2| \geq |a_1 - b_1| - \frac{1}{2}|a_1 - b_1| = \frac{1}{2}|a_1 - b_1|$. $\square$

To prove Theorem 1, let $P \subset \mathbf{R}^3$ be an n-point set. Suppose that none of the partially ordered sets $(P, \preceq_i)$ has an antichain of size larger than s. Then, using Dilworth's theorem [3] in its full strength, we have decompositions $P = P_{i1} \dot{\cup} P_{i2} \dot{\cup} \cdots \dot{\cup} P_{is}$, $i = 1, 2, 3$, where each P_{ik} is a chain in $(P, \preceq_i)$.

We define a 3-uniform hypergraph (system of triples) H. The vertex set $V(H) = [3] \times [s]$, and the edge set $E(H) = \{\{(1, k_1), (2, k_2), (3, k_3)\} : P_{1k_1} \cap P_{2k_2} \cap P_{3k_3} \neq \emptyset\}$. Every point $p \in P$ determines exactly one triple $e_p \in E(H)$ (since p lies in exactly one P_{1k_1}, exactly one P_{2k_2}, and exactly one P_{3k_3}), so there are n edges and $3s$ vertices. Moreover, we note that H has the following two properties:

(C1) Every two edges intersect in at most one vertex (in other words, an edge is uniquely determined by any two of its three vertices). This is because $|P_{ik} \cap P_{j\ell}| \leq 1$ whenever $i \neq j$.

(C2) There is no triangle, i.e., no three distinct pairwise intersecting edges. Some thought reveals that this is a re-formulation of Lemma 2.

Ruzsa and Szemerédi [7] proved, using the well-known Szemerédi's regularity lemma, that any system of triples on m vertices satisfying (C1) and (C2) has at most $o(m^2)$ triples. Therefore, in our setting, $n = o(s^2)$ and Theorem 1 follows. $\square$

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