On Homogeneous Graphs and Posets

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Abstract

We present a class $\mathcal{P}_\in$ of simple finite structures which induce the countable homogeneous universal poset. We also define the notion of a finitely presented countable structure and conjecture that every generic structure for a finitely axiomatizable class of structures is finitely presented. This is verified for undirected graphs, tournaments and posets. The structure $\mathcal{P}_\in$ extends Conway's surreal numbers and their linear ordering to posets.

1 Introduction

A countable partially ordered set P (for brevity from now on poset) is said to be *universal* if it contains any countable poset (as an induced subposet).

A poset P is said to be *homogeneous* if every partial isomorphism between finite subposets extends to an isomorphism (of the whole poset).

It is a classical model theory result that a homogeneous universal poset exists and that it is up to an isomorphism uniquely determined. This poset is naturally called *generic poset* and it will be denoted by $\mathcal{P}$. This paper is devoted to the study of $\mathcal{P}$. $\mathcal{P}$ can be constructed in a standard model

*Supported by a Grant LN00A56 of the Czech Ministry of Education

theoretic way as Fraïssé limit of all finite posets: we start with the singleton poset and at n-th step we add new vertices which extend the given poset in all possible (consistent) ways to a poset with $(n+1)$ vertices.

This procedure applies (as proved by Fraïssé) not only to posets but to structures in general and thus, in particular, the homogeneous universal (undirected) graph exists. This graph is called *Rado graph* $\mathcal{R}$. We state two of its striking properties which motivate the present paper (see the excellent survey by P. Cameron [4]):

1. $\mathcal{R}$ is isomorphic to the following graph $\mathcal{R}_\in$: vertices of $\mathcal{R}_\in$ are all finite sets (in some countable model of set theory) with edges of the form $\{A, B\}$ where either $A \in B$ or $B \in A$.
2. $\mathcal{R}$ is isomorphic to the following graph $\mathcal{R}_\mathbb{N}$: vertices of $\mathcal{R}_\mathbb{N}$ are all natural numbers with edges of the form $\{m, n\}$ where the m-th digit in the binary expansion of n is 1.

There are other explicit construction by means of quadratic residues and universal sequences, see [4]. It is remarkable that all these seemingly unrelated constructions define the same graph $\mathcal{R}$. The simplicity of these constructions motivated our notion of finitely presented structures:

Definition 1.1 *A countable structure $\mathcal{S}$ is* finitely presented *if there exists a finitely axiomatizable theory $\mathcal{T}$ and finitely axiomatizable theories $\mathcal{T}_R$ where R is any relation occurring in $\mathcal{S}$ such that all finite models of $\mathcal{T}$ in the countable set theory together with relations among them induced by theories $\mathcal{T}_R$ induce a structure isomorphic to $\mathcal{S}$.*

Let us state two particular cases of this definition explicitly:

A graph G is finitely presented if there exist finitely axiomatizable theories $\mathcal{T}_V$ and $\mathcal{T}_E$ such that the following graph $\mathcal{G}$ is isomorphic to G: vertices of $\mathcal{G}$ are all models of M of $\mathcal{T}_V$ (in a countable set theory); edges of $\mathcal{G}$ are all pairs of models $\{M, M'\}$ such that $T_E \models \{M, M'\}$.

Both constructions $\mathcal{R}_\in$ and $\mathcal{R}_\mathbb{N}$ above are obviously finite presentations of $\mathcal{R}$.

A poset P is finitely presented if there exists a finitely axiomatizable theories $\mathcal{T}_P, \mathcal{T}_\leq$ such that the following poset $\mathcal{P}$ is isomorphic to $\mathcal{P}$: vertices of $\mathcal{P}$ are all models M of $\mathcal{T}_P$ (in the finite set theory) with $M \leq M'$ iff $\mathcal{T}_\leq \models (M, M')$.

For posets the situation is more complicated than for graphs. In fact an explicit construction of the generic poset $\mathcal{P}$ by means of all finite models of a finitely aximatizable structure presented an open problem. The main result of this paper is a construction of a finitely presented poset $\mathcal{P}_{\in}$ which is isomorphic to $\mathcal{P}$. This is proved in Section 3.

We found the construction of $\mathcal{P}_{\in}$ in a broader context of study of finite presentation of homogeneous structures, homogeneous undirected and directed graphs, tournaments and posets. For these structures the classification programme has been completed in a series of difficult papers (see e.g. [10, 3, 20]) all based on the Fraïssé equivalent definition of homogeneous structures as amalgamation classes of finite structures. Particularly the homogeneous undirected graphs were characterized in [10]. We shall prove that all these graphs are finitely presented. For some graphs on Lachlan-Woodrow list (all finite examples, equivalences and Turán graphs) this is an easy exercise. For Rado graph this has been stated above. We prove in Section 2 that also generic graphs for the class $\mathrm{Forb}(K_k)$ of all finite graphs which do not contain a complete graph K_k, are finitely presented for every $k \geq 3$. Thus all homogeneous graphs are finitely presented (Corollary 2.1).

For tournaments and oriented graphs (two other structures with solved classification problem) the situation is different in that we have to construct a finite presentation even for the generic oriented graph and for the generic tournament. This is done in Section 2, Theorem 2.5 and in Section 5, Corollary 5.1. These constructions lie in the background of the construction of $\mathcal{P}_{\in}$ (in Section 3).

We further refine this construction to any oriented homogeneous graph of type $\mathrm{Forb}(T_1, \ldots, T_n)$ where T_i are (forbidden) tournaments. It follows that also all homogeneous tournaments are finitely axiomatizable (Corollary 5.1). As there are continuously many homogeneous oriented graphs we cannot expect finite presentation of all homogeneous oriented graphs. (See [22] for results for higher cardinality λ-categorical structures.) This shows that there are natural limits to the programme of representing homogeneous structures by means of simple structures.

The classification of homogeneous posets is easier than for undirected graphs. The classification was given by Schmerl [20] and it appears that apart from anti-chains, the set $\mathbb{Q}$ of all rationals, disjoint unions of $\mathbb{Q}$ and "blowing up" $\mathbb{Q}$, the only other homogeneous poset is the generic one.

Several examples of finitely presented linear orders and posets are easy to find:

- The set of all natural numbers $(\mathbb{N}, \leq)$ (proved by von Neumann);
- the set $\mathbb{Q}$ (by standard construction from $\mathbb{N}$);

- $P \times P'$ for finitely presented P and P';

- lexicographic product of P and P' for finitely presented P and P'.

It follows that most homogeneous posets are finitely presented. The only remaining case is the generic poset $\mathcal{P}$ which is shown to be finitely presented in Section 4 (by means of the structure $\mathcal{P}_{\in}$).

It seems that homogeneous structures are likely to be finitely presented. Intuitively a high degree of symmetry (such as homogeneity) perhaps leads to a "low entropy" and thus in turn perhaps to a concise representation. (Remark that "concise representations" of finite structures were studied from complexity point of view for graphs ([12, 21]) and posets ([6, 16]). These considerations and the above results perhaps suggest the following:

Conjecture 1.1 *Let $\mathcal{T}$ be a finitely axiomatizable theory. Let $\mathcal{M}$ be the class of all finite models of $\mathcal{T}$. Then the generic homogeneous universal structure corresponding to $\mathcal{M}$, if it exists, is finitely presented.*

On the other hand our main result (the construction of the structure $\mathcal{P}_{\in}$) may be viewed as an extension of surreal numbers of Conway [9] to posets. In Section 4 we exhibit this connection.

Our research was also motivated by trying to solve several problems related to universal posets represented by finite graphs with special properties. It has been proved in a different (category theory) context (see [17, 19]) that the class of all finite graphs ordered by the existence of homomorphism is the universal poset. (However note that neither this poset nor posets constructed in [17, 19] are homogeneous.) An alternative combinatorial proof of this result [13] was a starting point of this research.

The techniques of [17, 19] also do not extend to some of the basic subclasses of graphs such as planar graphs or graphs with bounded degrees. In fact these classes do not represent arbitrary subgroups [1] and monoids [2]. Yet by means of the techniques of this paper we can construct universal posets for both planar and bounded-degree graphs. This also solves a problem stated in [15]. This together with a deeper analysis of finitely presented homogeneous oriented graphs is going to appear in the sequel of this paper. Here we concentrate mostly on posets.

2 Homogeneous Directed Graphs

In this section we find a finite presentation of the directed homogeneous universal (generic) graph $\overrightarrow{\mathcal{R}}$ and also of some other homogeneous graphs. We shall study the homogeneous graphs by means of extension properties. This we briefly recall for completeness (see [18]).

Definition 2.1 *Let $\mathcal{C}$ be an isomorphism closed class of graphs, G a graph. We say that G has extension property for $\mathcal{C}$ if the following holds: For any pair of finite subgraphs $G', G'' \in \mathcal{C}$ and any embeddings $\varphi' : G' \hookrightarrow G$, $\varphi : G' \hookrightarrow G''$ there exists an embedding $\varphi'' : G'' \hookrightarrow G$ such that $\varphi'' \circ \varphi = \varphi'$.*

(An embedding is a isomorphism onto an induced subgraph.)

The extension property implies both universality and homogeneity of G (see [18]):

Lemma 2.1 *For any hereditary isomorphism closed class of finite graphs $\mathcal{C}$, every graph G having extension property for $\mathcal{C}$ is universal and homogeneous.*

This statement is a useful tool in proving that a finitely presented structure is generic. As a warm up we prove this for graph $\mathcal{R}_\in$:

Theorem 2.2 *$\mathcal{R}_\in$ has extension property for class of finite undirected graphs. Thus $\mathcal{R}_\in$ is isomorphic to the generic undirected graph $\mathcal{R}$.*

Proof. Let M_0 and M_1 be two disjoint finite sets of vertices of $R_\in$. (The subgraph of $R_\in$ induced by $M_1 \cup M_0$ plays the rôle of G' in the extension property.) We are looking for the vertex X of $\mathcal{R}_\in$ such that $Y \in X$ for every $Y \in M_1$ and $Y \notin X$ for every $Y \in M_0$. It suffices to put $X = M_1 \cup \{x\}$ where $M_1 \notin M_0$ satisfies $M_1 \cup \{x\} \notin M_0$ and $M_0 \notin M_1 \cup \{x\}$.

Thus $\mathcal{R}_\in$ has extension property and thus it is generic for the class of all finite undirected graphs. $\square$

Now we will proceed by analogy to the construction of the homogeneous universal directed graph $\overrightarrow{\mathcal{R}}$. In order to easily embed our construction into set theory, we first introduce alternative definition of ordered pair. In the rest of the paper we will use a fixed countable model of set theory $\mathfrak{M}$ containing single atomic element $\circlearrowleft$.

Definition 2.2 *For every set M we put*

$$M_L = \{A; A \in M, \acute{\bigcirc} \notin A\};$$

$$M_R = \{A; (A \cup \{\acute{\bigcirc}\}) \in M, A \neq \emptyset, \acute{\bigcirc} \notin A\}.$$

For any set A and B we will denote by $(\, A \mid B \,)$ the set

$$A \cup \{M \cup \{\acute{\bigcirc}\}; M \in B\}$$

For any M not containing $\acute{\bigcirc}$ holds $(\, M_L \mid M_R \,) = M$. Thus for the model $\mathfrak{M}$, the class of sets not containing $\acute{\bigcirc}$ represents the universum of the recursively nested ordered pairs.

Analogously to the relation $\mathcal{R}_\in$ and $\mathcal{R}_\mathbb{N}$ we can represent the ordered pairs by integers:

Definition 2.3 *Let M be any set not containing $\acute{\bigcirc}$. By* code of M, $c(m)$ *we denote the integer*

$$c(M) = \sum_{A \in M_L} 2^{2c(A)} + \sum_{A \in M_R} 2^{2c(B)+1}.$$

Notice that c is a bijection between the sets not containing $\acute{\bigcirc}$ and the integers. The predicate $X \in M_L$ is equivalent to test whether $2c(X)$-th digit of binary representation of $c(m)$ is 1 and similarly for $X \in M_P$. Thus all our constructions involved in construction of $\overrightarrow{\mathcal{R}}_\in$ based on these predicates can be expressed arithmetically.

Definition 2.4 *Define graph $\overrightarrow{\mathcal{R}}_\in$ as follows: The vertices are all sets not containing $\acute{\bigcirc}$. (M, N) is an arc of $\overrightarrow{\mathcal{R}}_\in$ iff either $M \in M_L$ or $N \in M_R$. (Recall, that we consider a fixed countable model $\mathfrak{M}$ of set theory containing an additional atomic element $\acute{\bigcirc}$.)*

Theorem 2.3 *$\overrightarrow{\mathcal{R}}_\in$ has extension property and thus it is the homogeneous universal directed graph for the class of all directed graphs. Thus $\overrightarrow{\mathcal{R}}_\in$ is isomorphic to $\overrightarrow{\mathcal{R}}$.*

Proof. We proceed analogously to the proof of Theorem 2.2: let M_-, M_+ and M_0 be three disjoint sets of vertices, where $M_0 \cap (M_- \cup M_+)$ is empty. We need to find vertex M with following properties:

1. For each $X \in M_-$ there is an edge from X to M.

2. For each $X \in M_+$ there is an edge from M to X.

3. For each $X \in (M_- \cup M_+ \cup M_0)$ there are no other edges from X to M or M to X then ones required by 2. and 3.

Fix any

$$x \notin \bigcup_{m \in M_- \cup M_+ \cup M_0} m.$$

Obviously vertex $M = (\, M_- \cup \{x\} \mid M_+ \,)$ has the specified property. $\square$

Thus generic graphs (both undirected and directed) are finitely presented. We can extend these representations to other homogeneous structures. We start with undirected graphs:

Definition 2.5 *By $\mathcal{R}_\mathcal{C}$ we denote homogeneous universal (i. e. generic) graph for class $\mathcal{C}$ of undirected graphs (if it exists). By $\overrightarrow{\mathcal{R}}_\mathcal{C}$ we denote homogeneous universal graph for class $\mathcal{C}$ of directed graphs. Such $\mathcal{R}_\mathcal{C}$ (and $\overrightarrow{\mathcal{R}}_\mathcal{C}$) is also called generic for $\mathcal{C}$.*

We denote by $\mathrm{Forb}(G)$ *the class of all finite graphs not containing G as an induced subgraph.*

The construction of graph $\mathcal{R}_{\mathrm{Forb}(K_k),\in}$, $k \geq 3$, is an extension of the construction of $\mathcal{R}_\in$ (recall that a finite set S is called *complete* for any $X, Y \in S$, $X \neq Y$ either $X \in Y$ or $Y \in X$):

Definition 2.6 *Define $\mathcal{R}_{\mathrm{Forb}(K_k),\in}$, $k \geq 3$ as follows: The vertices of $R_{\mathrm{Forb}(K_k),\in}$ are all sets which do not contain a complete subset with $k-1$ elements; two vertices of S and S' form an edge of $\mathcal{R}_{\mathrm{Forb}(K_k),\in}$ iff either $S \in S'$ or $S' \in S$.*

Thus $\mathcal{R}_{\mathrm{Forb}(K_k),\in}$ is the restriction of the graph $R_\in$ to the class of all sets without complete subset of size $k-1$.

Theorem 2.4 *$\mathcal{R}_{\mathrm{Forb}(K_k),\in}$ has extension property. Consequently $\mathcal{R}_{\mathrm{Forb}(K_k),\in}$ is the homogeneous universal undirected graph for the class* $\mathrm{Forb}(K_k)$.

Proof. $\mathcal{R}_{\mathrm{Forb}(K_k),\in}$ does not contain K_k: For contradiction, let us suppose that $V_1, V_2, \ldots, V_k$ are edges of complete graph. Without loss of generality we may assume that $V_i \in V_{i+1}$ for each $i = 1, 2, \ldots, k-1$. Since K_k is complete graph, $V_i \in V_n$ for each $i = 1, 2, \ldots, k-1$. It follows that

$\{V_1, \ldots, V_{i-1}\}$ represents the prohibited subset S. Thus V_n is not vertex of $\mathcal{R}_{\mathrm{Forb}(K_k)}$.

The extension property can be verified in analogical way as we did for $\mathcal{R}$ in Theorem 2.2. The constructed set satisfies the conditions required by definition $\mathcal{R}_{\mathrm{Forb}(K_k)}$. □

Corollary 2.1 *All homogeneous undirected graphs are finitely presented*

Proof. Clearly a graph G is finitely presented iff the complementary graph $\overline{G}$ is finitely presented. The statement follows from the discussion in section 1 and Theorem 2.2 and 2.4. □

Finally, we extend our construction of the generic directed graph $\overrightarrow{\mathcal{R}}_{\in}$ to the homogeneous graphs $\overrightarrow{\mathcal{R}}_{\mathrm{Forb}(T),\in}$ not containing a tournament T. This is slightly more technical (although it parallels the undirected case).

Put $T = (V, E)$ and for each $v \in V$ put

$$L(v) = \{v' \in V; (v', v) \in E\}$$

$$R(v) = \{v' \in V; (v, v') \in E\}$$

(observe that $L(v) \cup R(v) = V - \{v\}$.)

Our vertices will satisfy conditions $C_v(M)$ of the following type:

There are no sets $X_{v'}$, $v' \in L(v) \cup R(v)$ satisfying the following

i. $X_{v'} \in M_L$ for $v' \in L(v)$;

ii. $X_{v'} \in M_R$ for $v' \in R(v)$;

iii. For every arc $(v', v'') \in E$, $v', v'' \in L(v) \cup R(v)$ either $X_{v'} \in (X_{v''})_L$ or $X_{v''} \in (X_{v'})_R$.

In other words (condition *iii*) the sets $X_{v'}$, $v' \in L(v) \cup R(v)$ do not represent the tournament $T - \{v\}$ in $\overrightarrow{\mathcal{R}}_{\in}$.

Definition 2.7 *Denote by* $\overrightarrow{\mathcal{R}}_{\mathrm{Forb}(T),\in}$ *the directed graph* $\overrightarrow{\mathcal{R}}_{\in}$ *restricted to the class of all sets* M *which satisfy the condition* $C_v(M)$ *for every* $v \in V$.

Theorem 2.5 $\overrightarrow{\mathcal{R}}_{\mathrm{Forb}(T),\in}$ *is isomorphic to* $\overrightarrow{\mathcal{R}}_{\mathrm{Forb}(T)}$*, explicitly* $\overrightarrow{\mathcal{R}}_{\mathrm{Forb}(T),\in}$ *is homogeneous universal graph for class of all directed graphs not containing* T.

Proof. Proof is analogous to the proof of Theorem 2.4. □

This can be extended to classes Forb($\mathcal{T}$) for any finite set of tournaments (but clearly not to all classes Forb($\mathcal{T}$)). This together with a deeper analysis of finitely presented homogeneous directed graphs is going to appear elsewhere. In Section 5 we shall prove that all homogeneous tournaments are finitely presented. Here we concentrate mostly on posets.

3 Universal Homogeneous Structure $\mathcal{P}_\in$

In this section we further modify the finite presentation of $\overrightarrow{\mathcal{R}}_\in$ to a finite presentation of the generic poset $\mathcal{P}$. This structure will be denoted $\mathcal{P}_\in$. We use the same notation as in Section 2. Particularly we work in a fixed countable model $\mathfrak{M}$ of theory of finite set extended by a single atomic set $\acute{O}$. Also recall the following notations:

$$M_L = \{A; A \in M, \acute{O} \notin A\};$$

$$M_R = \{A; (A \cup \{\acute{O}\}) \in M, A \neq \emptyset, \acute{O} \notin A\}.$$

The following is our basic construction:

Definition 3.1 *Denote by $\mathcal{P}_\in$ the following relation:*
The elements of $\mathcal{P}_\in$ are all sets M with the following properties:

1. *(correctness)*

   i. $\acute{O} \notin M$;

   ii. $M_L \cup M_R \subset \mathcal{P}_\in$;

   iii. $M_L \cap M_R = \emptyset$.

2. *(ordering property)* $(\{A\} \cup A_R) \cap (\{B\} \cup B_L) \neq \emptyset$ *for each* $A \in M_L, B \in M_R$;

3. *(left completeness)* $A_L \subseteq M_L$ *for each* $A \in M_L$;

4. *(right completeness)* $B_R \subseteq M_R$ *for each* $B \in M_R$;

The relation of $\mathcal{P}_\in$ is denoted by $\leq$ and it is defined as follows: We put $M < N$ if:

$$(\{M\} \cup M_R) \cap (\{N\} \cup N_L) \neq \emptyset$$

We write $M \leq N$ if either $M < N$ or $M = N$.

The class $\mathcal{P}_\in$ is nonempty. It is of course $M = \emptyset = (\,\emptyset \mid \emptyset\,) \in \mathcal{P}_\in$. (Obviously correctness property holds. Because $M_L = \emptyset$, $M_R = \emptyset$, ordering property and completeness properties follow trivially.)

Here are a few examples of non-empty elements of the structure $\mathcal{P}_\in$ are:

$$(\,\emptyset \mid \emptyset\,)$$
$$(\,\emptyset \mid \{(\,\emptyset \mid \emptyset\,)\}\,)$$
$$(\,\{(\,\emptyset \mid \emptyset\,), (\,\emptyset \mid \{(\,\emptyset \mid \emptyset\,)\}\,)\} \mid \emptyset\,)$$

It is a non-trivial fact that $\mathcal{P}_\in$ is a poset. This will be proved after introducing some auxiliary notions:

Definition 3.2 *Any element $W \in (A \cup A_R) \cap (B \cup B_L)$ is called a* witness *of inequality $A < B$.*

The level of $A \in \mathcal{P}_\in$ *is defined as follows:*

$$\begin{aligned} l(\emptyset) &= 0; \\ l(A) &= max(l(B); B \in A_L \cup A_R) + 1 \textit{ for } A \neq \emptyset. \end{aligned}$$

We observe the following early facts (which follow directly from the above definitions):

Fact 1 *$X < A < Y$ for every $A \in \mathcal{P}_\in$, $X \in A_L$ and $Y \in A_R$.*

Fact 2 *$A \leq W^{AB} \leq B$ for any $A < B$ and witness W^{AB} of $A < B$.*

Fact 3 *Let $A < B$ and let W^{AB} to be witness of $A < B$. Then $l(W^{AB}) \leq \min(l(A), l(B))$ and either $l(W^{AB}) < l(A)$ or $l(W^{AB}) < l(B)$.*

First we prove transitivity of the strict inequality.

Lemma 3.1 *Relation $<$ is transitive for the class $\mathcal{P}_\in$.*

Proof. Assume that three elements A, B, C of $\mathcal{P}_\in$ satisfy $A < B < C$. We prove that $A < C$ holds as well. Let W^{AB} and W^{BC} to be witnesses of the inequalities $A < B$ and $B < C$ respectively. First we prove that $W^{AB} \leq W^{BC}$. We distinguish four cases (according to the definition of the witness):

1. $W^{AB} \in B_L$ and $W^{BC} \in B_R$. From Fact 1 follows that $W^{AB} < W^{BC}$.
2. $W^{AB} = B$ and $W^{BC} \in B_R$. Then W^{BC} is witness of the inequality $B < W^{BC}$ and thus $W^{AB} < W^{BC}$.

3. $W^{AB} \in B_L$ and $W^{BC} = B$. Inequality follows symmetrically to the previous case.

4. $W^{AB} = W^{BC} = B$.

In the last case B is the witness of the inequality $A < C$. Thus we may assume that $W^{AB} \neq W^{BC}$. Let W^{AC} be a witness of the inequality $W^{AB} < W^{BC}$. Finally we prove that W^{AC} is a witness of the inequality $A < C$. We distinguish three possibilities:

1. $W^{AC} = W^{AB} = A$.

2. $W^{AC} = W^{AB}$ and $W^{AC} \in A_R$.

3. $W^{AC} \in W_R^{AB}$, then also $W^{AC} \in A_R$ from the completeness property.

It follows that either $W^{AC} = A$ or $W^{AC} \in A_R$. Analogously $W^{AC} = C$ or $W^{AC} \in C_L$ and thus W^{AC} is the witness of inequality $A < C$. □

Lemma 3.2 *Relation $<$ is strongly antisymmetric on the class of elements of $\mathcal{P}_\in$.*

Proof. Assume that A and B, $A < B < A$, to be the counterexample with minimal $l(A) + l(B)$. Let W^{AB} be a witness of the inequality $A < B$ and W^{BA} a witness of reverse inequality. From Fact 2 it follows that $A \leq W^{AB} \leq B \leq W^{BA} \leq A \leq W^{AB}$. From the transitivity we know that $W^{AB} \leq W^{BA}$ and $W^{BA} \leq W^{AB}$.

Again we shall consider 4 possible cases:

1. $W^{AB} = W^{BA}$.

   From the disjointness of the sets A_L and A_R it follows that $W^{AB} = W^{BA} = A$. Symmetrically $W^{AB} = W^{BA} = B$ which is a contradiction.

2. Either $W^{AB} = A$ and $W^{BA} = B$ or $W^{AB} = B$ and $W^{BA} = A$.

   Then contradiction follows in both cases from the fact that $l(A) < l(B)$ and $l(B) < l(A)$ (By Fact 3).

3. $W^{AB} \neq A$, $W^{AV} \neq B$, $W^{AB} \neq W^{BA}$.

   Then $l(W^{AB}) < l(A)$ and $l(W^{AB}) < L(B)$. Additionally $l(W^{BA}) \leq l(A)$ and $l(W^{BA}) \leq l(B)$ and thus A and B is not the minimal counter example.

4. $W^{BA} \neq A$, $W^{BA} \neq B$, $W^{AB} \neq W^{BA}$.

   The contradiction follows symmetrically to the previous case from minimality of $l(A) + l(B)$.

□

Theorem 3.3 *$\leq$ is partial ordering on the class of elements of $\mathcal{P}_\in$.*

Proof. Reflexivity of the relation follows directly from the definition, transitivity and antisymmetry follows from Lemmas 3.1 and 3.2. □

Now we are ready to prove the main result of this section:

Theorem 3.4 *$\mathcal{P}_\in$ is the universal and homogeneous partially ordered class.*

First we show the following lemma:

Lemma 3.5 *$\mathcal{P}_\in$ has the extension property.*

Proof. Let M be a finite subset of the elements of $\mathcal{P}_\in$. We want to extend the partially ordered set induced by M by the new element X. This extension can be described by three subsets of M: M_- containing elements smaller than X, M_+ containing elements greater than X and M_0 containing elements incomparable with X. Since extended relation is partial order we have the following properties of these sets:

1. Any element of M_- is strictly smaller than any element of M_+;
2. $B \leq A$ for no $A \in M_-$, $B \in M_0$;
3. $A \leq B$ for no $A \in M_+$, $B \in M_0$;
4. M_-, M_+ and M_0 form a partition of M.

Put

$$\overline{M_-} = \bigcup_{B \in M_-} B_L \cup M_-;$$

$$\overline{M_+} = \bigcup_{B \in M_+} B_R \cup M_+.$$

We verify that the conditions 1., 2., 3., 4. still hold for $\overline{M_-}$, $\overline{M_+}$.

ad 1. We prove that any element of $\overline{M_-}$ is strictly smaller than any element of $\overline{M_+}$:

Let $A \in \overline{M_-}, A' \in \overline{M_+}$. We prove $A < A'$. By the definition of $\overline{M_-}$ there exists $B \in M_-$ such that either $A = B$ or $A \in B_L$. By the definition of $\overline{M_+}$ there exists $B' \in M_+$ such that either $A' = B'$ or $A' \in B'_R$. By the definition of $<$ we have $A \leq B$, $B < B'$ (by 1.) and $B' \leq A'$ again by the definition of $<$. It follows $A < A'$.

ad 2. We prove that $B \leq A$ for no $A \in \overline{M_-}$, $B \in M_0$:

Let $A \in \overline{M_-}, B \in M_0$ and let $A' \in M_-$ satisfies either $A = A'$ or $A \in A'_L$. We know that $B \not\leq A'$ and as $A \leq A'$ we have also $B \not\leq A$.

ad 3. We prove that $A \leq B$ for no $A \in \overline{M_+}$, $B \in M_0$;

We proceed similarly to *ad* 2.

ad 4. We prove that $\overline{M_-}$, $\overline{M_+}$ and M_0 are pairwise disjoint:

$\overline{M_-} \cap \overline{M_+} = \emptyset$ follows from 1. $\overline{M_-} \cap M_0 = \emptyset$ follows from 2. $\overline{M_+} \cap M_0 = \emptyset$ follows from 3.

It follows, that $A = (\, \overline{M_-} \mid \overline{M_+}\,)$ is element of $\mathcal{P}_\in$ with the desired inequalities to the elements in the sets M_- and M_+.

Obviously each element of M_- is smaller than A and each element of M_+ greater than A.

It remains to be shown that each $N \in M_0$ is incomparable with A. However we will run into problem here: it is possible that $A = N$. We can avoid this problem by first considering the set:

$$M' = \bigcup_{B \in M} B_R \cup M.$$

It is then easy to show that $B = (\, \emptyset \mid M'\,)$ is an element of $\mathcal{P}_\in$ strictly smaller than all elements of M.

Finally we construct the set $A' = (\, A_L \cup \{B\} \mid A_R\,)$. The set A' has the same properties with respect to the elements of the sets M_- and M_+ and differs from any set in M_0. It remains to be shown that A' is incomparable with N.

Assume for contrary, for example, that $N < A'$ and $W^{NA'}$ is the witness of the inequality. Then $W^{NA'} \in \overline{M_-}$ and $N \leq W^{NA'}$. Recall that $N \in M_0$. From 4. above and definition of A follows that $N < W^{NA'}$. From *ad* 2. above follows that there is no choice of elements such as $N < W^{NA'}$.

The case $N > A'$ is analogous. □

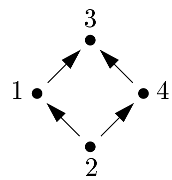

Figure 1: Partially ordered set P

Proof. Proof of Theorem 3.4 follows by combining Lemma 3.5 and Lemma 2.1. ☐

Example 3.1 *The above proof when applied to the poset P from Figure 1 (with the indicated order of elements) will proceed as follows:*

$$
\begin{aligned}
c(1) &= (\,\emptyset \mid \emptyset\,) \\
c(2) &= (\,\emptyset \mid \{(\,\emptyset \mid \emptyset\,)\}\,) \\
c(3) &= (\,\{(\,\emptyset \mid \emptyset\,), (\,\emptyset \mid \{(\,\emptyset \mid \emptyset\,)\}\,)\} \mid \emptyset\,) \\
c(4) &= (\,\{(\,\emptyset \mid \{(\,\emptyset \mid \emptyset\,)\}\,)\} \mid \{(\,\{(\,\emptyset \mid \emptyset\,), (\,\emptyset \mid \{(\,\emptyset \mid \emptyset\,)\}\,)\} \mid \emptyset\,)\}\,)
\end{aligned}
$$

Corollary 3.1 *All homogeneous posets are finitely presented.*

Proof. Using the Schmerl classification [20] and by remarks in Section 1 all homogeneous non-generic posets are finitely presented. The generic poset $\mathcal{P}$ is isomorphic to $\mathcal{P}_\in$ by Theorem 3.4.

It is not clear from Definition 3.1 that $\mathcal{P}_\in$ has finite representation (Definition 3.1 is recursive). We will now give the finite representation $\mathcal{P}_\mathcal{F}$:

Elements of $\mathcal{P}_\mathcal{F}$ are all finite partially ordered sets $(P, \leq_P)$ with the greatest element denoted by $m(P, \leq_P)$. In addition each $M \in P$ satisfies properties analogous to elements of $\mathcal{P}_\mathcal{F}$:

1. (correctness)

   i. 🍎 $\notin M$;

   ii. $M_L \cup M_R \subset P$;

   iii. $M_L \cap M_R = \emptyset$.

2. (ordering property) $(\{A\} \cup A_R) \cap (\{B\} \cup B_L) \neq \emptyset$ for each $A \in M_L, B \in M_R$;

3. (left completeness) $A_L \subseteq M_L$ for each $A \in M_L$;

4. (right completeness) $B_R \subseteq M_R$ for each $B \in M_R$;

The relation $<_P$ is the transitive closure of the set $\{(A,B); A \in B_L \cup B_R, B \in P\}$.

We prove that $P \subset \mathcal{P}_\in$ for each $(P,<_P)$. Suppose on the contrary that there is $A \in P$ such that $A \notin \mathcal{P}_\in$. Without loss of generality we may assume that there is no $B \in P$, $B \notin \mathcal{P}_\in$ such that $B <_P A$. From the definition of $<_P$ it follows that $C \in \mathcal{P}_\in$ for each $C \in A_L \cup A_R$. Thus for A we have 1. *ii*. in the definition above equivalent to the 1. *ii*. from the Definition 3.1. The rest of the definitions are equivalent, so we have $A \in \mathcal{P}_\in$.

The relation $\leq_{\mathcal{P}_\mathcal{F}}$ of $\mathcal{P}_\mathcal{F}$ is defined by comparison of the greatest elements:

$$(P,<_P) \leq_{\mathcal{P}_\mathcal{F}} (P',<_{P'}) \text{ iff } m(P,<_P) \leq m(P',<_{P'}) \text{ in } \mathcal{P}_\in.$$

This poset is finitely presented. We claim that the correspondence

$$\varphi : (P,<_P) \mapsto m(P,<_P)$$

is isomorphism of $\mathcal{P}_\mathcal{F}$ and $\mathcal{P}_\in$.

Clearly it suffices to prove that φ is bijective. This follows from the following two facts:

1. For each $(P,<_P)$ the set P contains all the elements of $\mathcal{P}_\in$ which appear in the construction of $m(P,<_P) \in \mathcal{P}_\in$. (This is the consequence of 1. *ii*. and both Definition 3.1 and definition above.)

2. For each $(P,<_P)$ the set P consists only of elements of $\mathcal{P}_\in$ which appear in the construction of $m(P,<_P)$.

   Let $A^1 <_P m(P,<_P)$. By definition of $<_P$ we have $A^1, A^2, \dots, A^t = m(P,<_P)$ such that $A^i \in A_L^{i+1} \cup A_R^{i+1}$. But as $m(P,<_P) \in \mathcal{P}_\in$ we aget also $A \in \mathcal{P}_\in$ by definition 3.1 *ii*.

So for different sets, the greatest elements are different and each $M \in \mathcal{P}_\in$ can be used as a greatest element to construct an element of $\mathcal{P}_\mathcal{F}$.

□

4 Conway's numbers

Definition 4.1 *A surreal number is a pair $x = \{x^L | x^R\}$, where every member of the sets x^L and x^L is a surreal number and every member of x^L is strictly lower than every member of x^R.*

We say that a surreal number x is lower than or equal to the surreal number y if and only if y is not less than any member of x^L and any member of y^R is not less than or equal to x.

We will denote the class of surreal numbers by $\mathbb{S}$.

The definition of surreal numbers is very close to the definition of $\mathcal{P}_\in$. Since the elements of $\mathcal{P}_\in$ are formally similar to $\mathbb{S}$, we can define new "Conway's inequality":

Definition 4.2 *For elements $A, B \in \mathcal{P}_\in$ we write $A \leq_{\mathbb{S}} B$, when there is no $l \in A_L$, $B \leq_{\mathbb{S}} l$ and no $r \in B_R$, $r \leq_{\mathbb{S}} X$.*

Theorem 4.1 *For any $A, B \in \mathcal{P}_\in$ $A < B$ implies $A <_{\mathbb{S}} B$.*

Proof. We proceed by induction on $l(A) + l(B)$.

For empty A and B the theorem holds as they are not comparable by $<$.

Let $A < B$ and W^{AB} be the witness. In the case $W^{AB} \neq A, B$, then $A <_{\mathbb{S}} W^{AB} <_{\mathbb{S}} B$ by induction. In the case $A \in B_L$, then $A <_{\mathbb{S}} B$ from definition of $<_{\mathbb{S}}$. □

Thus the surreal numbers $\mathbb{S}$ may be thought as a linear extension of $\mathcal{P}_\in$.

5 Concluding remarks

1. The above finite presentations can be extended to further classes of directed, oriented homogeneous graphs. This we intend to pursue in the near future.

2. In Theorem 3.4 we presented what we believe to be the first finite presentation of the generic homogeneous and universal poset.

One should stress that even the finite presentation of universal poset is a non-trivial question which presented a problem. The problem has been solved in category-theory context by [17] and [19]. However none of these structures is homogeneous. For example the extension properties of the class of finite graphs with the homomorphism order do not hold and also some difficult combinatorial problem (such as Hedetniemi's product conjecture) may be expressed as particular extension properties, [13].

3. We can also consider oriented graphs (i. e. antisymmetric relations). Let $\mathcal{O}$ denote the generic oriented graph. $\mathcal{O}$ has finite presentation $\mathcal{O}_\in$ which we obtain as a variant of $\overrightarrow{\mathcal{R}}_\in$: we say that M a is vertex of $\mathcal{O}_\in$, $M \in \overrightarrow{\mathcal{R}}_\in$ which satisfies $M_L \cap M_P = \emptyset$. (see Definition 2.4).

Further results of Section 2 may be modified accordingly.

4. The finite presentation of generic directed graph $\overrightarrow{\mathcal{R}}$ and of the generic oriented graph $\mathcal{O}$ may be used for finite presentation of the generic tournament $\mathcal{T}$.

Let $\mathcal{O}_\in$ be the finite presentation of $\mathcal{O}$ constructed in Remark 3. Denote by $\mathcal{O}_\mathbb{N}$ the arithmetic presentation of $\mathcal{O}_\in$. Explicitly, an integer n is a vertex of $\mathcal{O}_\mathbb{N}$ iff there exists an element M of $\mathcal{O}_\in$ such that $n = c(M)$. (Thus in addition to 2.3 we have that n does not contain 1's on both positions $2i$ and $2i+1$, $i \geq 1$.) Let n and n' be vertices of $\mathcal{O}_\in$. There is an edge from n to n' if and only if there are sets M and M' such as $c(M) = n$ and $c(M') = n'$ and there is edge from M to N' in $\mathcal{O}_\in$. Alternatively there is an edge from n to n' if there is 1 on $2n'$-th place of binary representation of n or on $(2n+1)$-th place of binary representation of n'.

We use the finite presentation $\mathcal{O}_\mathbb{N}$ of generic oriented graph $\mathcal{O}$ for the construction of a finite presentation $\mathcal{T}_\mathbb{N}$ of the generic tournament $\mathcal{T}$: An integer n is vertex of $\mathcal{T}_\mathbb{N}$ iff n is a vertex of $\mathcal{O}_\mathbb{N}$. The arcs of $\mathcal{T}_\mathbb{N}$ will be all arcs of $\mathcal{O}_\mathbb{N}$ together with pairs (n, n'), $n \leq n'$ for which (n', n) is not an arc of $\mathcal{O}_\mathbb{N}$.

$\mathcal{T}_\mathbb{N}$ is obviously a tournament. $\mathcal{T}_\mathbb{N}$ has the extension property by the same proof as above for Theorem 2.3 (the construction vertex M has the same properties in $\mathcal{T}_\mathbb{N}$ as in $\overrightarrow{\mathcal{R}}_\mathbb{N}$).

Thus we have:

Corollary 5.1 *All homogeneous tournaments are finitely presented.*

Proof. According to Lachlan's classification [11] (see also [3]) all homogeneous tournaments are C_3, $\mathbb{Q}$ (dense linear order), $S(2)$ (dense local order) and the generic tournament. Only $S(2)$ needs to be considered.

Intuitively, the tournament $S(2)$ can be seen as a circuit with edges forming a dense countable set of chords. The orientation is chosen in such a way that shorter chords are oriented clockwise.

One can check that $S(2)$ may be equivalently described as follows: The vertices of $S(2)$ are all rational numbers q, $0 \leq q < 1$. There is an arc (a, b) in $S(2)$ iff either $a < b < a + \frac{1}{2}$ or $a - 1 < b < a - \frac{1}{2}$.

□

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